

GROTHENDIECK-TO-LASCOUX EXPANSIONS

MARK SHIMOZONO AND TIANYI YU

ABSTRACT. We establish the conjecture of Reiner and Yong for an explicit combinatorial formula for the expansion of a Grothendieck polynomial into the basis of Lascoux polynomials. This expansion is a subtle refinement of its symmetric function version due to Buch, Kresch, Shimozono, Tamvakis, and Yong, which gives the expansion of stable Grothendieck polynomials indexed by permutations into Grassmannian stable Grothendieck polynomials. Our expansion is the K -theoretic analogue of that of a Schubert polynomial into Demazure characters, whose symmetric analogue is the expansion of a Stanley symmetric function into Schur functions. Our expansions extend to flagged Grothendieck polynomials.

1. INTRODUCTION

The Grothendieck polynomials \mathfrak{G}_w of Lascoux and Schützenberger [LasSc] are explicit polynomial representatives of the K -classes of structure sheaves of Schubert varieties in flag varieties. Reiner and Yong [ReY] conjectured an explicit combinatorial expansion of Grothendieck polynomials into the basis of Lascoux polynomials \mathfrak{L}_α [Las]. Our first main theorem (Theorem 1.1) gives a new¹ combinatorial formula for the Lascoux polynomials. This is used to prove our second main theorem (Theorem 1.9) which establishes the Reiner-Yong conjecture.

1.1. **Various expansions.** The Grothendieck-to-Lascoux expansion fits into a family of related expansions. The polynomials to be expanded are the cohomological and K -theoretic Schubert bases given by the Schubert polynomials \mathfrak{S}_w and the Grothendieck polynomials \mathfrak{G}_w respectively, and their symmetrized or stable versions, known as the Stanley symmetric functions F_w and Grothendieck symmetric functions (also known as stable Grothendieck polynomials) G_w . These are respectively expanded into type A Demazure characters (also called key polynomials) κ_α , Lascoux polynomials \mathfrak{L}_α , Schur functions s_λ , and Grassmannian Grothendieck symmetric functions G_λ .

$$\begin{array}{ccc}
 \mathfrak{S}_w & \xrightarrow{\text{symmetrize}} & F_w \\
 (a) \downarrow \text{expand} & & \text{expand} \downarrow (b) \\
 \kappa_\alpha & \xrightarrow{\text{symmetrize}} & s_\lambda
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathfrak{G}_w & \xrightarrow{\text{symmetrize}} & G_w \\
 (c) \downarrow \text{expand} & & \text{expand} \downarrow (d) \\
 \mathfrak{L}_\alpha & \xrightarrow{\text{symmetrize}} & G_\lambda
 \end{array}$$

cohomology

K -theory

Using the formalism of connective K -theory (equivalently, introducing a harmless grading parameter β into the Grothendieck polynomial), as we do in this article, all

Date: May 2021.

¹meaning, “not stated explicitly in the literature”. See Remark 1.2.

expansions specialize to their K -theoretic or cohomological counterparts by setting β to -1 or 0 respectively.

In chronological order, expansion (b) was established by Edelman and Greene [EG] via a Schensted-type insertion algorithm for reduced words. The expansion (a) was found by Lascoux and Schützenberger and proved in [RS]. Expansion (d) was established by Buch, Kresch, Shimozono, Tamvakis, and Yong [BKSTY] via Hecke insertion, which takes Hecke words as input. Expansion (c) is the topic of this article.

The expansion coefficients have geometric significance. The Stanley-to-Schur coefficients of the expansion (b) coincide with large rank affine Stanley to affine Schur coefficients [LLS, Prop. 9.17], which in turn coincide with Gromov-Witten invariants for the flag variety via Peterson’s Quantum Equals Affine Theorem [Pet] [LamSh] [LLMSSZ, Part 3, §10]. In another important application, specializing w to a Zelevinsky permutation, one obtains the classes of cohomological quiver loci, which were shown by Buch and Fulton [BF] to specialize to virtually all known variants of type A Schubert polynomials. In this context, (a) and (b) give the expansion of cohomology classes of equioriented type A quiver loci into sums of products of Schur functors evaluated at differences of bundles, each bundle coming from a node of the quiver [KMS, Theorem 7.14]. The expansion coefficients, called quiver constants, were the subject of Buch and Fulton’s Factor Sequence Conjecture [BF], which proposed a combinatorial formula for the quiver constants as the number of sequences of tableaux resulting from certain iterated factoring and multiplying operations. A variant of this conjecture was proved in [KMS, Cor. 8.23].

Expansions (c) and (d) give analogous expansions in K -theory [B2] [Mi]. That is, for Zelevinsky permutations one obtains the expansion of K -theory classes of equioriented type A quiver loci into sums of products of stable Grothendieck polynomials in differences of bundles.

The nonsymmetric expansions are subtle refinements of their symmetric counterparts. In the symmetric expansions there is a set of tableaux in which each tableau T in the set, gives a copy of s_λ or G_λ where λ is the shape of T . There is a corresponding term κ_α or \mathfrak{L}_α in the nonsymmetric expansion, but an additional datum must be supplied: a composition or extremal weight α in the symmetric group orbit of λ ; see (1.15) through (1.18). Such constructions assigning a composition to a tableau go by the general name of *key*. In the crystal graph of semistandard Young tableaux of shape λ , the left and right keys of the tableau T of shape λ are given by the final and initial directions of the corresponding Littelmann path whose highest weight vector is the directed line segment from the origin to λ . The initial direction indicates the smallest Demazure crystal containing the given tableau.

1.2. Grothendieck and Lascoux polynomials. The group $S_+ = \bigcup_{n \geq 1} S_n$ acts on $R = \mathbb{Z}[\beta][x_1, x_2, \dots]$ by permuting the variables: for $i \geq 1$ let s_i exchange x_i and x_{i+1} . We define the following operators on R , where an element $f \in R$ (or its

fraction field) denotes the operator of left multiplication by f .

$$(1.1) \quad \partial_i = (x_i - x_{i+1})^{-1}(1 - s_i)$$

$$(1.2) \quad \pi_i = \partial_i x_i$$

$$(1.3) \quad \partial_i^{(\beta)} = \partial_i(1 + \beta x_{i+1})$$

$$(1.4) \quad \pi_i^{(\beta)} = \partial_i^{(\beta)} x_i.$$

All satisfy the braid relations for S_+ .

We have the operator identity

$$(1.5) \quad \partial_i f = \partial_i(f) + s_i(f)\partial_i \quad \text{for all } f \in R.$$

The operators satisfy the quadratic relations

$$(1.6) \quad \partial_i^2 = 0$$

$$(1.7) \quad \pi_i^2 = \pi_i$$

$$(1.8) \quad (\partial_i^{(\beta)})^2 = -\beta \partial_i^{(\beta)}$$

$$(1.9) \quad (\pi_i^{(\beta)})^2 = \pi_i^{(\beta)}.$$

Let $w_0^{(n)} \in S_n$ be the long element and $\rho^{(n)} = (n-1, n-2, \dots, 1, 0)$. For $w \in S_n$ the β -Grothendieck polynomial is defined by [\[LasSc\]](#)

$$(1.10) \quad \mathfrak{G}_w^{(\beta)} = \begin{cases} x^{\rho^{(n)}} & \text{if } w = w_0^{(n)} \\ \partial_i^{(\beta)} \mathfrak{G}_{ws_i}^{(\beta)} & \text{if } ws_i > w. \end{cases}$$

Since the $\partial_i^{(\beta)}$ satisfy the braid relations, $\mathfrak{G}_w^{(\beta)}$ is well-defined for $w \in S_n$. It is also well-defined for $w \in S_+$, that is, unchanged under the standard embedding $S_n \rightarrow S_{n+1}$ for all $n \geq 1$. The Schubert \mathfrak{S}_w and Grothendieck polynomials \mathfrak{G}_w are defined by

$$(1.11) \quad \mathfrak{S}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=0}$$

$$(1.12) \quad \mathfrak{G}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=-1}.$$

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a composition (sequence of nonnegative integers, almost all 0). The Lascoux polynomial $\mathfrak{L}_\alpha^{(\beta)}$ is defined by [\[Las\]](#)

$$(1.13) \quad \mathfrak{L}_\alpha^{(\beta)} = \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition} \\ \pi_i^{(\beta)} \mathfrak{L}_{s_i \alpha}^{(\beta)} & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

The Demazure character κ_α is defined by

$$(1.14) \quad \kappa_\alpha = \mathfrak{L}_\alpha^{(\beta)}|_{\beta=0}.$$

Given a composition $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ let α^+ be the unique partition in the S_n -orbit of α . For $w \in S_n$ and $w_0 \in S_n$ the long element we have the symmetrizations

$$(1.15) \quad G_{\alpha^+}^{(\beta)}(x_1, \dots, x_n) = \pi_{w_0}^{(\beta)}(\mathfrak{L}_\alpha)$$

$$(1.16) \quad s_{\alpha^+}(x_1, \dots, x_n) = \pi_{w_0}(\kappa_\alpha)$$

$$(1.17) \quad G_w(x_1, \dots, x_n) = \pi_{w_0}^{(\beta)}(\mathfrak{G}_w(x))$$

$$(1.18) \quad F_w(x_1, \dots, x_n) = \pi_{w_0}(\mathfrak{S}_w(x)).$$

1.3. New tableau formula for Lascoux polynomials. Given the definition of a certain kind of tableau which involves entries in a totally ordered set, we say “reverse” to mean the same definition but with the total order reversed. So a *reverse semistandard Young tableau* (RSSYT) is a tableau in which the entries weakly decrease along rows from left to right and strictly decrease along columns from top to bottom.

For a partition λ a *reverse set-valued tableau* (RSVT) T of shape λ is a filling of the boxes of λ by finite subsets of $\mathbb{Z}_{>0}$ satisfying the following. For the box $s \in \lambda$ let $T(s)$ be the set which occupies the box s in T .

- (1) $\min(T(s)) \geq \max(T(t))$ if the box t is immediately right of the box s in λ .
- (2) $\min(T(s)) > \max(T(t))$ if the box t is immediately below the box s in λ .

This is the reverse of Buch’s set-valued tableaux [B].

Given a RSVT T , let $L(T)$ be the RSSYT obtained from T by replacing every entry $T(s)$ by its largest value $\max(T(s))$.

The *weight* $\text{wt}(T)$ of a tableau T is the composition whose i -th part is the total number of times i appears in T .

The *left key* $K_-(T)$ of a RSSYT T is a composition computed by the algorithm in §3.2. Up to an order-reversal bijection this is equivalent to the right key of a SSYT defined by Lascoux and Schützenberger [LasSc3]; our algorithm is a variant of that of Willis [Wil]. Let $|\alpha| = \sum_{i \geq 1} \alpha_i$. Let RSVT_λ be the set of reverse set-valued tableaux of shape λ . For $T \in \text{RSVT}_\lambda$ let $\text{ex}(T) = |\text{wt}(T)| - |\lambda|$.

Our first main theorem is:

Theorem 1.1. *For any composition α*

$$(1.19) \quad \mathfrak{L}_\alpha^{(\beta)} = \sum_{\substack{T \in \text{RSVT}_{\alpha^+} \\ K_-(L(T)) \leq \alpha}} \beta^{\text{ex}(T)} x^{\text{wt}(T)}$$

Here \leq indicates the quotient of Bruhat order on the orbit $S_+\alpha$.

Another way to state the \leq relation on compositions is the following. A *key tableau* (or just key) is a SSYT² of partition shape such that the j -th column, viewed as a set, contains the $(j+1)$ -th for all j . There is a bijection $\alpha \mapsto \text{key}(\alpha)$ from compositions to key tableaux where $\text{key}(\alpha)$ is the unique SSYT of shape α^+ and weight α . Its j -th column consists of the numbers $\{i \mid \alpha_i \geq j\}$ for all j .

Then for compositions α and β , $\alpha \leq \beta$ if and only if $\alpha^+ = \beta^+$ and $\text{key}(\alpha)$ is entrywise less than or equal to $\text{key}(\beta)$.

Theorem 1.1 is proved in §3.

Remark 1.2. There have been a number of conjectural combinatorial formulas for Lascoux polynomials, such as the K-Kohnert move rule of Ross and Yong [RoY] ([Kir, Footnote on p. 19] for the general β version), the set-valued skyline filling formula of Monical [Mon], and a set-valued tableau (SVT) rule of Pechenik and Scrimshaw [PeSc] which requires the fairly involved Lusztig involution on a crystal structure on SVTs in addition to an entrywise minimum and the usual right key of a SSYT. Buciumas, Scrimshaw and Weber [BSW] proved the last two of these rules using solvable lattice models. In response to a previous version of this article, Travis Scrimshaw kindly informed us that Theorems 1.1 and 1.3 are implicit in [BSW]: see the proof of [BSW, Theorem 4.4]. We feel it is worthwhile to state these theorems

²Sometimes we will write a key tableau as a RSSYT depending on context.

in their simplest and most explicit form. We note that the naive nonreversed SVT analogue of the RSSYT formula does not yield the Lascoux polynomial.

Equation (1.19) can be restated using only RSSYTs, avoiding SVT altogether. Let RSSYT_λ be the set of reverse semistandard tableaux of shape λ .

Theorem 1.3. *For any composition α*

$$(1.20) \quad \mathfrak{L}_\alpha^{(\beta)} = \sum_{\substack{T \in \text{RSSYT}_{\alpha^+} \\ K_-(T) \leq \alpha}} x^{\text{wt}(T)} \prod_{(s,k)} (1 + \beta x_k)$$

where the product runs over pairs (s, k) where s is a box of α^+ , $k < T(s)$, and replacing the s -th entry of T by k results in a RSSYT.

Remark 1.4. The naive nonreversed SSYT analogue of Theorem 1.3 also does not give the Lascoux polynomial.

Remark 1.5. The condition on the pairs (s, k) should be compared with the formula for Grothendieck polynomials indexed by vexillary permutations in part 3 of the second Corollary in section 1.2 of [KMY].

1.4. Stable limit of Lascoux polynomials. For a fixed composition α consider the limit $\lim_{N \rightarrow \infty} \mathfrak{L}_{(0^N, \alpha)}^{(\beta)}$ in which more zero parts are prepended to α . In Theorem 1.1 it is evident that the limit depends only on $\lambda = \alpha^+$ and it is given by removing the left key condition. By the definition of Lascoux polynomial, this limit can be computed by $\pi_i^{(\beta)}$ operators on x^λ . One may show that

$$\mathfrak{L}_{w_0^{(n)}}^{(\beta)} = G_\lambda^{(\beta)}(x_1, \dots, x_n) \quad \text{for } \lambda = (\lambda_1, \dots, \lambda_n)$$

where $G_\lambda^{(\beta)}$ is the Grassmannian Grothendieck symmetric function, which will be defined in §1.6.

We deduce that the above limit coincides with the symmetric series $G_\lambda^{(\beta)}$ on one hand and the generating function of RSSYT of shape λ on the other. This is the reversed version of Buch's SVT formula for $G_\lambda^{(\beta)}$ [B]; see (1.30).

1.5. Fomin-Kirillov monomial formula. Our point of departure is the explicit monomial expansion of Grothendieck polynomials due to Fomin and Kirillov [FK].

The 0-Hecke monoid \mathcal{H} is the quotient of the free monoid of words on the alphabet $\mathbb{Z}_{>0}$ by the relations

$$(1.21) \quad i(i+1)i \equiv_H (i+1)i(i+1)$$

$$(1.22) \quad ii \equiv_H i$$

$$(1.23) \quad ij \equiv_H ji \quad \text{for } |i-j| \geq 2.$$

\mathcal{H} acts on S_+ by

$$i * w = \begin{cases} s_i w & \text{if } s_i w > w \\ w & \text{if } s_i w < w. \end{cases}$$

Given a word $u \in \mathcal{H}$ define its *associated permutation* by $u * \text{id} \in S_+$. For $w \in S_+$ let \mathcal{H}_w be the words in \mathcal{H} with associated permutation w . The subsets $\mathcal{H}_w \subset \mathcal{H}$ are the \equiv_H -equivalence classes.

Lemma 1.6. *$u \in \mathcal{H}_w$ if and only if $\text{rev}(u) \in \mathcal{H}_{w^{-1}}$.*

For $a \in \mathcal{H}_w$ let $\text{ex}(a) = \text{length}(a) - \ell(w)$, the *excess* of the length of a above the minimum possible, the Coxeter length $\ell(w)$ of w .

The following is merely the definition in [BJS] but with both words reversed, which is better suited to our use of decreasing tableaux.

Definition 1.7. [BJS] A pair of words (a, i) is *compatible* if they satisfy

- (1) a, i are words of positive numbers with the same length.
- (2) i is weakly decreasing
- (3) $i_j = i_{j+1}$ implies $a_j < a_{j+1}$.

A compatible pair (a, i) is *bounded* if

$$(1.24) \quad i_j \leq a_j \quad \text{for all } j.$$

Let \mathcal{C} be the set of all compatible pairs, \mathcal{C}^b those that are bounded, \mathcal{C}_w the pairs $(a, i) \in \mathcal{C}$ such that $a \in \mathcal{H}_w$, and $\mathcal{C}_w^b = \mathcal{C}^b \cap \mathcal{C}_w$.

The following monomial expansion of β -Grothendieck polynomials is due to Fomin and Kirillov [FK].

$$(1.25) \quad \mathfrak{G}_w^{(\beta)} = \sum_{(a,i) \in \mathcal{C}_{w^{-1}}^b} \beta^{\text{ex}(a)} x^{\text{wt}(i)}$$

When $\beta = 0$ this is the Billey-Jockusch-Stanley formula for Schubert polynomials [BJS].

For $w \in S_n$ and a positive integer N let $1^N \times w$ be the permutation of S_{n+N} obtained by adding N fixed points before w . The β -Grothendieck symmetric function is defined by

$$(1.26) \quad G_w^{(\beta)} = \lim_{N \rightarrow \infty} \mathfrak{G}_{1^N \times w}$$

It lives in a completion of the ring of symmetric functions over $\mathbb{Z}[\beta]$. The Stanley and Grothendieck symmetric functions are defined by

$$(1.27) \quad F_w = G_w^{(\beta)}|_{\beta=0}$$

$$(1.28) \quad G_w = G_w^{(\beta)}|_{\beta=-1}.$$

It follows from (1.25) and the definitions that

$$(1.29) \quad G_w^{(\beta)} = \sum_{(a,i) \in \mathcal{C}_{w^{-1}}} \beta^{\text{ex}(a)} x^{\text{wt}(i)}.$$

Comparing with (1.25) just the boundedness condition has been dropped.

1.6. $G_w^{(\beta)}$ to $G_\lambda^{(\beta)}$ via Hecke insertion: restriction of compatible pairs according to w . The code $c(w)$ of a permutation is the sequence (c_1, c_2, \dots) such that $c_i = |\{j \mid 1 \leq j < w(i) \text{ and } w^{-1}(j) > i\}|$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ the *Grassmannian Grothendieck symmetric function* $G_\lambda^{(\beta)}$ is by definition equal to $G_w^{(\beta)}$ where w is the permutation with code $(\lambda_k, \dots, \lambda_2, \lambda_1, 0, 0, \dots)$.

Buch [Bu] showed that the $\mathbb{Z}[\beta]$ -span of the $G_w^{(\beta)}$ for $w \in S_+$, has basis given by the $G_\lambda^{(\beta)}$ and proved the increasing version of the following:

$$(1.30) \quad G_\lambda^{(\beta)} = \sum_{T \in \text{RSSYT}_\lambda} \beta^{\text{ex}(T)} x^{\text{wt}(T)}$$

For $\beta = 0$ this becomes the RSSYT formula for the Schur function s_λ .

To find the coefficients of the $G_w^{(\beta)}$ to $G_\lambda^{(\beta)}$ expansion, the *Hecke insertion* algorithm was developed in [BKSTY] in the language of *increasing tableaux*, which strictly increase along rows from left to right and strictly increase along columns from top to bottom. In §2.3 we recall these definitions but use the variant of Hecke insertion for *decreasing tableaux*, which are the “reverse” of increasing: they strictly decrease along rows from left to right and strictly decrease along columns from top to bottom.

It was not explicitly stated in [BKSTY] but all ingredients are there to define a Hecke Robinson-Schensted-Knuth (RSK) bijection called *Insert* (and its inverse bijection *RevInsert*)

$$\mathcal{C} \begin{array}{c} \xrightarrow{\text{Insert}} \\ \xleftarrow{\text{RevInsert}} \end{array} \bigsqcup_\lambda (\text{Dec}_\lambda \times \text{RSVT}_\lambda) =: \mathcal{T}$$

$$(a, i) \begin{array}{c} \xrightarrow{\hspace{2cm}} \\ \xleftarrow{\hspace{2cm}} \end{array} (P, Q)$$

where Dec_λ is the set of decreasing tableaux of shape λ . Note that the set \mathcal{T} as defined by the above diagram, consists of pairs (P, Q) of tableaux of the same partition shape with P decreasing and Q reverse set-valued.

Let $(P, Q) = \text{Insert}(a, i)$. By Proposition 2.18 the bijection satisfies

$$(1.31) \quad \text{rev}(a) \equiv_K P$$

$$(1.32) \quad \text{wt}(Q) = \text{wt}(i)$$

where the relation \equiv_K is defined in §4.1. Note that \equiv_K refines the relation \equiv_H of §1.5. By Lemma 1.6 and (1.31) the bijection *Insert* restricts to a bijection

$$(1.33) \quad \mathcal{C}_{w^{-1}} \leftrightarrow \bigsqcup_\lambda (\text{Dec}_\lambda^w \times \text{RSVT}_\lambda) =: \mathcal{T}_w$$

where $\text{Dec}_\lambda^w = \{T \in \text{Dec}_\lambda \mid \text{word}(T) \in \mathcal{H}_w\}$ and $\text{word}(T)$ is defined in §2.1; see [BKSTY] for the increasing tableau version. Taking the generating function of both sides we obtain

$$(1.34) \quad G_w^{(\beta)} = \sum_\lambda |\text{Dec}_\lambda^w| G_\lambda^{(\beta)}.$$

1.7. $\mathfrak{S}_w^{(\beta)}$ to $\mathfrak{L}_\alpha^{(\beta)}$ by Hecke insertion and keys: restriction to bounded compatible pairs. The original definition of right key of a SSYT was given by Lascoux and Schützenberger [LasSc3], who used it to give a tableau rule for a type A Demazure character. This definition is phrased in terms of Knuth equivalence. Willis [Wil] gave an alternative formulation of the right key of a SSYT. We give an algorithm to compute the right key $K_+(T)$ of a RSSYT T in Definition 4.10, which is essentially equivalent to Willis’. However we find a novel use for this construction: we show that when it is applied to the special kind of RSSYT given by a decreasing tableau, it is compatible with Hecke insertion and the K-Knuth equivalence \equiv_K . In other words, the same construction plays a similar role for two different equivalence relations on words. The *K-jeu-de-taquin* (Kjdt) of Thomas and Yong [TY] may be used (following [ReY] for increasing tableaux) to give another definition of right key of decreasing tableau, and two definitions of right key of decreasing tableau are shown to coincide (Proposition 4.12). For our amusement we give a third formulation of right key for a decreasing tableau in §6 using the

“transpose” of Hecke reverse insertion; in the context of semistandard tableaux and Knuth equivalence, this formulation is analogous to the original definition of Lascoux and Schützenberger.

Let \mathcal{T}^b be the subset of pairs $(P, Q) \in \mathcal{T}$ such that $K_+(P) \geq K_-(L(Q))$. With \mathcal{T}_w defined as in (1.33) let $\mathcal{T}_w^b = \mathcal{T}^b \cap \mathcal{T}_w$.

In §4 we show the following (Theorem 4.2):

Theorem 1.8. *Insert restricts to a bijection $\mathcal{C}^b \cong \mathcal{T}^b$.*

Intersecting with the bijection (1.33), Insert restricts to a bijection

$$(1.35) \quad \mathcal{C}_{w^{-1}}^b \cong \mathcal{T}_w^b \quad \text{for every } w \in S_+.$$

Using Theorem 1.1 we obtain our second main theorem, the Grothendieck-to-Lascoux expansion via decreasing tableaux.

Theorem 1.9.

$$(1.36) \quad \mathfrak{G}_w^{(\beta)} = \sum_{\lambda} \sum_{P \in \text{Dec}_{\lambda}^w} \mathfrak{L}_{\text{wt}(K_+(P))}^{(\beta)}.$$

Proof. We have

$$\begin{aligned} \mathfrak{G}_w^{(\beta)} &= \sum_{(a,i) \in \mathcal{C}_{w^{-1}}^b} \beta^{\text{ex}(a)} x^{\text{wt}(i)} \\ &= \sum_{(P,Q) \in \mathcal{T}_w^b} \beta^{\text{ex}(Q)} x^{\text{wt}(Q)} \\ &= \sum_{\lambda} \sum_{P \in \text{Dec}_{\lambda}^w} \sum_{\substack{Q \in \text{RSVT}_{\lambda} \\ K_-(Q) \leq K_+(P)}} \beta^{\text{ex}(Q)} x^{\text{wt}(Q)} \\ &= \sum_{\lambda} \sum_{P \in \text{Dec}_{\lambda}^w} \mathfrak{L}_{\text{wt}(K_+(P))}^{(\beta)} \end{aligned}$$

using (1.25), the extra- and weight-preserving bijection (1.35), and Theorem 1.1. \square

1.8. Connecting with the Reiner-Yong conjecture. The Reiner-Yong conjecture asserts:

Conjecture 1.10. [ReY]

$$(1.37) \quad \mathfrak{G}_w^{(\beta)} = \sum_{\lambda} \sum_{P \in \text{Inc}_{\lambda}^{w^{-1}}} \mathfrak{L}_{\text{wt}(K_-(P))}^{(\beta)}$$

where Inc_{λ}^w is similar to Dec_{λ}^w except that the tableaux are increasing and $K_-(P)$ is the left key construction on the increasing tableau P using the $Kjdt$.

Proof. By Propositions 2.10 and 5.2 the map $T \mapsto T^{\sharp}$ defines a bijection $\text{Dec}_{\lambda} \rightarrow \text{Inc}_{\lambda}$ such that $K_+(T) = K_-(T^{\sharp})$ and $\text{word}(T^{\sharp}) \equiv_K \text{rev}(\text{word}(T))$. Using Lemma 1.6 we see that $T \in \mathcal{H}_w$ if and only if $T^{\sharp} \in \mathcal{H}_{w^{-1}}$. Thus the bijection restricts to a bijection $\text{Dec}_{\lambda}^w \cong \text{Inc}_{\lambda}^{w^{-1}}$ as required. \square

1.9. Cohomological case. Say that a word is *reduced* if it is a reduced word for some permutation in S_+ . Say that a tableau is reduced if its word is. Let $\mathcal{C}^b(0)$ denote the set of compatible pairs (a, i) such that a is reduced. Let $\mathcal{T}^b(0)$ be the set of pairs (P, Q) of tableaux of the same partition shape such that P is a reduced decreasing tableau and Q is a RSSYT with $K_+(P) \geq K_-(Q)$. By setting $\beta = 0$ in Theorem 1.8 we obtain the following.

Theorem 1.11. (1) *The decreasing analogue of Lascoux and Schützenberger’s right nil key of a reduced decreasing tableau P (see left nil key for reduced increasing tableaux in [RS]) agrees with $K_+(P)$.*
 (2) *The restriction of Insert gives a bijection $\mathcal{C}^b(0) \cong \mathcal{T}^b(0)$. In this case Insert is computed by Edelman-Greene column insertion into reduced decreasing tableaux, recorded by RSSYT.*

This recovers the Schubert to Demazure expansion [LasSc2] [RS].

Remark 1.12. The Demazure crystal structure on SSYT was essentially known to Lascoux and Schützenberger [LasSc3]. Theorem 1.11 clarifies the Demazure crystal structure in the Schubert expansion: it is pulled back via Edelman-Greene column insertion from the Lascoux-Schützenberger Demazure crystal structure on the semistandard Q -tableau. See also [AS].

1.10. Plactic variant. Let $\tilde{\mathcal{C}}$ be the set of Knuth biwords, which are defined by changing the third condition of Definition 1.7 to: $i_j = i_{j+1}$ implies $a_j \leq a_{j+1}$. Let $\tilde{\mathcal{T}}$ be the set of pairs (P, Q) of RSSYT of the same partition shape. Let $\tilde{\mathcal{C}}^b$ be the bounded compatible pairs in $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{T}}^b$ the pairs $(P, Q) \in \tilde{\mathcal{T}}$ such that $K_+(P) \geq K_-(Q)$ using the classical definition of left and right key in [LasSc3] but adapted to RSSYT.

Theorem 1.13. *The value-reversed version of column insertion RSK yields a bijection $\tilde{\mathcal{C}}^b \cong \tilde{\mathcal{T}}^b$.*

See Remark 4.23 for more details. This result applies to the representation theory of the general linear group over a non-archimedean local field [GLS].

1.11. Flagged Grothendieck to Lascoux. In this subsection we extend our expansion to flagged Grothendieck polynomials.

In the literature there is a definition of flagged Grothendieck polynomial whose generality extends to the case of 321-avoiding permutations [Mat]; see [KMY2] for the case of vexillary permutations. For 321-avoiding permutations there is a monomial tableau formula and a determinantal formula.

We use a divided difference definition of flagged Grothendieck polynomial from [LLS2] which is valid for any permutation. This flagged Grothendieck polynomial has an explicit monomial expansion given in Proposition 1.14.

The main result of this subsection is a Lascoux polynomial expansion of flagged Grothendiecks.

A *flag* is a sequence of integers $f = (f_1, f_2, \dots, f_n)$ which is weakly increasing, satisfies $f_i \geq i$ for all i , and $f_n = n$. Let $f_{\min} = (1, 2, \dots, n)$ and $f_{\max} = (n, n, \dots, n)$ be the minimum and maximum flags respectively. Given a flag f , define the permutation $\sigma_f \in S_n$ as follows. For the minimum flag f_{\min} let $\sigma_{f_{\min}} = \text{id}$. For $f \neq f_{\min}$ there is an index j such that $f_j > j$; take the minimum such. Define $\sigma_f = s_i \sigma_f$

where $i + 1 = f_j$ and f' is obtained from f by replacing the $i + 1$ by i . The *flagged Grothendieck polynomial* is defined by $\mathfrak{G}_{w,f}^{(\beta)} = \pi_{\sigma_f}^{(\beta)}(\mathfrak{G}_w^{(\beta)})$.

The flagged Grothendieck polynomials have the following explicit monomial expansion.

Proposition 1.14. [LLS2]

$$(1.38) \quad \mathfrak{G}_{w,f}^{(\beta)} = \sum_{\substack{(a,i) \in \mathcal{C}_w \\ i_k \leq f_{a_k}}} \beta^{\text{ex}(a)} x^{\text{wt}(i)}$$

Remark 1.15. Note that only the bound $i_k \leq a_k$ in (1.25) has been changed to $i_k \leq f_{a_k}$.

The flagged Grothendieck polynomials interpolate between Grothendieck polynomials and their symmetric counterparts.

Corollary 1.16. For $w \in S_n$

$$(1.39) \quad \pi_{w_0}^{(\beta)}(\mathfrak{G}_w^{(\beta)}) = G_w^{(\beta)}(x_1, \dots, x_n).$$

Proof. We have

$$\begin{aligned} \pi_{w_0}^{(\beta)}(\mathfrak{G}_w^{(\beta)}) &= \pi_{w_0}^{(\beta)}(\mathfrak{G}_{w,f_{\min}}^{(\beta)}) \\ &= \mathfrak{G}_{w,f_{\max}}^{(\beta)} \\ &= G_w^{(\beta)}(x_1, \dots, x_n) \end{aligned}$$

where the last equality holds by the equality of (1.38) with (1.29) with x_i set to 0 for $i > n$. \square

Define the Demazure action \circ of S_+ on compositions by

$$(1.40) \quad s_i \circ \alpha = \begin{cases} s_i(\alpha) & \text{if } \alpha_i > \alpha_{i+1} \\ \alpha & \text{otherwise.} \end{cases}$$

Theorem 1.9 implies the following.

Corollary 1.17.

$$(1.41) \quad \mathfrak{G}_{w,f}^{(\beta)} = \sum_{\lambda} \sum_{P \in \text{Dec}_{\lambda}^w} \mathfrak{G}_{\sigma_f \circ \text{wt}(K_+(P))}^{(\beta)}$$

Acknowledgements. The authors thank Alex Yong for helpful conversations and especially for sharing with us the details of his conjecture with Vic Reiner. M. S. thanks Tomoo Matsumura for help related to flagged Grothendieck polynomials. T. Y. thanks Brendon Rhoades for helpful conversations.

2. BACKGROUND

2.1. Partitions, tableaux, words. A skew shape is *normal* (resp. *antinormal*) if it is empty or has a unique northwestmost (resp. southeastmost) corner. Given a skew shape D let D^* be a skew shape obtained by 180 degree rotation of D .

For a tableau T let $\text{word}(T)$ be the *column-reading word* of T , obtained by reading the first column of T from bottom to top, then the second column of T from bottom to top, and so on. Thus the word of a column of an increasing (resp. decreasing) tableau is a decreasing (resp. increasing) word. We shall often make no

distinction between a column, its word, and the underlying set. Similarly we may write T and mean $\text{word}(T)$.

2.2. Thomas and Yong's K -theoretic jeu-de-taquin (Kjdt).

2.2.1. *Kjdt.* We will define the Kjdt using the language of increasing tableaux as it is more natural. But the same definitions apply to decreasing tableaux in the same way; one is just labeling the chains of partitions differently.

Say that the skew shapes D and E are *successive* if there are partitions $\nu \subset \mu \subset \lambda$ such that $D = \mu/\nu$ and $E = \lambda/\nu$. In that case let $D + E = \lambda/\nu$. We also say that E *extends* D .

A *rook strip* is a skew shape that is both a *horizontal strip* (has at most one box in each column) and a *vertical strip* (has at most one box in each row). Say that a skew shape is *thin* if it is the sum of two successive rook strips. Note that a skew shape is thin if and only if it has no 2×2 subdiagram and has at most two boxes in any row and column. When decomposing a thin shape into a sum of successive rook strips the only choice is for isolated boxes, which can be in either the inner or outer rook strip.

Let D and E be successive rook strips. The *switch* of (D, E) is the unique successive pair of rook strips (E', D') such that $E' + D' = D + E$, every isolated box of D is in D' and every isolated box of E is in E' .

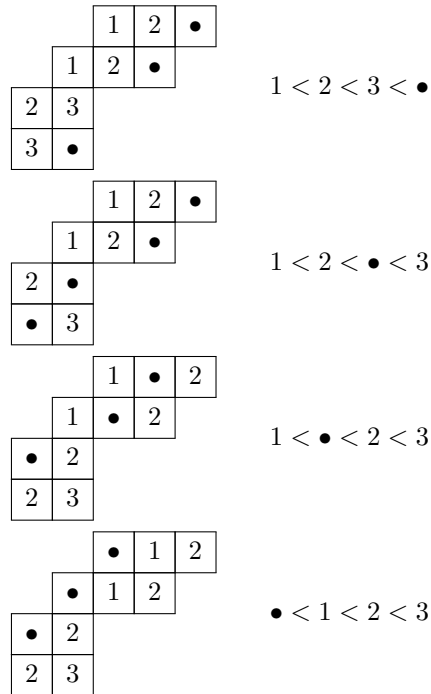
Example 2.1. In the following the boxes of D and D' are filled with \circ and those of E and E' are filled with \bullet .

$$D + E = \begin{array}{cccc} & & & \bullet \\ & & \circ & \\ & \circ & \bullet & \\ \circ & & & \end{array} \leftrightarrow \begin{array}{cccc} & & & \bullet \\ & & \bullet & \\ & \bullet & \circ & \\ \circ & & & \end{array} = E' + D'$$

An increasing skew tableau T of shape D can be viewed as a sequence of successive rook strips D_1, D_2, \dots, D_n with $D = D_1 + D_2 + \dots + D_n$: the boxes of D_i are filled with i . Let E be a rook strip which extends D . The *forward Kjdt* $J_E^{\searrow}(T)$ is the increasing tableau defined by switching E past all the D_i . More precisely let $E^{(n)} = E$ and define $\text{switch}(D_i, E^{(i)}) = (E^{(i-1)}, D'_i)$ for all i going from n down to 1. Then $J_E^{\searrow}(T)$ is the increasing tableau defined by the successive rook strips D'_1, D'_2, \dots, D'_n . The *vacated rook strip* $V_E^{\searrow}(T)$ is by definition $E^{(0)}$.

Example 2.2. Here is an example of $J_E^{\searrow}(T)$ where the boxes of $E^{(i)}$ are filled with \bullet and D_i and D'_i are filled with i 's. E is filled with \bullet in the first diagram. T and T' are the skew increasing tableaux in the first and last diagrams respectively, where

one ignores the \bullet 's.



Note that we can think of all the intermediate “tableaux” as being increasing with respect to the total orders indicated to the right.

Suppose T and U are increasing tableaux whose shapes are successive. Let T have rook strip decomposition $D_1 + D_2 + \cdots + D_n$ and let U be $E_1 + E_2 + \cdots + E_m$. Then define $J_U^{\searrow}(T)$ to be the increasing tableau given by switching T past E_1 using $J_{E_1}^{\searrow}$ then past E_2 by $J_{E_2}^{\searrow}$ etc. Let $V_U^{\searrow}(T)$ be the sequence of vacated rook strips. Both are increasing tableaux.

Example 2.3. Using T from Example 2.2 let U be the increasing skew tableau filled with a 's and b 's. T is first switched past the a 's as in Example 2.2 and then past

the b 's.

		1	2	a		
	1	2	a	b		
2	3	b				
3	a					
		a	1	2		
	a	1	2	b		
a	2	b				
2	3					
		a	1	2		
	a	1	2	b		
a	2	b				
2	3					
		a	1	b		
	a	1	b	2		
a	b	2				
2	3					
		a	b	1		
	a	b	1	2		
a	b	2				
2	3					

$1 < 2 < 3 < a < b$

$a < 1 < 2 < 3 < b$

$a < 1 < 2 < b < 3$

$a < 1 < b < 2 < 3$

$a < b < 1 < 2 < 3$

The final resting place of the numbers is $J_U^{\searrow}(T)$ and that of the letters is $V_U^{\searrow}(T)$.

Instead of switching into rook strips that are to the outside of our skew increasing tableau, we can switch into rook strips lying on the inside. Given a rook strip D and increasing tableau U of a shape E which extends D , let $J_D^{\nwarrow}(U)$ be the increasing skew tableau obtained by switching D_1 with the first rook strip of U , then switching with the second rook strip of U , and so on. Let $V_D^{\nwarrow}(U)$ be the rook strip vacated by this process. Here U moves to the northwest. Similarly if T and U are skew increasing tableaux whose shapes are successive, we may define $J_T^{\nwarrow}(U)$ by switching U to the northwest, past the last strip of T , then the next to last, and so on. Let $V_T^{\nwarrow}(U)$ be the increasing tableau defined by the sequence of rook strips vacated by this process.

It is equivalent to think of the inner tableau as a sequence of switching instructions to apply to the outer, as it is to think of the outer tableau as a sequence of switching instructions to apply to the inner. Both give the same result.

Theorem 2.4. [TY, Theorem 3.1] *Let T and U be increasing tableaux whose shapes are successive. Then $J_U^{\searrow}(T) = V_T^{\nwarrow}(U)$ and $V_U^{\searrow}(T) = J_T^{\nwarrow}(U)$.*

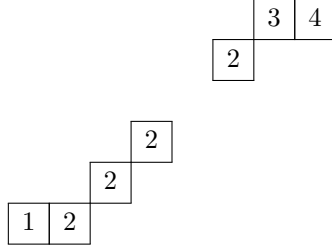
Let T and U be increasing tableaux whose shapes are successive. The *infusion* of the pair (T, U) is the pair $(V_U^{\searrow}(T), J_U^{\searrow}(T)) = (J_T^{\nwarrow}(U), V_T^{\nwarrow}(U))$.

Remark 2.5.

- More generally is known (see [BSS] [H] for jeu-de-taquin for semistandard tableaux) that infusion can be computed by any sequence of switches which “shuffles” the alphabets of the inner and outer tableaux. In Example 2.3 this means starting with $1 < 2 < 3 < a < b$, ending with $a < b < 1 < 2 < 3$ and always using a total order on these 5 values with $1 < 2 < 3$ and $a < b$.
- Since a single switch is an involution it follows that infusion is involutive.

2.2.2. *K-Pieri property of Kjdt.* A *horizontal K-Pieri t-strip* (called a *t-Pieri filling* in [TY]) is sequence of t nonempty successive rook strips D_1, D_2, \dots, D_t such that $D_1 + D_2 + \dots + D_t$ is a horizontal strip and the boxes of D_{i+1} are to the right of the boxes of D_i for all $1 \leq i < t$. This can be depicted by an increasing tableau in which the boxes of D_i are filled with i . A *vertical K-Pieri strip* is the transpose analogue.

Example 2.6. A horizontal K -Pieri 4-strip is pictured below as an increasing tableau.



Proposition 2.7. [TY] *The Kjdt operations J_T^{\nwarrow} and J_U^{\searrow} send horizontal (resp. vertical) K -Pieri t -strips, to horizontal (resp. vertical) K -Pieri t -strips.*

2.2.3. *Two special rectification orders.* In the infusion of the pair (T, U) , as computed by $J_T^{\nwarrow}(U)$ (resp. $J_U^{\searrow}(T)$), the tableau T (resp. U) is called a *rectification order*; the tableau T (resp. U) is viewed as the sequence of instructions for the moving of U (resp. T) to the northwest (resp. southeast). We require two special kinds of tableaux for this purpose.

Given a partition λ let $\text{CSS}(\lambda)$ be the *column superstandard tableau*, the tableau of shape λ with first column is filled with 1 through λ'_1 from top to bottom, second column filled with $\lambda'_1 + 1$ through $\lambda'_1 + \lambda'_2$, and so on. For $\lambda' = (\lambda'_1, \dots, \lambda'_k)$ let $\text{CR}(\lambda)$ be the *column reading tableau*, the unique tableau of shape λ built from the empty tableau by placing the numbers 1 through λ'_k at the ends of the first λ'_k rows, then placing the next λ'_{k-1} numbers at the ends of the first λ'_{k-1} rows, and so on.

We make similar definitions of column superstandard and column reading antitableaux $\text{CSS}(\lambda^*)$ and $\text{CR}(\lambda^*)$, which can also be obtained respectively from $\text{CSS}(\lambda)$ and $\text{CR}(\lambda)$ by rotation by 180 degrees and complementing.

We shall sometimes use decreasing versions of these special tableaux.

Example 2.8. For $\lambda = (4, 3, 2)$ we have $\lambda' = (3, 3, 2, 1)$ and

$$\begin{array}{l} \text{CSS}(\lambda) = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 7 & 9 \\ \hline 2 & 5 & 8 & \\ \hline 3 & 6 & & \\ \hline \end{array} \\ \text{CR}(\lambda) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & 9 & & \\ \hline \end{array} \\ \text{CSS}(\lambda^*) = \begin{array}{|c|c|c|c|} \hline & & 4 & 7 \\ \hline & 2 & 5 & 8 \\ \hline 1 & 3 & 6 & 9 \\ \hline \end{array} \\ \text{CR}(\lambda^*) = \begin{array}{|c|c|c|c|} \hline & & 1 & 4 \\ \hline & 2 & 5 & 7 \\ \hline 3 & 6 & 8 & 9 \\ \hline \end{array} \end{array}$$

2.2.4. K -rectification and anti-rectification. In studying the Kjdt for most situations it does not matter whether a tableau is increasing or decreasing because the tableaux are specified by sequences of rook strips and only the labeling of the strips is different. For technical reasons we will use different rectification orders in the following definitions for increasing versus decreasing tableaux.

- Definition 2.9.*
- (1) Given an increasing (resp. decreasing) tableau T of shape $D = \lambda/\mu$ define its K -rectification by $J^{\leftarrow}(T) = J_S^{\leftarrow}(T)$ where $S = \text{CR}(\mu)$ (resp. $S = \text{CSS}(\mu)$).
 - (2) Let R be a tight $(\ell(\lambda) \times \lambda_1)$ rectangle placed around λ . Then R/λ is an antinormal skew shape. Define the K -anti-rectification³ of the increasing (resp. decreasing) tableau T by $J^{\rightarrow}(T) = J_U^{\rightarrow}(T)$ where $U = \text{CSS}(R/\lambda)$ (resp. $U = \text{CR}(R/\lambda)$).
 - (3) Given a decreasing tableau T of partition shape define $T^\sharp = J^{\rightarrow}(T)^*$.
 - (4) Given an increasing tableau T of partition shape define $T^\flat = J^{\leftarrow}(T^*)$.

Proposition 2.10. *The map $T \mapsto T^\sharp$ is a bijection $\text{Dec}_\lambda \rightarrow \text{Inc}_\lambda$ with inverse $T \mapsto T^\flat$. Moreover $T^\sharp = J^{\leftarrow}(T^*)$ and $T^\flat = J^{\rightarrow}(T)^*$.*

Proof. It follows from Proposition 2.7 that infusion sends a CSS tableau to a CR tableau whose shape is the 180-degree rotation, and vice versa. This implies that $T \mapsto T^\sharp$ and $T \mapsto T^\flat$ are shape-preserving. They are mutually inverse since $*$ is involutive and infusion is involutive. The alternate descriptions of the maps hold since switching commutes with 180 degree rotation. \square

Example 2.11. Let T be the following decreasing tableau T . We illustrate how to compute T^\sharp . The first step is to anti-rectify T with respect to the rectification order U . $J^{\rightarrow}(T)$ is the result of the anti-rectification.

$$\begin{array}{l} T = \begin{array}{|c|c|c|c|c|} \hline 9 & 8 & 7 & 5 & 3 \\ \hline 7 & 5 & 4 & 3 & \\ \hline 4 & 2 & 1 & & \\ \hline 3 & & & & \\ \hline 1 & & & & \\ \hline \end{array} \\ U = \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 5 \\ \hline 3 & 6 & 8 & a \\ \hline 4 & 7 & 9 & b \\ \hline \end{array} \\ J^{\rightarrow}(T) = \begin{array}{|c|c|c|c|c|} \hline & & & & 7 \\ \hline & & & & 5 \\ \hline & & 8 & 7 & 3 \\ \hline & 9 & 5 & 4 & 2 \\ \hline 9 & 7 & 3 & 2 & 1 \\ \hline \end{array} \end{array}$$

³We feel this name is more descriptive as the result has antinormal shape. The original name is “reverse K -rectification” [TY].

Then we rotate $J^{\searrow}(T)$ and obtain T^\sharp :

$$T^\sharp = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 7 & 9 \\ \hline 2 & 4 & 5 & 9 & \\ \hline 3 & 7 & 8 & & \\ \hline 5 & & & & \\ \hline 7 & & & & \\ \hline \end{array}$$

Next, we illustrate how to compute $(T^\sharp)^\flat$. We rotate T^\sharp and get $(T^\sharp)^*$. Then we rectify $(T^\sharp)^*$ using U' . The result $J^{\nwarrow}((T^\sharp)^*)$ is $(T^\sharp)^\flat$.

$$(T^\sharp)^* = \begin{array}{|c|c|c|c|} \hline & & & 7 \\ \hline & & & 5 \\ \hline & & 8 & 7 & 3 \\ \hline & 9 & 5 & 4 & 2 \\ \hline 9 & 7 & 3 & 2 & 1 \\ \hline \end{array} \quad U' = \begin{array}{|c|c|c|c|} \hline 1 & 5 & 8 & a \\ \hline 2 & 6 & 9 & b \\ \hline 3 & 7 & & \\ \hline 4 & & & \\ \hline \end{array} \quad J^{\nwarrow}((T^\sharp)^*) = \begin{array}{|c|c|c|c|c|} \hline 9 & 8 & 7 & 5 & 3 \\ \hline 7 & 5 & 4 & 3 & \\ \hline 4 & 2 & 1 & & \\ \hline 3 & & & & \\ \hline 1 & & & & \\ \hline \end{array}$$

Readers can check $(T^\sharp)^\flat = T$ in this case.

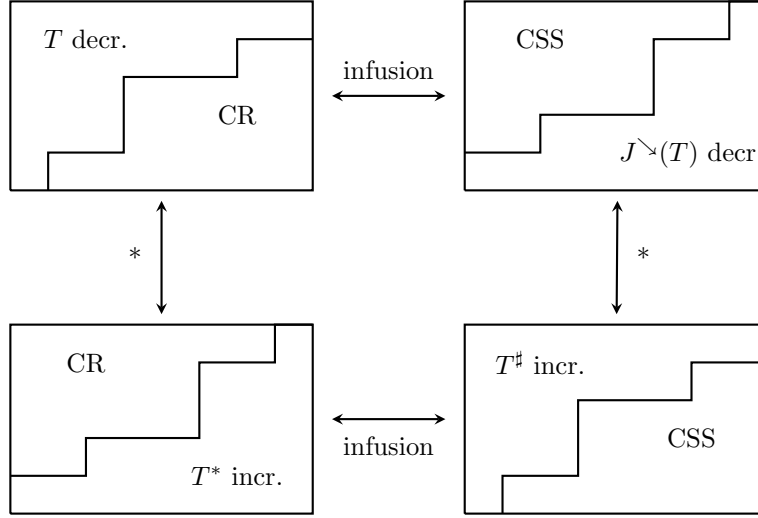
- Remark 2.12.* (1) Unlike jeu-de-taquin for semistandard tableaux, $J_S^{\nwarrow}(T)$ may depend on the rectification order S [TY, Ex. 1.3]. Thus for the well-definedness of $J^{\nwarrow}(T)$, it is necessary to specify S .
- (2) The λ and μ defining a skew shape λ/μ are not unique; one may add several rows and columns to the top and left of the diagrams of μ and λ simultaneously and get the same difference of partition diagrams. Using Proposition 2.7 it can be shown that $J^{\nwarrow}(T)$ depends only on the set of boxes in λ/μ and not on the pair (λ, μ) .
- (3) This definition of $J^{\searrow}(T)$ was used in [ReY].

2.2.5. *Keys defined via Kjdt for increasing/decreasing tableaux.* For a tableau T let $T_{\leq j}$ be the tableau consisting of the first j columns of T . $T_{\geq j}$ is defined similarly.

Let T be an increasing (resp. decreasing) tableau of partition shape. The *left key* $K_-(T)$ of the increasing tableau T is defined by the condition that its j -th column is equal to the first column of $J^{\searrow}(T_{\leq j})$ for all j . The *right key* $K_+(T)$ of the decreasing tableau T is defined by the condition that its j -th column is equal to the last column of $J^{\searrow}(T_{\geq j})$ for all j . We note that J^{\searrow} is defined using a different reverse rectification order for increasing versus decreasing tableaux. However, it is ultimately shown that these keys are independent of the rectification order; see Proposition 4.12 for decreasing tableaux.

2.3. **Hecke insertion.** Let T be a decreasing tableau of partition shape λ and x a positive integer. The (column) *Hecke insertion* of x into T is defined as follows. It maps the pair (x, T) to a triple (P, c, α) . P will be a decreasing tableau which either has shape λ in which case we set $\alpha = 0$ or differs by adding a single box, in which case we set $\alpha = 1$. We write $P = (x \xrightarrow{H} T)$. c will be the box of P where the algorithm ends.

The algorithm first inserts x into column 1 of T . This may output a number. If so, the output number is then inserted to the next column. The algorithm repeats

FIGURE 1. From decreasing to increasing: $T \mapsto T^\sharp$

until an insertion to a column has no output. To describe the insertion of x to a column of T , we consider two cases:

- Case 1: x is less than or equal to all entries in this column. Then the algorithm makes no output. In addition, it appends x to the bottom of this column as long as the result is a decreasing tableau. c is set to be this newly appended box. Otherwise the column is unchanged and c is set to be the rightmost box in the row that contains the bottom entry of this column.
- Case 2: Otherwise let y be the smallest value in this column such that $y < x$. Then the algorithm outputs y from the column. In addition it replaces y by x as long as the result is a decreasing tableau. Otherwise the column remains unchanged.

We use the word “contraction” when $\alpha = 0$.

This algorithm has an inverse called reverse (column) Hecke insertion, which maps a triple (P, c, α) to a pair (T, x) . Here, P is a decreasing tableau of partition shape. c is an entry on P that is at the end of its row and its column. α is 0 or 1.

The algorithm behaves in the following way: First let y be the number in the box c . If $\alpha = 1$, we remove this box. If $\alpha = 0$, we do not remove it. In either case, the algorithm “reverse inserts” y into the previous column. When a value y is “reverse inserted” into a column, the algorithm finds the largest y' in that column such that $y' > y$. It replaces y' by y as long as the result is a decreasing tableau. Otherwise it does nothing to the column. In either case, y' is reverse inserted into the previous column. If there is no column on the left, the algorithm lets $x = y'$ and T is the resulting tableau. Then it terminates.

Lemma 2.13. [BKSTY] *Hecke insertion and reverse Hecke insertion are inverses of each other.*

2.3.1. *Pieri property of Hecke insertion.* Hecke insertion has the following Pieri property:

Lemma 2.14. [BKSTY, Lemma 2] *Let T be a decreasing tableau. Let x_1, x_2 be two positive integers. Hecke insert x_1 into T with result (T_1, c_1, α_1) and then Hecke insert x_2 into T_1 , with result (T_2, c_2, α_2) . Then c_2 is strictly to the right of c_1 if and only if $x_1 < x_2$.*

2.3.2. *Set-valued recording tableaux for Hecke insertion and Hecke RSK.* Let $\mathcal{T} := \bigsqcup_{\lambda} (\text{Dec}_{\lambda} \times \text{RSVT}_{\lambda})$. Now we describe the bijections between \mathcal{C} and \mathcal{T} . First, we recursively define $\text{Insert} : \mathcal{C} \mapsto \mathcal{T}$.

Definition 2.15. Take $(a, i) \in \mathcal{C}$. If a is the empty string, then $\text{Insert}(a, i)$ is the pair of two empty tableaux. Now assume a has positive length. Let $a = a'x, i = i'y$, where x, y are positive numbers. Now let $(P', Q') = \text{Insert}(a', i')$. We Hecke insert x into P' and get (P, c, α) . If $\alpha = 1$, we append y to Q' at the corresponding position of c . Otherwise, we add y to the entry in Q' that corresponds to c . We let Q be the resulting RSVT. Then $\text{Insert}(a, i) := (P, Q)$.

Insert is well-defined by Lemma 2.14.

Now we recursively define the map $\text{RevInsert} : \mathcal{T} \mapsto \mathcal{C}$.

Definition 2.16. Take $(P, Q) \in \mathcal{T}$. If P is the empty tableau, $\text{RevInsert}(P, Q)$ is the pair of two empty words. Now assume P is non-empty. Let y be the smallest number in Q . We pick the rightmost y , and remove this number from Q . Let Q' be the resulting RSVT. If this y is the only number in its entry, we set $\alpha = 1$. Otherwise, $\alpha = 0$. Then we invoke reverse Hecke insertion on the corresponding entry in P with α . Let (P', x) be the output. Finally, we let $(a', i') = \text{RevInsert}(P', Q')$. Then $\text{RevInsert}(P, Q) := (ax, iy)$.

RevInsert is similarly well-defined.

Lemma 2.17. *Insert and RevInsert are inverses.*

Proof. Follows from Lemmas 2.13 and 2.14. □

Also, these two maps have the following property:

Proposition 2.18. *Take $(a, i) \in \mathcal{C}$ and let $(P, Q) = \text{Insert}(a, i)$.*

$$(2.1) \quad \text{rev}(a) \equiv_K P$$

$$(2.2) \quad \text{wt}(Q) = \text{wt}(i)$$

Proof. Equation (2.2) holds by definition. The relation (2.1) holds by [BS, Thm. 6.2]. □

3. RSVT RULE FOR LASCoux POLYNOMIALS

In this section, we give a combinatorial rule for Lascoux polynomials involving tableaux. Let T be a RSVT. Let $L(T)$ be the RSSYT obtained by picking the largest number in each entry. Then we have

Theorem 3.1. *For any composition α*

$$(3.1) \quad \mathfrak{L}_{\alpha}^{(\beta)} = \sum_{K - (L(T)) \leq \alpha} \beta^{\text{ex}(T)} x^{\text{wt}(T)}$$

where T runs over the RSVT of shape α^+ .

Example 3.2. The following RSVTs contribute to $\mathfrak{L}_{(1,0,2)}^{(\beta)}$:

2	1						
1							
2	2	2	21				
1		1					
3	1	32	1				
1		1					
3	2	3	21	32	2	32	21
1		1		1		1	
3	3	3	31	3	32	3	321
1		1		1		1	

Thus, we may write $\mathfrak{L}_{(1,0,2)}^{(\beta)}$ as

$$\begin{aligned} & x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 \\ & + \beta(x_1^2 x_2^2 + 2x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_1 x_2 x_3^2) \\ & + \beta^2(x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2) \end{aligned}$$

Next, we want to rewrite Theorem 3.1 as a rule involving RSSYT, instead of RSVT. We need a definition:

Definition 3.3. Fix a RSSYT T of shape λ . Define $WT(T)$ by

$$x^{WT(T)} = \sum_{\substack{T' \in \text{RSVT}_\lambda \\ L(T')=T}} \beta^{\text{ex}(T')} x^{\text{wt}(T')}.$$

Then Theorem 3.1 can be rewritten as:

Theorem 3.4. For any composition α

$$\mathfrak{L}_\alpha^{(\beta)} = \sum_{\substack{T \in \text{RSVT}_{\alpha^+} \\ K_-(T) \leq \alpha}} x^{WT(T)}$$

It is clear that Theorem 3.4 and Theorem 3.1 are equivalent. Readers may still insist that Theorem 3.4 involves RSVTs, since they appear in how we defined $x^{WT(T)}$. The following lemma resolves this issue:

Lemma 3.5. For any RSSYT of shape λ

$$(3.2) \quad x^{WT(T)} = x^{\text{wt}(T)} \prod_{(s,k)} (1 + \beta x_k)$$

where (s, k) runs over pairs such that s is a box in λ , k is less than the value of T in that box, and replacing the s -th entry of T by k results in a RSSYT.

Proof of Lemma 3.5. Consider the following way of turning T into a RSVT in $L^{-1}(T)$. Let a be an entry in T . Let b be the entry on its right and $b = 1$ if such an entry does not exist. Let c be the entry below a and $c = 0$ if such an entry does not exist. We turn a into $\{a\}$, and then add some numbers to this set. We

may add any k such that $a > k$, $k > c$ and $k \geq b$. Not adding this k will contribute 1 and adding this k will contribute an βx_k . Thus, each such k contributes $(1 + \beta x_k)$. Clearly, the choices are independent and any element in $L^{-1}(T)$ can be obtained this way. \square

In the rest of this section, we show Theorem 3.4 using only RSSYT's. The idea is to show the sum in Theorem 3.4 satisfies the defining recursion of Lascoux polynomials. Now, we fix an i throughout the rest of this section. In subsection 3.1, we partition all RSSYT appearing in the formula for a Lascoux polynomial, into several families. In subsection 3.2, we investigate the left keys of RSSYT's in a family. In subsection 3.3, we derive a few identities regarding the operators π_i and $\pi^{(\beta)}$. Finally, in subsection 3.4, we study the weight-generating function over RSSYT's in a family, and particularly how the operators apply to them. Then we can prove Theorem 3.4.

3.1. Partitioning RSSYT's. Let T be a RSSYT. We classify its i and $i + 1$ into 3 categories: “ignorable”, “frozen”, and “free”. First, we find all pairs of $i + 1$ and i that appear in the same column. We pair them and say they are “ignorable”. Next, we find non-ignorable i and $i + 1$ such that:

- (1) i is on the left of $i + 1$.
- (2) Any column between them must have an ignorable pairs.

We pair them and say they are “frozen”. Other non-ignorable i and $i + 1$ are called “free”.

Example 3.6. When $i = 3$, consider the following RSSYT:

6	6	6	6	4	4
5	4	3	3		
4	3				

The red entries are ignorable and blue entries are frozen. Other 3 and 4 are free.

Based on this labelling, we may partition RSSYT's into families.

Definition 3.7. A *family* is an equivalence class under the transitive closure of the following: two RSSYT's are related if they differ by changing a single i into an $i + 1$ (or vice versa) where the changed letters are free in both tableaux.

Example 3.8. Consider the reverse tableau in the previous example. Its family also includes:

6	6	6	6	4	3	6	6	6	6	4	3	6	6	6	6	4	3
5	4	3	3			5	4	3	3			5	4	4	3		
3	3					4	3					4	3				

6	6	6	6	4	4	6	6	6	6	4	4
5	4	3	3			5	4	4	3		
3	3					4	3				

However, the following is in another family:

6	6	6	6	4	3
5	4	4	3		
3	3				

Given an RSSYT, how can we enumerate its family? Clearly, we can only change its free entries. We also need to make sure they are still free after our changes. In other words, assume a and b are two free entries. If a is on the left of b and all columns between them have ignorable pairs, then we cannot change a into i and b into $i + 1$. This criterion leads to the following definition.

Definition 3.9. Let T be a RSSYT. We partition its free i and $i + 1$ into “blocks”. Two entries are in the same block iff all columns between them have ignorable pairs.

Thus, to enumerate the family of a RSSYT T , we just replace entries in each block by a weakly decreasing (from left to right) sequence of i and $i + 1$. The reader may check the enumeration of the family in the previous example.

3.2. Families and left keys. This subsection aims to describe the left keys of a family. This idea is formalized in the following lemma:

Lemma 3.10. *Let \mathcal{F} be a family. Then its elements can have at most 2 different left keys. If they all have the same left key γ , then $\gamma_i \geq \gamma_{i+1}$.*

If they have two different left keys, then they must be γ and $s_i\gamma$, where $\gamma_i > \gamma_{i+1}$. In this case, we also have:

- (1) $T \in \mathcal{F}$ has left key γ iff T 's leftmost block only has i .
- (2) All columns before the first block must have ignorable pairs.

Before proving the lemma, we need to introduce an algorithm that computes the left key. The algorithm is introduced in section 5 of [Wil]. Here we describe this algorithm in a slightly different way.

Definition 3.11. Given two columns C_1, C_2 such that C_1C_2 is a RSSYT, we define the column $C_1 \triangleleft C_2$ as follows. Assume $C_2 = \{a_1 < a_2 < \dots < a_m\}$. We find the smallest b_1 in C_1 such that $b_1 \geq a_1$. Then we find the smallest b_2 in C_1 such that $b_2 \geq a_2$ and $b_2 > b_1$. Similarly, we find b_3, \dots, b_m . Let $C_1 \triangleleft C_2 = \{b_1 < b_2 < \dots < b_m\}$.

More generally suppose C_1, C_2, \dots, C_k are the columns of a RSSYT. Observe that the following expression is well-defined when $j \leq k$

$$C_j \triangleleft \dots \triangleleft C_k := C_j \triangleleft (C_{j+1} \triangleleft \dots \triangleleft C_k)$$

where the base case is

$$C_k \triangleleft \dots \triangleleft C_k := C_k$$

With this definition, we may compute column k of $K_-(T)$, where T is a RSSYT. Let the first k columns of T be C_1, \dots, C_k . Then column k of $K_-(T)$ is by definition $C_1 \triangleleft \dots \triangleleft C_k$.

To study this algorithm, we need to classify columns of T . Each column can be labeled as follows:

- (1) Type 1 column: It has neither i nor $i + 1$.
- (2) Type 2 column: It has i but no $i + 1$.

- (3) Type 3 column: It has $i + 1$ but no i .
- (4) Type 4 column: It has both i and $i + 1$.

Now we make several observations.

Lemma 3.12. *If C_1 has type 4 and C_2 does not have type 3, then $C_1 \triangleleft C_2$ cannot have type 3.*

Proof. Assume $C_1 \triangleleft C_2$ has type 3. Then we must pick $i + 1$ in C_1 for some m in C_2 . Moreover, i in C_1 is never picked. Thus, m must be $i + 1$ and C_2 cannot have i . C_2 has type 3, contradiction. \square

Lemma 3.13. *Let T be a RSSYT with no free $i + 1$. Assume $\gamma = K_-(T)$. Then $\gamma_i \geq \gamma_{i+1}$.*

Proof. Let C_1, C_2, \dots be columns of T . Consider column k of $K_-(T)$. We only need to prove it cannot have type 3.

Suppose C_1, \dots, C_k all have type 4. Then Lemma 3.12 guarantees $C_1 \triangleleft \dots \triangleleft C_k$ cannot have type 3. Otherwise, we can find $j \leq k$ such that C_1, \dots, C_{j-1} have type 4 and C_j does not have type 4. Since T has no free $i + 1$, C_j must have type 1 or 2. Then $C_j \triangleleft \dots \triangleleft C_k$ must also have type 1 or 2. By Lemma 3.12, C_1, \dots, C_{j-1} cannot turn it into type 3. \square

Lemma 3.14. *Assume C_2 has type 2. We change its i into $i + 1$ and obtain C'_2 . Assume $C_1 C'_2$ is a RSSYT. Then,*

- (1) *If C_1 has type 4, then $C_1 \triangleleft C_2 = C_1 \triangleleft C'_2$, or $C_1 \triangleleft C'_2$ is obtained from $C_1 \triangleleft C_2$ by changing an i into $i + 1$.*
- (2) *If C_1 has type 1 or 3, then $C_1 \triangleleft C_2 = C_1 \triangleleft C'_2$.*

Proof. We do a case study based on the type of C_1 .

- (1) Assume C_1 has type 4. When we consider i in C_2 , there are 3 possibilities: The i in C_1 is picked; or a number larger than it is picked; or the i is still available.

In the first 2 cases, clearly this i in C_2 behaves as if it is an $i + 1$. Then $C_1 \triangleleft C_2 = C_1 \triangleleft C'_2$. In the last case, i in C_2 picks i , and $i + 1$ in C'_2 picks $i + 1$. Our claim is clear.

- (2) Assume C_1 has type 1 or 3. Clearly the i in C_2 behaves as if it is an $i + 1$, so $C_1 \triangleleft C_2 = C_1 \triangleleft C'_2$. \square

Lemma 3.15. *Let T be a RSSYT. Assume column j of T has a free i , which is the leftmost free i in its block. We change this i into $i + 1$ and get T' . If $\gamma = K_-(T)$, then $K_-(T') = \gamma$ or $s_i \gamma$. Moreover, if the latter case happens, we must have:*

- (1) *The i we changed is in the leftmost block of T .*
- (2) *Each of column $1, \dots, j - 1$ of T has ignorable pairs.*

Proof. Let C_1, C_2, \dots be the columns of T . Let D_1, D_2, \dots be the columns of T' . Consider column k of $K_-(T)$ and $K_-(T')$. If $k < j$, then clearly they are the same. Now assume $k > j$. Let $C = C_{j+1} \triangleleft \dots \triangleleft C_k$. Because the i in column j is free, we know that C_{j+1}, \dots, C_k all have type 4, or the leftmost non-type-4 column among them has type 1 or 2. Similar to the proof of Lemma 3.13, C cannot have type 3. Next, we compare $C_j \triangleleft C$ and $D_j \triangleleft C$. If i in C_j is picked by x in C , then this x

will pick $i + 1$ in D_j . Thus, $D_j \triangleleft C$ is obtained by changing i in $C_j \triangleleft C$ into $i + 1$. If i in C_j is not picked, the $i + 1$ in D_j will not be picked. Then $C_j \triangleleft C = D_j \triangleleft C$.

Consequently, if $k \geq j$, $C_j \triangleleft \cdots \triangleleft C_k$ agrees with $D_j \triangleleft \cdots \triangleleft D_k$, or the latter differs from the former by changing an i into $i + 1$. In Lemma 3.14, we showed this difference might be preserved or corrected by type 4 columns. If C_1, \dots, C_{j-1} all have type 4, then we know column k of $K_-(T)$ agrees with column k of $K_-(T')$, or the latter differs from the former by changing an i into $i + 1$. Otherwise, we let l be the largest such that $l < j$ and C_l does not have type 4. Since the i in column j of T is the leftmost i in its block, C_l must have type 1 or 3. By Lemma 3.14,

$$C_l \triangleleft \cdots \triangleleft C_k = D_l \triangleleft \cdots \triangleleft D_k$$

Thus, each column of $K_-(T')$ either agrees with the corresponding column in $K_-(T)$, or differs by changing an i into $i + 1$. Since $K_-(T')$ is a key, we have $K_-(T') = \gamma$ or $s_i \gamma$. In the latter case we know C_1, \dots, C_{j-1} have type 4. Our claims are immediate. \square

Now we may prove Lemma 3.10.

Proof. First pick T from \mathcal{F} that has no free $i + 1$. Assume $\gamma = K_-(T)$. By Lemma 3.13, $\gamma_i \geq \gamma_{i+1}$.

Then we enumerate other elements in \mathcal{F} by changing free i in T into $i + 1$. As long as we do not change the first block, the left key will still be γ . Once we change the first i in the first block, the left key might be fixed, or turned into $s_i \gamma$. The latter case is possible only when all columns before the first blocks have ignorable pairs. After that, no matter which i we change, the left key will be fixed. \square

3.3. π_i and $\pi_i^{(\beta)}$. In this subsection, we derive some basic facts about π_i and $\pi_i^{(\beta)}$. Define $X_i = x_i(1 + \beta x_{i+1})$ and $X_{i+1} = x_{i+1}(1 + \beta x_i)$. Then we have

- (1) $s_i(X_i) = X_{i+1}$
- (2) $\pi_i(f) = \partial_i(x_i f)$ and $\pi_i^{(\beta)}(f) = \partial_i(X_i f)$
- (3) $\partial_i(X_i) = \partial_i(x_i) = 1$.

The following lemma describes how ∂_i acts on a product of several x_i and X_i :

Lemma 3.16. *Assume we have u_1, \dots, u_n , where each u_j is either x_i or X_i . Then*

$$\partial_i(u_1 \dots u_n) = \sum_{j=1}^n s_i(u_1 \dots u_{j-1}) u_{j+1} \dots u_n$$

For instance,

$$\partial_i(x_i X_i x_i X_i) = X_i x_i X_i + x_{i+1} x_i X_i + x_{i+1} X_{i+1} X_i + x_{i+1} X_{i+1} x_{i+1}$$

Proof. Notice:

$$\begin{aligned} \partial_i(u_1 \dots u_n) &= \partial_i(u_1) u_2 \dots u_n + s_i(u_1) \partial_i(u_2 \dots u_n) \\ &= u_2 \dots u_n + s_i(u_1) \partial_i(u_2 \dots u_n) \end{aligned}$$

Then the proof is finished by induction. \square

Corollary 3.17. *Assume we have u_1, \dots, u_n , where each u_j is either x_i or X_i . Then*

$$(3.3) \quad \pi_i(u_1 \dots u_n) = u_1 \dots u_n + x_{i+1} \sum_{j=1}^n s_i(u_1 \dots u_{j-1}) u_{j+1} \dots u_n$$

$$(3.4) \quad \pi_i^{(\beta)}(u_1 \dots u_n) = u_1 \dots u_n + X_{i+1} \sum_{j=1}^n s_i(u_1 \dots u_{j-1}) u_{j+1} \dots u_n$$

3.4. $x^{WT(T)}$ and Family. In this subsection, we investigate how $x^{WT(T)}$ works and how it changes within a family. More explicitly, the goal is to understand: $\sum_{T \in \mathcal{F}} x^{WT(T)}$ where \mathcal{F} is a family.

The first step is to understand what governs the power of $(1 + \beta x_j)$ in $x^{WT(T)}$. Based on our definition, each row can have at most one entry that contributes $(1 + \beta x_j)$ for a fixed j . How is it determined whether a row has such a contributor? The following lemma answers this question. To make it concise, we adopt the following convention throughout the rest of this section: a 0 is appended below each column in a RSSYT.

Lemma 3.18. *A row has an entry that contributes $(1 + \beta x_j)$ iff we can find an entry j' on this row such that:*

- (1) $j' > j$
- (2) The entry below j' is less than j .

Proof. Assume an entry m contributes $1 + \beta x_j$. Then clearly $m > j$ and the entry below m is less than j . The row of m clearly satisfies the requirement.

Conversely, assume a row has j' that satisfies the two requirements. Moreover, we pick the rightmost j' among all such j' on this row. Then the entry to the right of j' either does not exist or is at most j . Changing this j' to j will make T a valid anti-SSYT. Thus, this entry contributes $(1 + \beta x_j)$. \square

With this lemma, we may ascribe contributions of $(1 + \beta x_j)$ to rows, instead of entries. However, we would like to ascribe contributions of $(1 + \beta x_i)$ and $(1 + \beta x_{i+1})$ to specific entries, but the rule is different from our previous criterion. If a row contributes $(1 + \beta x_i)$, then we may find the leftmost entry on this row satisfying:

- (1) It is larger than i .
- (2) The entry below it is less than i .

We say this entry contributes an $(1 + \beta x_i)$. Similarly, if a row contributes $(1 + \beta x_{i+1})$, then we may find the rightmost entry on the row below satisfying:

- (1) It is less than $i + 1$.
- (2) The entry above it is larger than $i + 1$.

We say this entry contributes an $(1 + \beta x_{i+1})$. To illustrate our new ‘‘contribution system’’, consider the following example:

Example 3.19.

6	6	6	6	4	4
5	4	4	3	0	0
4	3	0	0		
0	0				

When $i = 3$, each blue 4 contributes $x_4(1 + \beta x_3)$. The red 3 contributes $x_3(1 + \beta x_4)$.

Now we fix an arbitrary family \mathcal{F} throughout this subsection. Take any $T \in \mathcal{F}$. Let m be the number of blocks in T . Then we may break $x^{WT(T)}$ into a product:

$$x^{WT(T)} = g^T f_1^T \dots f_m^T$$

Here, f_j^T is the contribution of the j^{th} block in T from left to right. g^T contains the contribution of $x_i, x_{i+1}, (1 + \beta x_i)$ and $(1 + \beta x_{i+1})$ from all other entries. It also contains powers of x_j and $(1 + \beta x_j)$ with $j \neq i$ or $i + 1$. Next, we analyze these polynomials. Let us start with g^T :

Lemma 3.20. g^T is invariant within the family. Moreover, $s_i g^T = g^T$.

Proof. Clearly, changing free entries will not affect powers of x_j and $(1 + \beta x_j)$ with $j \neq i$ or $i + 1$. Let us focus on powers of $x_i, x_{i+1}, (1 + \beta x_i)$ and $(1 + \beta x_{i+1})$. Each ignorable pair contributes $x_i x_{i+1}$. Now, consider a frozen i . The column on its right must have an ignorable pair or a frozen $i + 1$. In either case, if we look at the entry the entry above it and the entry on its top right:

$$\begin{array}{|c|c|} \hline a & b \\ \hline i & \\ \hline \end{array}$$

We must have $a > i + 1 \geq b$. Thus, a frozen i always contributes $x_i(1 + \beta x_{i+1}) = X_i$. Similarly, a frozen $i + 1$ always contributes X_{i+1} . Thus, each frozen pair contributes $X_i X_{i+1}$.

Now, we still need to look at contributions of $(1 + \beta x_i)$ and $(1 + \beta x_{i+1})$ by entries that are not i or $i + 1$. Assume j is an entry that contributes $(1 + \beta x_{i+1})$ and j is not i or $i + 1$. Let j' be the entry above j . Then $j < i$ and $j' > i + 1$. There is a k' on the row of j' such that k' contributes $(1 + \beta x_i)$. Also, k' is weakly left of j' . The diagram looks like:

$$\begin{array}{|c|c|c|} \hline k' & \dots & j' \\ \hline k & \dots & j \\ \hline \end{array}$$

with $k' > i + 1$ and $k < i$. We pair this j with k' . Similarly, given such k' , we can find its corresponding j . In other words, we pair $(1 + \beta x_i)$ contributors with $(1 + \beta x_{i+1})$ contributors that are not i or $i + 1$. This pairing is clearly invariant under changing free entries. \square

Due to this result, we may change our notation g^T into $g^{\mathcal{F}}$, since it only depends on \mathcal{F} . The next step is to study each f_j^T . Clearly, a free i contributes either x_i or X_i . How can we determine its contribution? Consider the following lemma:

Lemma 3.21. Choose a free i in T . If it is not the last entry in its block, then it contributes x_i iff it is contiguous to the next free i . If it is the last entry in its block, then it contributes x_i iff one of the following happens:

(1) It is in the highest row.

(2) There is a b on its top right: $\begin{array}{|c|} \hline b \\ \hline i \\ \hline \end{array}$ with $b > i + 1$.

Proof. First, assume i is not the last entry in its block. We study the entry on its right:

- (1) The column on its right has ignorable pair. Then we look at $\begin{array}{|c|c|} \hline a & b \\ \hline i & \\ \hline \end{array}$ where our chosen i is red. We must have $a > i + 1 \geq b$. This i contributes X_i .
- (2) The column on its right has a free i and this free i is in the same row. Then we have $\begin{array}{|c|} \hline a \\ \hline i & i \\ \hline \end{array}$ with $a > i + 1$, or our chosen i is in the top row. In either case, it contributes x_i .
- (3) The column on its right has a free i and this free i is not on the same row as our chosen i . Then we have $\begin{array}{|c|c|} \hline a & b \\ \hline i & \\ \hline \end{array}$ with $a > i + 1$ and $b \leq i$. Our chosen i contributes X_i .

Now assume i is the last entry in its block. If it is in the top row, then it clearly contributes x_i . Otherwise, we look at: $\begin{array}{|c|c|} \hline a & b \\ \hline i & \\ \hline \end{array}$ We know $a > i + 1$. If b exists and $b > i + 1$, then clearly our i contributes x_i . Otherwise, our i contributes X_i . \square

Similarly, for $i + 1$, we have:

Lemma 3.22. *Choose a free $i + 1$ in T . If it is not the first entry in its block, then it contributes x_{i+1} iff it is contiguous to the previous free $i + 1$. If it is the first entry in its block, then it contributes x_{i+1} iff there is an a on its lower left with $a < i$.*

$$\begin{array}{|c|} \hline i+1 \\ \hline a \\ \hline \end{array}$$

We omit the proof since it is basically the same as the previous one.

Now we understand how the free entries contribute. Clearly, the contribution of one block is independent from other blocks. This implication allows us to simplify $\sum_{T \in \mathcal{F}} x^{WT(T)}$. In this family \mathcal{F} , there are $a_j + 1$ ways to fill the block j , where a_j is the number of entries in block j . Let f_j^l be the contribution of this block when the number of $(i + 1)$'s is l . l ranges between 0 and a_j . Then we have the following:

$$\sum_{T \in \mathcal{F}} x^{WT(T)} = g^{\mathcal{F}} \prod_{j=1}^m \left(\sum_{l=0}^{a_j} f_j^l \right)$$

Then we have

Lemma 3.23.

$$\sum_{l=0}^{a_j} f_j^l = \pi_i(f_j^0) \text{ or } \pi_i^{(\beta)}(f_j^0)$$

Moreover, take any $T \in \mathcal{F}$ such that its j^{th} block has an $i + 1$. Then we are in the second case iff the first $i + 1$ in the j^{th} block of T contributes X_{i+1} .

Proof. First, assume block j only has i . Let u_p be the contribution of the p^{th} free entry. Then $f_j^0 = u_1 \dots u_{a_j}$ and each $u_p = X_i$ or x_i .

We change the first free i into $i + 1$. By Lemma 3.21, this change only affects the first entry's contribution. Then $f_j^1 = vu_2 \dots u_{a_j}$ with $v = x_{i+1}$ or X_{i+1} . If $a_j = 1$, we are done by Corollary 3.17. Otherwise, we change the second free i into $i + 1$. The second $i + 1$ contributes x_{i+1} iff it is contiguous to the first free entry. Also, $u_1 = x_i$ iff the first entry is contiguous to the second entry. Thus,

we know the second entry contributes $s_i u_1$. $f_j^2 = v s_i(u_1) u_3 \dots u_{a_j}$. Continuing this argument, we have $f_j^l = v s_i(u_1 \dots u_{l-1}) u_{l+1} \dots u_{a_j}$. The proof is finished by invoking Corollary 3.17. \square

By this result, $\sum_{l=0}^{a_j} f_j^l$ must be symmetric in i and $i+1$. Recall that we have shown $g^{\mathcal{F}}$ is symmetric in i and $i+1$. Thus, $\sum_{T \in \mathcal{F}} x^{WT(T)}$ is symmetric in i and $i+1$. Finally, we have enough results to prove Theorem 3.4.

Proof. Let α be a weak composition with $\alpha_i > \alpha_{i+1}$. Let $A := \{T \in \mathcal{F} : K_-(T) \leq \alpha\}$ and $B := \{T \in \mathcal{F} : K_-(T) \leq s_i \alpha\}$.

We only need to show

$$(3.5) \quad \pi_i^{(\beta)} \left(\sum_{T \in A} x^{WT(T)} \right) = \sum_{T \in B} x^{WT(T)}$$

This is clearly true when $B = \emptyset$. Now assume $B \neq \emptyset$. If $A = B$, then $A = B = \mathcal{F}$, (3.5) is true since $\sum_{T \in \mathcal{F}} x^{WT(T)}$ is symmetric in x_i and x_{i+1} .

Finally, assume A is a proper subset of B . We can find γ with $\gamma_i > \gamma_{i+1}$ such that elements in A has left key γ and elements in B has left key $s_i \gamma$. Then $s_i \gamma \leq s_i \alpha$ and $\gamma \leq \alpha$. By Lemma 3.10, A has elements whose first block only has i . We have:

$$\sum_{T \in A} x^{WT(T)} = \left(g^{\mathcal{F}} \prod_{j=2}^m \left(\sum_{l=0}^{a_j} f_j^l \right) \right) f_1^0$$

Take $T \in B$. Consider its $i+1$ in the first block. There are two possibilities: It is in the first column, or the column on its left has an ignorable pair. In either case, this $i+1$ contributes X_{i+1} , so

$$\pi_i^{(\beta)}(f_1^0) = \sum_{l=0}^{a_1} f_1^l$$

Finally, letting $f = g^{\mathcal{F}} \prod_{j=2}^m (\sum_{l=0}^{a_j} f_j^l)$, we have

$$\begin{aligned} \pi_i^{(\beta)} \left(\sum_{T \in A} x^{WT(T)} \right) &= \pi_i^{(\beta)}(f f_1^0) \\ &= f \pi_i^{(\beta)}(f_1^0) \\ &= f \sum_{l=0}^{a_1} f_1^l \\ &= \sum_{T \in B} x^{WT(T)} \end{aligned}$$

\square

4. COMPATIBLE WORD RULE FOR LASCOUX POLYNOMIALS

In this section we give another rule for Lascoux polynomials involving compatible pairs. Recall the set \mathcal{C} from §1.5 and \mathcal{T} from §2.3.2. We would like to focus on smaller subsets of them.

Definition 4.1. Let P be a decreasing tableau. Let \mathcal{C}_P be the set consisting of all $(a, i) \in \mathcal{C}$ such that:

- (1) $a_j \geq i_j$ for all j , and
- (2) When we insert a into an empty decreasing tableau using Hecke insertion, we get P .

Correspondingly, we define \mathcal{T}_P to be the set consisting of all $(P, Q) \in \mathcal{T}$ such that $K_-(L(Q)) \leq K_+(P)$.

Then we can introduce our main result of this section:

Theorem 4.2. *The restriction of Insert to \mathcal{C}_P and the restriction of RevInsert to \mathcal{T}_P , give inverse bijections between \mathcal{C}_P and \mathcal{T}_P .*

Using this result, we have:

Theorem 4.3.

$$\mathfrak{Q}_{K_+(P)}^{(\beta)} = \sum_{(a,i) \in \mathcal{C}_P} \beta^{\text{ex}(a)} x^{\text{wt}(i)}$$

Proof. By Theorem 3.1, we have

$$\mathfrak{Q}_{K_+(P)}^{(\beta)} = \sum_{(P,Q) \in \mathcal{T}_P} \beta^{\text{ex}(Q)} x^{\text{wt}(Q)}$$

Then the proof is finished by applying RevInsert on the summands and invoking Theorem 4.2. \square

The rest of this section aims to prove Theorem 4.2. More specifically, we only need to show $\text{Insert}(\mathcal{C}_P) \subseteq \mathcal{T}_P$ and $\text{RevInsert}(\mathcal{T}_P) \subseteq \mathcal{C}_P$.

4.1. Right key of decreasing tableau. In this subsection, we investigate the right key of a decreasing tableau. First, we introduce an efficient algorithm that computes the right key.

4.1.1. Right key via \star -action. We start with a definition:

Definition 4.4. Let \star denote the following right action of the monoid of words with letters in the set $\mathbb{Z}_{>0}$, on the set of subsets of $\mathbb{Z}_{>0}$.

Let $S \subseteq \mathbb{Z}_{>0}$ and let $m \in \mathbb{Z}_{>0}$. Let m' be the smallest number in S of value at least m . If m' does not exist, we let $S \star m = S \sqcup \{m\}$. Otherwise, we define $S \star m = (S - \{m'\}) \sqcup \{m\}$.

More generally, if $w = w_1 \dots w_n$ is a word of positive integers, we define $S \star w = (S \star w_1) \star (w_2 \dots w_n)$, and $S \star w = S$ if w is the empty word.

Example 4.5. We have:

$$\begin{aligned} \emptyset \star 3414 &= \{1, 4\} \\ \{3, 4, 7\} \star 3414 &= \{1, 4, 7\} \\ \{3, 4, 7\} \star 3141 &= \{1, 4, 7\} \\ \{3, 4, 7\} \star 3411 &= \{1, 4, 7\} \end{aligned}$$

We will use this action to introduce our right key algorithm. Before that, we need to define a relation on words called K-Knuth equivalence, which is first introduced in [BS].

Definition 4.6. The K-Knuth relations are:

$$(4.1) \quad u a a v \equiv_K u a v$$

$$(4.2) \quad u a b a v \equiv_K u b a b v$$

$$(4.3) \quad u b a c v \equiv_K u b c a v \quad (a < b < c)$$

$$(4.4) \quad u a c b v \equiv_K u c a b v \quad (a < b < c)$$

where u, v are words and $a < b < c$ are positive numbers.

The K-Knuth equivalence \equiv_K is the transitive and symmetric closure of these four relations.

The Knuth equivalence relation \equiv differs from \equiv_K by removing the relation (4.1) and replacing the braid-like relation (4.2) by the relations

$$(4.5) \quad u b b a v \equiv u b a b v$$

$$(4.6) \quad u b a a v \equiv u a b a v.$$

Remark 4.7. It follows from the definitions that $S \star w$ may be obtained by taking the single column semistandard tableau defined by S , applying the usual Schensted column insertion of w_1 , then w_2 , up to w_n , and keeping only the first column. In particular, applying the well-known fact [Kn] that Knuth-equivalent words have the same Schensted insertion tableau, letting $\text{rev}(w)$ be the reverse of the word w , if $\text{rev}(w) \equiv \text{rev}(w')$ then $S \star w = S \star w'$.

Words in the same K-Knuth class have the same \star action.

Lemma 4.8. $S \star w = S \star w'$ if $w \equiv_K w'$.

Proof. We may assume that u and v are the empty word. The result is obvious when $w = aa$ and $w' = a$. If w and w' are related by (4.3) or (4.4) then (4.8) holds by Remark 4.7. For (4.2) we have

$$S \star a b a = S \star a a b = S \star a b = S \star a b b = S \star b a b$$

where the middle two equalities hold by the case of (4.1) and the first and last hold by the reverses of the Knuth relations (4.6) and (4.5). \square

The \star product is monotonic under left multiplication.

Lemma 4.9. For any words w and v , $\emptyset \star v \subseteq \emptyset \star w v$ as subsets.

Proof. We perform an induction on the length of v . When v has length 0, the claim is trivial.

Now assume $v = v'x$ where $x \in \mathbb{Z}_{>0}$. By the induction hypothesis, $\emptyset \star v' \subseteq \emptyset \star w v'$. Next we consider the action of x . If $\emptyset \star v'x \not\subseteq \emptyset \star w v'x$ the only possibility is that y in the latter set is replaced by x , but y in the former set is not. If this happens, x must replace a number smaller than y in the former set, say z . However, z is also in the latter set and x cannot replace y . Contradiction. Thus, $\emptyset \star v'x \subseteq \emptyset \star w v'x$. \square

We use the \star action to define the right key of a RSSYT and in particular a decreasing tableau.

Definition 4.10. For a RSSYT P of partition shape, we define its right key $K_+(P)$ to be the RSSYT whose j -th column is the column given by $\emptyset \star \text{word}(P_{\geq j})$ where $P_{\geq j}$ is the decreasing tableau obtained by removing the first $j - 1$ columns of P .

Remark 4.11. By Lemma 4.9 $K_+(P)$ is a key.

4.1.2. *Right key via Kjdt.* One may also define the right key of a decreasing tableau using Kjdt. This is the decreasing analogue of the definition of left key of increasing tableau given in [ReY]. We prove the implicit suggestion in [ReY] that the rectification order is irrelevant.⁴

Proposition 4.12. (1) *For any decreasing tableau T of partition shape, the rightmost column of the Kjdt anti-rectification of T with respect to an **arbitrary** rectification order, is equal to $\emptyset \star \text{word}(T)$. In particular this column does not depend on the rectification order.*

(2) *For any decreasing tableau P , the right key of P is the key tableau whose j -th column is the rightmost column of any Kjdt anti-rectification of $P_{\geq j}$.*

Proof. We only prove part (1) as it immediately implies part (2). Let T' be any Kjdt anti-rectification of T . By Theorem 6.2 of [BS], we know $\text{word}(T') \equiv_K \text{word}(T)$. By Lemma 4.8, $\emptyset \star \text{word}(T') = \emptyset \star \text{word}(T)$. Since T' is a decreasing tableau of antinormal shape, $\emptyset \star \text{word}(T')$ agrees with the rightmost column of T' . \square

4.1.3. *Right key and Hecke insertion.* We determine the precise change in the right key of a decreasing tableau under the operation of Hecke insertion of a single value.

We know the following lemma from Theorem 6.2 of [BS].

Lemma 4.13. *Let P' be a decreasing tableau, x a value, and $P = (x \xrightarrow{H} P')$.*

Then $x \text{word}(P') \equiv_K \text{word}(P)$.

Then we have:

Lemma 4.14. *Let P' be a decreasing tableau. Say the insertion $P = (x \xrightarrow{H} P')$ ends at column c . Then $K_+(P)$ and $K_+(P')$ agree everywhere except at column c . Moreover, if the insertion causes a contraction, then $K_+(P) = K_+(P')$.*

Proof. Clearly, P and P' agree on column k if $k > c$. Thus, $K_+(P)$ and $K_+(P')$ agree on column k .

Now assume $k \leq c$. Let w (resp. w') be the column word of P (resp. P') starting at the bottom of column k . Thus, either $w = w'$ or $w \equiv_K yw'$ for some number y , where y is the number inserted to column k . The former case directly implies $K_+(P)$ and $K_+(P')$ agree on column k . Now we consider the latter case. Column k of $K_+(P)$ (resp. $K_+(P')$) consists of numbers $\emptyset \star w$ (resp. $\emptyset \star w'$). If $k < c$, then we know the first character in w' is less than y . Thus, $\emptyset \star w = \emptyset \star yw' = \emptyset \star w'$. Finally, assume $k = c$ and a contraction occurs. Then clearly $w = w'$. Thus, $K_+(P)$ and $K_+(P')$ agree on column c . \square

What can we say about the changes at column c of $K_+(P')$ if no contraction occurs? This is answered by the following lemma:

Lemma 4.15. *Keep the notation from the previous lemma and assume contraction does not occur. Let C and C' be column c of $K_+(P)$ and $K_+(P')$ respectively. Let D be column $c + 1$ of $K_+(P)$. Then as sets $C = C' \sqcup \{e\}$ where e is the smallest number in C that is not in D .*

Example 4.16. In the following examples we do not distinguish between a column and its underlying set.

(1) If $C = \{1, 3, 4, 6, 7\}$ and $D = \{1, 3, 7\}$ then $C' = \{1, 3, 6, 7\}$.

⁴We have since learned that this was previously proved in [GMP+, Prop. 44].

- (2) If $C = \{1, 3, 4, 6, 7\}$ and D is empty then $C' = \{3, 4, 6, 7\}$.

Proof. We prove the lemma by induction on the number of entries to the right of C . If there are no such entries, then column c is the rightmost column. Thus, C (resp. C') agrees with column c of P (resp. P'). C' has all but the smallest number in C , so the claim holds.

Now let's assume there are entries to the right of C . Let m be the last number in $\text{word}(P)$. We may pretend as if m does not exist in P and P' and compute C, C' and D . By the inductive hypothesis, $C' = \{s_1 < \dots < s_n\}$ and $C = \{s_1 < \dots < s_i < e < s_{i+1} < \dots < s_n\}$, where e is the extra number. Moreover, s_1, \dots, s_i are the i smallest numbers in D . Now we consider the effect of m . There are two cases:

- (1) This ignored m is in column $c + 1$. Then column $c + 1$ is the last column of P . Now we let m act on C, C' and D . m simply adds itself to D . If m changes s_j into m in C , then s_j is also changed into m in C' . Our claim clearly holds. Otherwise, e in C is changed into m . Then s_{i+1} in C' is changed to m . Then s_{i+1} becomes the "extra number". Numbers less than it in C are s_1, \dots, s_i and m . They are all in D .
- (2) This ignored m is not in column $c + 1$. Now we let m act on C, C' and D . Assume m changes s_j into m in C with $j > i$. Then s_j is also changed into m in C' . For D , m will not change s_1, \dots, s_i . Our claim still holds. Now if m changes s_j into m in C with $j \leq i$, then m also changes s_j in C' and D into m . Our claim still holds. Finally, assume m changes e into m in C . Then m will change s_{i+1} in C' . For D , m will change a number other than s_1, \dots, s_i . Then s_{i+1} becomes the "extra number". Numbers less than it in C are s_1, \dots, s_i and m . They are all in D .

□

This lemma leads to the following.

Corollary 4.17. *Entries in C' are entrywise weakly less than corresponding entries in C .*

4.2. Left key of RSSYT. In this subsection, we are going to derive some results about the left key. Our results are analogous to the results in the previous subsection. We start with a result similar to Lemma 4.14.

Lemma 4.18. *Take $(a, i) \in \mathcal{C}$. Assume $a = a'x$ and $i = i'y$, where $x, y \in \mathbb{Z}_{>0}$. Let $(P, Q) = \text{Insert}((a, i))$ and $(P', Q') = \text{Insert}((a', i'))$. Assume the insertion of x ends at column c . Then $K_-(L(Q))$ and $K_-(L(Q'))$ agree everywhere except at column c . If the insertion of x causes a contraction, then $K_-(L(Q)) = K_-(L(Q'))$.*

Proof. When a contraction occurs, clearly $L(Q) = L(Q')$ and the conclusion is trivial. Now assume there's no contraction. Then $L(Q)$ is obtained by appending a number y beneath column c of $L(Q')$. Then clearly the first $c - 1$ columns of $K_-(L(Q))$ and $K_-(L(Q'))$ agree. Consider column c' where $c' > c$. We know any number in column c' of $L(Q)$ is strictly larger than y . Thus, when we compute column c' of $K_-(L(Q))$, this y will be ignored. □

As in the previous subsection, we would like to know what happens at column c of $K_-(L(Q))$ and $K_-(L(Q'))$ when a contraction does not occur. The following lemma is an analogue of 4.15:

Lemma 4.19. *Keep the notation from the previous lemma and assume contraction does not occur. Let C and C' be column c of $K_-(L(Q))$ and $K_-(L(Q'))$ respectively. Then $C = C' \sqcup \{e\}$ where for $c = 1$, $e = y$ and for $c \geq 2$, e is the smallest number in D that is not in C' where D is column $c - 1$ of $K_-(L(Q))$.*

Example 4.20. We have the following examples:

- (1) If $C' = \{1, 3, 6, 7\}$ and $D = \{1, 3, 4, 6, 7, 8\}$ then $C = \{1, 3, 4, 6, 7\}$.
- (2) If C' is empty and $D = \{1, 3, 4, 6, 7, 8\}$ then $C = \{1\}$.

Proof. If $c = 1$, our claim is immediate. Thus, we assume $c \geq 2$. Let C_1, C_2, \dots be the columns of $L(Q)$ and let C'_1, C'_2, \dots be the columns of $L(Q')$. Then $C = C_1 \triangleleft \dots \triangleleft C_c$, $C' = C'_1 \triangleleft \dots \triangleleft C'_c$ and $D = C_1 \triangleleft \dots \triangleleft C_{c-1}$. We prove our claim by induction on c .

For the base case, we assume $c = 2$. Then $D = C_1$. Assume $C'_2 = \{s_1 < \dots < s_m\}$. Then $C_2 = \{y < s_1 < \dots < s_m\}$. Assume $C_1 = \{t_1 < t_2 < \dots\}$. Then we know $y \leq t_1$. When we compute $C'_1 \triangleleft C'_2$, consider two cases:

- (1) Case 1: s_j picks t_j for all $j \in [m]$. When we compute $C_1 \triangleleft C_2$, y picks t_1 . $s_1 < s_2 \leq t_2$, so s_1 picks t_2 . Consequently, s_j picks t_{j+1} for all $j \in [m]$. Thus, $C'_1 \triangleleft C'_2$ contains t_1, \dots, t_m while $C_1 \triangleleft C_2$ contains t_1, \dots, t_m, t_{m+1} . Our claim is immediate.
- (2) Case 2: Otherwise let j be the smallest such that s_j in C'_2 does not pick t_j . Thus, $C'_1 \triangleleft C'_2$ has t_1, \dots, t_{j-1} but does not have t_j . When we compute $C_1 \triangleleft C_2$, similar to the previous case, y, s_1, \dots, s_{j-1} will pick t_1, \dots, t_j . Then s_j will make the same choice as in C'_2 . Since then, numbers in C_2 will make the same choices as in C'_2 . Thus, $C_1 \triangleleft C_2$ has all numbers in $C'_1 \triangleleft C'_2$ together with t_j . Our claim is proved.

Now we do the inductive step. We first ignore C_1 and compute C, C' and D . By the inductive hypothesis, we assume $C' = \{s_1 < \dots < s_n\}$ and $C = \{s_1 < \dots < s_i < e < s_{i+1} < \dots < s_n\}$, where e is the extra number. Assume $D = \{t_1 < t_2 < \dots\}$. Then we know $t_j = s_j$ for all $j \leq i$ and $e = t_{i+1}$. Now we consider the effect of C_1 . We need to study $C_1 \triangleleft C, C_1 \triangleleft C'$, and $C_1 \triangleleft D$. s_1, \dots, s_i make the same choices in all 3 scenarios. Notice that since $K_-(L(Q))$ is a key, numbers in $C_1 \triangleleft C$ must appear in $C_1 \triangleleft D$. Thus, when we study $C_1 \triangleleft C$, we can ignore numbers in C_1 not picked by D . The same is true for $C_1 \triangleleft C'$. Now we study two cases:

- (1) s_j in C' and t_j in D make the same choices for all $i < j \leq m$. In $C_1 \triangleleft C$, e picks what t_1 picks. Consequently, s_j picks what t_{j+1} picks for all $i < j \leq m$. Our claim is immediate.
- (2) Case 2: Otherwise let j be the smallest such that s_j in C' does not pick what t_j picks. Thus, s_{i+1}, \dots, s_{j-1} in C' make the same choices as t_{i+1}, \dots, t_{j-1} . When we compute $C_1 \triangleleft C$, similar to the previous case, $e, s_{i+1}, \dots, s_{j-1}$ will make the same choices as t_{i+1}, \dots, t_j . Then s_j will make the same choice as in C' . Since then, numbers in C will make the same choices as in C' . Our claim is proved. □

Corollary 4.21. *Values in C' are less than or equal to corresponding values in C .*

4.3. Proof of Theorem 4.3. Now we can prove Theorem 4.3. It is enough to prove the following:

- Lemma 4.22.** (1) Take $(a, i) \in \mathcal{C}$. Assume $a = a'x$ and $i = i'y$, where $x, y \in \mathbb{Z}_{>0}$. Let $(P, Q) = \text{Insert}((a, i))$ and $(P', Q') = \text{Insert}((a', i'))$. Assume $K_+(P') \geq K_-(L(Q'))$. Assume the insertion of x ends at column c and does not cause a contraction. Then column c of $K_+(P)$ is entrywise greater than or equal to column c of $K_-(L(Q))$.
- (2) Take $(P, Q) \in \mathcal{T}_P$. Find the smallest number in Q and break ties by picking the rightmost. Suppose it is y in column c , living in an entry that only contains it. We remove that entry in Q and invoke Hecke reverse insertion on the corresponding entry of P with $\alpha = 1$. Let x be the output. Then we must have $x \geq y$. Moreover, assume we get (P', Q') after the process. Then column c of $K_+(P')$ is entrywise greater than or equal to the column c of $K_-(L(Q'))$.

Why these two statements are enough? Well, we can use induction to prove $\text{Insert}(\mathcal{C}_P) \subseteq \mathcal{T}_P$ and $\text{RevInsert}(\mathcal{T}_P) \subseteq \mathcal{C}_P$. For the former, we keep all notations in the first part of Lemma 4.22. Then clearly $(a', i') \in \mathcal{C}_{P'}$. By induction on length of a , we may assume $K_+(P') \geq K_-(L(Q'))$ and need to show $K_+(P) \geq K_-(L(Q))$. The only place where things can go wrong is at column c with no contraction occurs. Thus, studying this case is enough. The latter is similar. Part 2 of Lemma 4.22 guarantees $(P', Q') \in \mathcal{T}_{P'}$. Then by inductive hypothesis, $\text{RevInsert}(P', Q') \in \mathcal{C}_{P'}$. Appending x, y respectively makes the pair in \mathcal{C}_P .

Now, we prove Lemma 4.22:

Proof. We begin with the first part. Keep all notation and assumptions in the first part of the lemma. We proceed by considering two cases. First assume $c = 1$. Then column 1 of $K_-(L(Q))$ (resp. $K_-(L(Q'))$) agrees with column 1 of $L(Q)$ (resp. $L(Q')$). Thus, column 1 of $K_-(L(Q))$ is obtained by appending y on the bottom of column 1 of $K_-(L(Q'))$. Since column 1 of $K_+(P')$ is entry-wise less than or equal to column 1 of $K_+(P)$, we only need to worry about the new entry on the bottom. Clearly, the bottom entry in column 1 of $K_+(P)$ equals a number in P , which is a number in a . It cannot be smaller than y , which is the smallest number in i . Thus, we are done.

Now assume $c > 1$. Let $t_1 < \dots < t_m$ be the numbers in column c of $K_+(P')$. Let $s_1 < \dots < s_m$ be the numbers in column c of $K_-(L(Q'))$. Then we know $t_j \geq s_j$. Let e be the extra number in column c of $K_-(L(Q))$ and assume it is the i^{th} smallest number in this column. By Lemma 4.19, if $j \leq i$, we have the following:

$$\begin{aligned}
& j^{\text{th}} \text{ smallest number in column } c \text{ of } K_-(L(Q)) \\
&= j^{\text{th}} \text{ smallest number in column } c-1 \text{ of } K_-(L(Q)) \\
(4.7) \quad &\leq j^{\text{th}} \text{ smallest number in column } c-1 \text{ of } K_+(P) \\
&\leq j^{\text{th}} \text{ smallest number in column } c \text{ of } K_+(P)
\end{aligned}$$

Now if $j > i$, the j^{th} smallest number in column c of $K_-(L(Q))$ is s_{j-1} . The j^{th} smallest number in column c of $K_+(P)$ is either at least t_{j-1} . Clearly, our inequality still holds.

Now we prove the second part. Keep all notation and assumptions in the second part of the lemma. First, $y \leq x$ is immediate: y is the smallest number in Q . If an entry in P is less than y , than the right key on that entry is also less than y , which is a contradiction.

Now we only need to prove the bound about keys. Let $t_1 < \cdots < t_m$ be the numbers in column c of $K_+(P)$. Let $s_1 < \cdots < s_m$ be the numbers in column c of $K_-(L(Q))$. Then we know $t_j \geq s_j$. Assume t_i is not in column c of $K_+(P')$. Take any $j < i$, we have:

$$\begin{aligned}
 & j^{\text{th}} \text{ smallest number in column } c \text{ of } K_+(P') \\
 (4.8) \quad &= j^{\text{th}} \text{ smallest number in column } c+1 \text{ of } K_+(P') \\
 &\geq j^{\text{th}} \text{ smallest number in column } c+1 \text{ of } K_-(L(Q')) \\
 &\geq j^{\text{th}} \text{ smallest number in column } c \text{ of } K_-(L(Q'))
 \end{aligned}$$

Now, if $j \geq i$, then the j^{th} smallest number in column c of $K_+(P')$ is t_{j+1} . The j^{th} smallest number in column c of $K_+(L(Q'))$ is s_{j+1} or s_j . Our equality holds in either case. \square

Remark 4.23. There is a semistandard analogue of Theorem 1.8. To state this result, we need to modify a few definitions:

- (1) In the definition of Insert, RevInsert and \mathcal{C}_P , ‘‘Hecke column insertion’’ is replaced by ‘‘RSK column insertion’’.
- (2) We change \mathcal{T} to be the set of pairs (P, Q) , where P and Q are RSSYT of the same shape.
- (3) In the definition of \mathcal{T}_P , $L(Q)$ is replaced by Q and the $K_+(P)$ is the classical right key of a RSSYT.

With these definitions, Theorem 4.2 is the semistandard analogue. It can be proved by exactly the same argument.

5. K-THEORETIC ANALOGUE OF THE REVERSE COMPLEMENT MAP

In this section we investigate the map $T \mapsto T^\sharp$ of Proposition 2.10. It is a bijection from decreasing tableaux to increasing tableaux. Recall the map is defined as follows. Take a decreasing tableau T . Find the smallest rectangle that contains T . Then we do several ‘‘iterations’’ of Kjdt to anti-rectify T . In each iteration, we perform Kjdt at the leftmost empty space of each row, from top to bottom. Then we rotate the result by 180 degrees and obtain T^\sharp .

Example 5.1. We recompute $J^{\searrow}(T)$ from Example 2.11 using the above iterations. We start with the decreasing tableau T , along with the rectification order (a tableau giving the order that boxes are occupied during the computation of $J^{\searrow}(T)$), and another tableau whose entry i means that the given box is occupied during the i -th iteration.

$$T = \begin{array}{|c|c|c|c|c|} \hline 9 & 8 & 7 & 5 & 3 \\ \hline 7 & 5 & 4 & 3 & \\ \hline 4 & 2 & 1 & & \\ \hline 3 & & & & \\ \hline 1 & & & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 5 \\ \hline 3 & 6 & 8 & a \\ \hline 4 & 7 & 9 & b \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 1 & 2 \\ \hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

Let $T^{(i)}$ be the skew tableau just before the i -th iteration. So $T = T^{(1)}$ and $T^{(m)} = J^{\searrow}(T)$ where m is the number of columns of T .

$T^{(2)}, T^{(3)}, T^{(4)}$ and $T^{(5)}$ are listed below:

9	8	7	5
7	5	4	3
4	2	1	
3			
9	1		

8	7	5
5	4	3
4	2	1
	9	3
9	7	1

		7	5
		5	3
	8	4	1
	9	5	2
9	7	3	1

				7
				5
	8	7	3	
	9	5	4	2
9	7	3	2	1

Once we rotate $T^{(5)}$ by 180 degrees, we get:

$$T^\sharp = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 7 & 9 \\ \hline 2 & 4 & 5 & 9 & \\ \hline 3 & 7 & 8 & & \\ \hline 5 & & & & \\ \hline 7 & & & & \\ \hline \end{array}$$

Consider the rightmost columns of the tableaux $T^{(5)}, T^{(4)}, \dots, T^{(1)}$ in that order. We get

7	5	5	5	3
5	3	3	3	
3	1	1		
2				
1				

Lemma 5.9 implies that this is $K_+(T)$.

Now, we have:

Proposition 5.2. *Let T be a decreasing tableau. Then*

$$K_+(T) = K_-(T^\sharp)$$

Example 5.3. In Example 5.1, $K_+(T)$ and $K_-(T^\sharp)$ are both $(3, 1, 5, 0, 4, 0, 1)$.

The rest of this section aims to prove it. We start with a few definitions.

Definition 5.4. Let T be a decreasing skew tableau. We use $T_{\leq c, \leq r}$ to denote the decreasing skew tableau obtained by keeping the first c columns and first r rows of T . Analogously, we define similar notations such as $T_{\geq c, \leq r}$.

Definition 5.5. We generalize $K_+(T)$ to decreasing skew tableaux T . Let $K_+(T)$ be the normal shape RSSYT whose c -th column equals $\emptyset \star w(T_{\geq c, < \infty})$ for all c .

Example 5.6. Let $T^{(2)}, \dots, T^{(5)}$ be the tableaux in Example 5.1. Their right keys are:

7	7	5	5	5
5	5	3	3	3
3	3	1	1	
2	2			
1	1			

7	7	7	5	5
5	5	5	3	3
3	3	3	1	1
2	2	2		
1	1	1		

7	7	7	7	5
5	5	5	5	3
3	3	3	3	1
2	2	2	2	
1	1	1	1	

7	7	7	7	7
5	5	5	5	5
3	3	3	3	3
2	2	2	2	2
1	1	1	1	1

Remark 5.7. We make a few observations about this generalization.

- (1) $K_+(T)$ is a key.
- (2) The rightmost column of $K_+(T)$ and T must agree.
- (3) When T is of normal shape, this definition agrees with our previous definition.

Let us look at Example 5.6 carefully. $T^{(4)}$ is obtained from $T^{(3)}$ by applying Kjdt on column 4 twice. Their right keys are the same except at column 4. Moreover, column 4 of $T^{(4)}$ agrees with column 3 of $T^{(3)}$. This phenomenon is captured in the next result:

Lemma 5.8. *Let T be a decreasing skew tableau. Assume its c -th column ends at row r_1 , while its $(c+1)$ -th column ends at row r_2 . Assume $r_1 > r_2$. We do Kjdt at the leftmost position of row $r_2 + 1, \dots, r_1$ and let T' be the result. Then $K_+(T)$ and $K_+(T')$ agree everywhere except at column $c+1$. Moreover, column $c+1$ of $K_+(T')$ agrees with column c of $K_+(T)$.*

Proof. Notice: $T_{>c+1, <\infty} = T'_{>c+1, <\infty}$. Thus, $K_+(T)_{>c+1, <\infty} = K_+(T')_{>c+1, <\infty}$.

Now assume $i \leq c+1$ and column i of T ends at row j . Column i of $K_+(T)$ has numbers $\emptyset \star w(T_{\geq i, <\infty}) = \emptyset \star w(T_{\geq i, \leq j})$. Now we look at $w(T_{\geq i, \leq j})$. It is also $w(T_{=i, \leq j})w(T_{>i, \leq j})$. Because $T_{\leq i, \leq j}$ is reverse rectified, $\emptyset \star w(T_{\leq i, \leq j}) = \emptyset \star w(T_{=i, \leq j})$. Thus, column i of $K_+(T)$ has numbers: $\emptyset \star w(T_{\leq i, \leq j})w(T_{>i, \leq j}) = \emptyset \star w(T_{<\infty, \leq j})$

Next, if $i \leq c$, then we know column i of T' also ends at row j . Similar to above, column i of $K_+(T')$ has numbers $\emptyset \star w(T'_{<\infty, \leq j})$. Notice that $T'_{<\infty, \leq j}$ is obtained from $T_{<\infty, \leq j}$ by a sequence of Kjdt moves. Thus, their column words are K-Knuth equivalent, so they yield the same results when acting on \emptyset .

Finally, we study column $c+1$. Column $c+1$ of $K_+(T')$ is $\emptyset \star w(T'_{<\infty, \leq r_1})$. Column c of $K_+(T)$ is $\emptyset \star w(T_{<\infty, \leq r_1})$. Clearly $T'_{<\infty, \leq r_1}$ is obtained from $T_{<\infty, \leq r_1}$ by K-jdt moves, so we are done. \square

Then we can describe how the right key changes during the map $T \mapsto T^\sharp$. In Example 5.6, each iteration changes the right key in the following way: It copies column c to column $c+1$, where c goes from 4 to 1. This pattern holds in general:

Lemma 5.9. *Let T be a decreasing tableau. Let C_1, \dots, C_m be columns of $K_+(T)$. Assume we finished t iterations while computing T^\sharp . Then the right key of the current skew-shape tableau has columns: $C_1, C_1, \dots, C_1, C_2, \dots, C_{m-t}$*

Proof. We prove by induction. When $t = 0$, the claim is trivial. Now assume the claim holds after t iterations. Assume now column i ends at r_i . During the $(t+1)^{th}$ iteration, the algorithm performs Kjdt at the leftmost position of row $r_m + 1, r_m + 2, \dots, r_1$. How do these moves affect the right key? By Lemma 5.8, it will copy column c of the right key to column $c+1$, where c goes from $m-1$ to 1. Thus, after this iteration, the right key becomes: $C_1, C_1, \dots, C_1, C_2, \dots, C_{m-t-1}$. \square

Proof of Proposition 5.2. Let C_1, \dots, C_m be the columns of $K_+(T)$. By Lemma 5.9, after t iterations of the computation of T^\sharp , the rightmost column of the resulting skew tableau (call it $T^{(t+1)}$) is C_{m-t} . By definition we also have $T^{(m)} = J^{\searrow}(T)$.

On the other hand consider the computation of the $(m-t)$ -th column C of $K_-(T^\sharp)$. By definition we take the first $m-t$ columns of T^\sharp within a tight rectangle, moving to the southeast via a Kjdt which infuses with a column superstandard

tableau having $m - t - 1$ columns. C is the leftmost column of the result. Rotating by 180 degrees, this is the same as taking $(T^\sharp)^* = J^{\searrow}(T)$, using the last $m - t - 1$ columns in a column superstandard Kjdt to move it to the northwest, and taking the rightmost column C . However the $m - t - 1$ iterations which slide into these columns are precisely undoing the last $m - t - 1$ iterations in the computation of $J^{\searrow}(T)$. Therefore C equals the rightmost column of $T^{(t+1)}$, which is C_{m-t} , as required. \square

Example 5.10. We illustrate this argument on column 3 of $K_+(T)$, where T is the tableau in Example 5.1. By definition, here's how we compute column 3 of $K_-(T^\sharp)$. We start with T^\sharp . Then we Kjdt to anti rectify T^\sharp . We stop once the first 3 columns are anti rectified:

			1	7	9
			3	9	
			1	2	4
			3	4	5
			5	7	8

Then the red numbers form column 3 of $K_-(T^\sharp)$. Notice that when we rotate this tableau, we get $T^{(3)}$ in Example 5.1, whose rightmost column is column 3 of $K_+(T)$.

6. ALTERNATIVE DESCRIPTIONS OF $K_+(T)$

This section provides some alternative descriptions of the right key $K_+(T)$, where T is a decreasing tableau.

Proposition 6.1. *Let T be a decreasing tableau. Let $T_{\geq j}$ be the decreasing tableau obtained by removing the first $j - 1$ columns of T . Then the column C computed by the following procedures all agree with column j of $K_+(T)$.*

- (1) $C = \emptyset \star \text{word}(T_{\geq j})$.
- (2) C is the rightmost column of a **arbitrary** anti-rectification of $T_{\geq j}$.
- (3) Assume column j of T has m numbers. We conjugate $T_{\geq j}$ and obtain $T'_{\geq j}$. Then we invoke Hecke reverse column insertion at the end of column $m, \dots, 1$ with $\alpha = 1$. Then C is the set of output numbers.

Proof. Let T be a decreasing tableau. Proposition 4.12 shows that the first two methods for computing $K_+(T)$ agree. Next, we show procedure 3 agrees with procedure 1. We may assume $j = 1$. Thus we need to prove: Assume we conjugate T and get T' . Invoke reverse insertion on each column of T' from right to left. Show the output agree with $\emptyset \star \text{word}(T)$.

Prove by induction on m , the number of rows in T . When $m = 1$, our claim is trivial. Now we do the inductive step. Assume both column 1 and column 2 of T has m entries. Notice that the result of reverse insertions and $\emptyset \star \text{word}(T)$ are not affected if we ignore column 1 of T . Thus, by keep removing columns on the left, we may assume T has only one column with m entries. Now, we invoke reverse Hecke insertion at the end of column m of T' and let x_m be the output. We conjugate the resulting tableau and get P . Then $\text{word}(T') \equiv_K x_m \text{word}(P')$. Define the row word of T to be the following: Read entries of T from bottom to top, within each column from left to right. Then $\text{rev}(\text{word}(T'))$ is the row word of T . By Theorem

5.4 of [BS] the row word of T is K-Knuth equivalent to $\text{word}(T)$. Thus, we know

$$\emptyset \star \text{word}(T) = \emptyset \star \text{rev}(\text{word}(T')) = \emptyset \star \text{rev}(\text{word}(P'))x_m$$

Similarly, $\text{rev}(\text{word}(P'))$ is the row word of P , so

$$\emptyset \star \text{word}(P) = \emptyset \star \text{rev}(\text{word}(P'))$$

Next, invoke reverse insertion at column $m-1, m-2, \dots, 1$ of P' and get numbers x_{m-1}, \dots, x_1 . Since P has $m-1$ rows, our inductive hypothesis says

$$\emptyset \star \text{word}(P) = \{x_{m-1}, \dots, x_1\}$$

By Lemma 2.14, $x_m > x_{m-1} > \dots > x_1$. Thus,

$$\begin{aligned} \emptyset \star \text{word}(T) &= \emptyset \star \text{rev}(\text{word}(P'))x_m \\ &= \emptyset \star \text{word}(P)x_m \\ &= \{x_{m-1}, \dots, x_1\} \star x_m \\ &= \{x_m, \dots, x_1\} \end{aligned}$$

□

REFERENCES

- [AS] S. Assaf and A. Schilling. A Demazure crystal construction for Schubert polynomials. *Algebr. Comb.* 1 (2018) no. 2, 225–247.
- [B] A. Buch. A Littlewood-Richardson rule for the K-theory of Grassmannians. *Acta Math.* 189 (2002), no. 1, 37–78.
- [B2] A. Buch. Alternating signs of quiver coefficients. *J. Amer. Math. Soc.* 18 (2005), no. 1, 217–237.
- [BF] A. Buch and W. Fulton. Chern class formulas for quiver varieties. *Invent. Math.* 135 (1999), no. 3, 665–687.
- [BJS] S. Billey, W. Jockusch, and R. Stanley. Some combinatorial properties of Schubert polynomials. *J. Algebraic Combin.* 2 (1993), no. 4, 345–374.
- [BKSTY] A. Buch, A. Kresch, M. Shimozono, H. Tamvakis, and A. Yong. Stable Grothendieck polynomials and K-theoretic factor sequences. *Math. Ann.* 340 (2008), no. 2, 359–382.
- [BS] A.S. Buch and M.J. Samuel, 2016. K-theory of minuscule varieties. *Journal für die reine und angewandte Mathematik*, 2016 (719), pp. 133–171.
- [BSS] G. Benkart, J. Stroomer, and F. Sottile. Tableau switching: algorithms and applications. *J. Combin. Theory Ser. A* 76 (1996), no. 1, 11–43.
- [BSW] V. Buciumas, T. Scrimshaw and K. Weber. Colored five-vertex models and Lascoux polynomials and atoms. *J. Lond. Math. Soc. (2)* 102 (2020), no. 3, 1047–1066.
- [Bu] A. Buch. A Littlewood-Richardson rule for the K -theory of Grassmannians. *Acta Math.* 189 (2002), no. 1, 37–78.
- [EG] P. Edelman and C. Greene. Balanced tableaux. *Adv. in Math.* 63 (1987), no. 1, 42–99.
- [FK] S. Fomin and A. N. Kirillov. Grothendieck polynomials and the Yang-Baxter equation. *Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique*, DIMACS, 1994, 183–189, Piscataway, NJ.
- [GMP+] C. Gaetz, M. Mastrianni, R. Patrias, H. Peck, C. Robichaux, D. Schwein, K. Y. Tam. K-Knuth equivalence for increasing tableaux. *Electron. J. Combin.* 23 (2016), no. 1, Paper 1.40, 37 pp.
- [GLS] M. Gurevich, E. Lapid, and M. Shimozono. Robinson-Schensted-Knuth correspondence in the representation theory of the general linear group over a non-archimedean local field. To appear in *Represent. Theory*.
- [H] M. Haiman. On mixed insertion, symmetry, and shifted Young tableaux. *J. Combin. Theory Ser. A* 50 (1989), no. 2, 196–225.
- [Kir] A. N. Kirillov. Notes on Schubert, Grothendieck and key polynomials. *SIGMA Symmetry Integrability Geom. Methods Appl.* 12 (2016), Paper No. 034, 56 pp.

- [KMS] A. Knutson, E. Miller, and M. Shimozono. Four positive formulae for type A quiver polynomials. *Invent. Math.* 166 (2006), no. 2, 229–325.
- [KMY] A. Knutson, E. Miller, and A. Yong. Tableau complexes. *Israel J. Math.* 163 (2008), 317–343.
- [KMY2] A. Knutson, E. Miller, and A. Yong. Gröbner geometry of vertex decompositions and of flagged tableaux. *J. Reine Angew. Math.* 630 (2009), 1–31.
- [Kn] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.* 34 (1970) no. 3, 709–727.
- [LamSh] T. Lam and M. Shimozono. Quantum cohomology of G/P and homology of affine Grassmannian. *Acta Math.* 204 (2010), no. 1, 49–90.
- [Las] A. Lascoux. Schubert & Grothendieck: un bilan bidécennal, *Sém. Lothar. Combin.* 50 (2003/04), Art. B50i.
- [LasSc] A. Lascoux and M. P. Schützenberger. Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux. *C. R. Acad. Sci. Paris Sér. I Math.* 295, 11 (1982), 629–633.
- [LasSc2] A. Lascoux M. P. Schützenberger. Tableaux and non-commutative Schubert polynomials. *Funct. Anal. Appl.* 23 (1989), 63–64.
- [LasSc3] A. Lascoux M. P. Schützenberger. Keys and standard bases. *Invariant theory and tableaux* (Minneapolis, MN, 1988), 125–144, IMA Vol. Math. Appl., 19, Springer, New York, 1990.
- [LLMSSZ] T. Lam, Luc Lapointe, J. Morse, A. Schilling, M. Shimozono, and M. Zabrocki. k -Schur functions and affine Schubert calculus. *Fields Institute Monographs*, 33. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2014.
- [LLS] T. Lam, S. J. Lee, and M. Shimozono. Back stable Schubert calculus. *Compos. Math.* 157 (2021), no. 5, 883–962.
- [LLS2] T. Lam, S. J. Lee, and M. Shimozono. Back stable K -theory Schubert calculus. In preparation.
- [Mat] T. Matsumura. Flagged Grothendieck polynomials. *J. Algebraic Combin.* 49 (2019), no. 3, 209–228.
- [Mi] E. Miller, Alternating formulas for K -theoretic quiver polynomials. *Duke Math. J.* 128 (2005), no. 1, 1–17.
- [Mon] C. Monical. Set-valued skyline fillings. *Sém. Lothar. Combin.* 78B (2017), Art. 35, 12 pp.
- [PeSc] O. Pechenik and T. Scrimshaw. K -theoretic crystals for set-valued tableaux of rectangular shapes. *Sém. Lothar. Combin.* 84B (2020), Art. 19, 12 pp.
- [Pet] D. Peterson. Quantum cohomology of G/P . Lecture notes, MIT, 1997.
- [ReY] V. Reiner and A. Yong. The “Grothendieck-to-Lascoux” conjecture. preprint arXiv:2102.12399v1, 2021.
- [RS] V. Reiner and M. Shimozono. Key polynomials and a flagged Littlewood-Richardson rule. *J. Combin. Theory Ser. A* 70 (1995), no. 1, 107–143.
- [RoY] C. Ross and A. Yong. Combinatorial rules for three bases of polynomials. *Sém. Lothar. Combin.* 74 (2015), Article B74a, 11pp.
- [TY] H. Thomas and A. Yong. A jeu de taquin theory for increasing tableaux, with applications to K -theoretic Schubert calculus. *Algebra Number Theory* 3 (2009), no. 2, 121–148.
- [Wil] M. Willis. A Direct Way to Find the Right Key of a Semistandard Young Tableau. *Annals of Combinatorics*, 17, no. 2, p. 393–400, 2013.

DEPARTMENT OF MATHEMATICS, 460 MCBRYDE HALL, VIRGINIA TECH, 255 STANGER ST.,
BLACKSBURG, VA, 24601, USA

Email address: mshimo@math.vt.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CA,
92093, USA

Email address: tiy059@ucsd.edu