Grothendieck-to-Lascoux expansions

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Outline

1. Introducing 8 polynomials

2. The Grothendieck-to-Lascoux expansions

Operators

Define operators on $\mathbb{Z}[\beta][x_1, x_2, \ldots, x_n]$:

$$\begin{aligned}
\partial_i(f) &= (x_i - x_{i+1})^{-1}(f - s_i f) \\
\pi_i(f) &= \partial_i(x_i f) \\
\partial_i^{(\beta)}(f) &= \partial_i(f + \beta x_{i+1} f) \\
\pi_i^{(\beta)}(f) &= \partial_i^{(\beta)}(x_i f).
\end{aligned}$$

- $\partial_1(x_1^2) = x_1 + x_2$
- $\pi_1(x_1^2) = x_1^2 + x_1x_2 + x_2^2$
- $\partial_1^{(\beta)}(x_1^2) = x_1 + x_2 + \beta x_1 x_2$
- $\pi_1^{(\beta)}(x_1^2) = x_1^2 + x_1x_2 + x_2^2 + \beta x_1^2 x_2 + \beta x_1 x_2^2$

Lascoux polynomials and key polynomials

For weak composition α,

$$\mathfrak{L}_{\alpha}^{(\beta)} := \begin{cases} x^{\alpha} & \text{if } \alpha \text{ is a partition} \\ \pi_i^{(\beta)} \mathfrak{L}_{s_i\alpha}^{(\beta)} & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

$$\kappa_{\alpha} := \begin{cases} x^{\alpha} & \text{if } \alpha \text{ is a partition} \\ \pi_i(\kappa_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

Fact: $\kappa_{\alpha} = \mathfrak{L}_{\alpha}^{(\beta)}|_{\beta=0}$.

- $\mathfrak{L}_{210}^{(\beta)} = x_1^2 x_2$
- $\mathfrak{L}_{120}^{(\beta)} = \pi_1^{(\beta)}(\mathfrak{L}_{210}^{(\beta)}) = x_1^2 x_2 + x_1 x_2^2 + \beta x_1^2 x_2^2$

β-Grothendieck Polynomials and Schubert Polynomials

For $w \in S_{n+1}$,

$$\mathfrak{G}_w^{(\beta)} := \begin{cases} x_1^n x_2^{n-1} \cdots x_n & \text{if } w = (n+1, n, \ldots, 1) \\ \partial_i^{(\beta)}(\mathfrak{G}_{ws_i}^{(\beta)}) & \text{if } ws_i > w. \end{cases}$$

$$\mathfrak{S}_w := \begin{cases} x_1^n x_2^{n-1} \cdots x_n & \text{if } w = (n+1, n, \ldots, 1) \\ \partial_i(\mathfrak{S}_{ws_i}) & \text{if } ws_i > w. \end{cases}$$

Fact: $\mathfrak{S}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=0}$.

- $\mathfrak{G}_{321}^{(\beta)} = x_1^2 x_2$
- $\mathfrak{G}_{312}^{(\beta)} = \partial_2^{(\beta)}(\mathfrak{G}_{321}^{(\beta)}) = x_1^2$
- $\mathfrak{G}_{132}^{(\beta)} = \partial_1^{(\beta)}(\mathfrak{G}_{312}^{(\beta)}) = x_1 + x_2 + \beta x_1 x_2$

Symmetrization

Let $\pi_{w_0} := (\pi_{n-1} \ldots \pi_2 \pi_1)^{n-1}$.

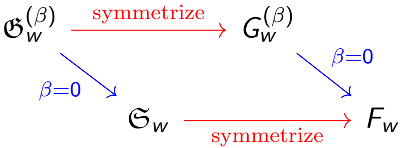
$$\pi_{w_0}^{(\beta)}(\mathfrak{L}_\alpha^{(\beta)}) = G_{\alpha^+}^{(\beta)} \qquad \text{(Grass. Symm. Groth. polynomials)}$$

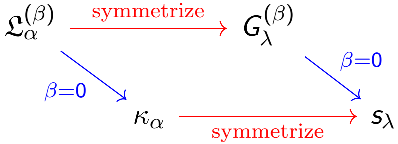
$$\pi_{w_0}(\kappa_\alpha) = s_{\alpha^+} \qquad \text{(Schur polynomials)}$$

$$\pi_{w_0}^{(\beta)}(\mathfrak{G}_w^{(\beta)}) = G_w^{(\beta)} \qquad \text{(Symm. Groth. polynomials)}$$

$$\pi_{w_0}(\mathfrak{S}_w) = F_w \qquad \text{(Stanley Symmetric Functions)}$$

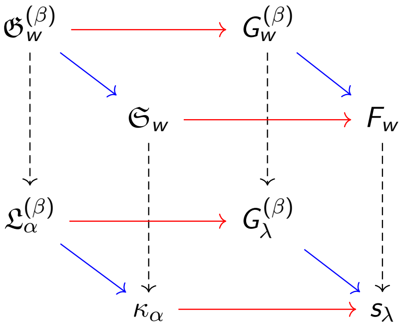
Quick review





Next, we connect these two diagrams.

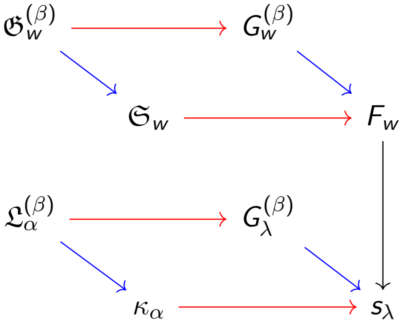
Expansions



$\mathfrak{G}_w^{(\beta)}$
$G_w^{(\beta)}$
$\mathfrak{S}_w$
F_w
$\mathfrak{L}_\alpha^{(\beta)}$
$G_\lambda^{(\beta)}$
κ_α
s_λ

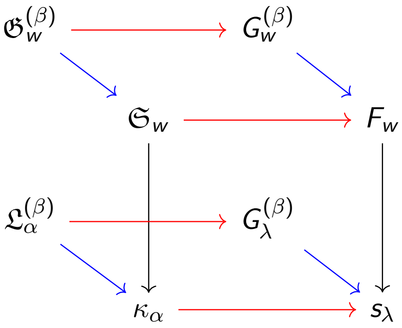
F_w into s_λ

In 1987, Edelman and Greene expanded F_w into s_λ.



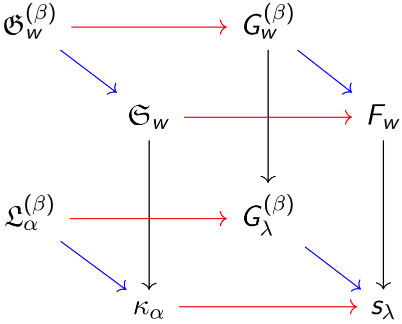
$\mathfrak{S}_w$ into κ_α

In 1995, Reiner and Shimozono expanded $\mathfrak{S}_w$ into κ_α. This expansion was first stated by Lascoux and Schützenberger in 1989.



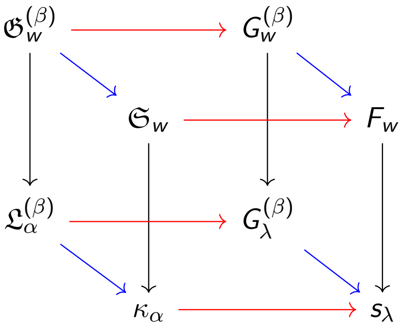
$G_w^{(\beta)}$ into $G_\lambda^{(\beta)}$

In 2008, Buch, Kresch, Shimozono, Tamvakis, Yong expanded $G_w^{(\beta)}$ into $G_\lambda^{(\beta)}$.



$\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$

In 2021, Shimozono and Yu expanded $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$. This expansion was conjectured by Reiner and Yong.



Keys

A *key* is a filling of a Young diagram where each number in column j is also in column $j-1$ and each column is decreasing. Keys are in bijection with weak compositions:

$$(1,0,3,2) \Longleftrightarrow \begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

Let $\mathrm{key}(\cdot)$ send a weak composition to its corresponding key.

Decreasing tableaux

The following P is an example of a decreasing tableau.

5	4	1
3	2	
2		

Each decreasing tableau is associated with a key called its "right key". $K_+(P) =$

4	4	1
3	1	
1		

Reversed set-valued tableaux (RSVT)

The following T is an example of a RSVT

54	4	1
3	321	
21		

Then $\mathrm{wt}(Q) = (3, 2, 2, 2, 1)$, $\mathrm{ex}(Q) = 4$.
Each RSVT is associated with a key called its "left key". $K_{-}(Q) =$

5	5	3
3	3	
2		

RSVT rule for $\mathfrak{L}_\alpha^{(\beta)}$

Theorem (Buciumas, Scrimshaw, Weber 2020; Shimozono, Y 2021)

$$\mathfrak{L}_\alpha^{(\beta)} = \sum_T \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)}$$

where T is a RSVT s.t. $K_-(T) \leq \mathrm{key}(\alpha)$.

Example $\mathfrak{L}^{(\beta)}_{(1,0,2)}$

$$\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 31 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 21 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 32 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 21 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 32 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 32 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 321 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 32 & 21 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\begin{aligned}\mathfrak{L}^{(\beta)}_{(1,0,2)} &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 \\ &+ \beta(x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_3^2 + x_1x_2x_3^2) \\ &+ \beta^2(x_1^2x_2^2x_3 + x_1^2x_2x_3^2)\end{aligned}$$

Compatible words

Definition (Billey, Jockusch, Stanley 1993)

A pair of words (a, i) with the same length is compatible if they satisfy

- *i is weakly decreasing*
- *$i_j = i_{j+1}$ implies $a_j < a_{j+1}$.*

A compatible word (a, i) is *bounded* if $a_j \geq i_j$ for all j.
Each word a is associated with a permutation, denoted as $[a]_H$.

Combinatorial formula for $\mathfrak{G}_w^{(\beta)}$ and $G_w^{(\beta)}$

Theorem (Fomin, Kirillov 1994)

$$\mathfrak{G}_w^{(\beta)}(x_1, \ldots, x_n) = \sum_{\substack{(a,i) \text{ compatible} \\ (a,i) \text{ bounded} \\ [a]_H = w^{-1}}} x^{\mathrm{wt}(i)} \beta^{\ell(i) - \ell(w)},$$

Hecke Insertion

Let $\mathcal{C}$ be the set of all compatible words.
Let $\mathcal{T}$ be the set of all (P, Q) such that, P is decreasing, Q is a RSVT, and P, Q have the same shape.

Theorem (Buch,Kresch,Shimozono,Tamvakis,Yong 2008)

Hecke insertion is a bijection from $\mathcal{C}$ to $\mathcal{T}$. If we insert (a, i) and get (P, Q), then

- $[a]_H = [P]_H$.
- $\mathrm{wt}(i) = \mathrm{wt}(Q)$.

Expand $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$

Let $\mathcal{C}^B_{w^{-1}} := \{(a,i) \in \mathcal{C} : [a]_H = w^{-1}, \text{bounded}\}$.
Let $\mathcal{T}^B_{w^{-1}} := \{(P,Q) \in \mathcal{T} : [P]_H = w^{-1}, K_+(P) \geq K_-(Q)\}$.

Theorem (Shimozono, Y 2021)

Hecke insertion restricts to a bijection from $\mathcal{C}^B_{w^{-1}}$ to $\mathcal{T}^B_{w^{-1}}$.

$$
\begin{aligned}
\mathfrak{G}_w^{(\beta)} &= \sum_{(a,i)\in\mathcal{C}^B_{w^{-1}}} x^{\mathrm{wt}(i)} \beta^{\ell(i)-\ell(w)} \\
&= \sum_{(P,Q)\in\mathcal{T}^B_{w^{-1}}} x^{\mathrm{wt}(Q)} \beta^{\mathrm{ex}(Q)+|\mathrm{shape}(Q)|-\ell(w)} \\
&= \sum_{P} \beta^{|\mathrm{shape}(P)|-\ell(w)} \sum_{Q} x^{\mathrm{wt}(Q)} \beta^{\mathrm{ex}(Q)} \\
&= \sum_{P} \beta^{|\mathrm{shape}(P)|-\ell(w)} \mathfrak{L}^{(\beta)}_{K_+(P)}
\end{aligned}
$$

Expand $\mathfrak{G}_w^{(\beta)}$ into $\mathfrak{L}_\alpha^{(\beta)}$ example

Let $w = 31524$. Then P can be:

4	3	1
2		

4	3
2	1

4	3	1
2	1	

There right keys are:

3	1	1
1		

3	3
1	1

3	3	1
1	1	

Thus, we have $\mathfrak{G}_w^{(\beta)} = \mathfrak{L}_{301}^{(\beta)} + \mathfrak{L}_{202}^{(\beta)} + \beta\mathfrak{L}_{302}^{(\beta)}$.

Thanks for listening!!

- M. Shimozono, and T Yu. Grothendieck to Lascoux expansions. arXiv preprint arXiv:2106.13922 (2021).