Harmonic bases for generalized coinvariant algebras

Tianyi Yu
(Joint with Brendon Rhoades and Zehong Zhao)

UCSD

Outline

1. The classical coinvariant algebra R_n and its harmonic space V_n

2. The generalized coinvariant algebra $R_{n,\lambda}$

3. Describe the harmonic space and construct a harmonic basis for $R_{n,\lambda}$.

Classical coinvariant algebra

Let I_n be an ideal of $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ defined as

$$I_n := \langle e_1, \ldots, e_n \rangle$$

where e_d is the elementary symmetric polynomial of degree d.

The classical coinvariant ring R_n is the associated quotient ring

$$R_n := \mathbb{Q}[\mathbf{x}_n]/I_n$$

Some properties of R_n

1. **Artin**: The following set of monomials:

$$\{x_1^{i_1} \dots x_n^{i_n} : 0 \leq i_j \leq n-j\}$$

   descends to a basis of R_n.

2. **Chevalley**: R_n is isomorphic to the regular representation $\mathbb{Q}[\mathfrak{S}_n]$ as ungraded $\mathfrak{S}_n$-modules.

3. **Lusztig-Stanley**:

$$\mathrm{grFrob}(R_n; q) = \sum_{w = w_1 \dots w_n} q^{maj(w)} x_{w_1} \dots x_{w_n}$$

Defining the harmonic space

Take $f \in \mathbb{Q}[\mathbf{x}_n]$. Let ∂f be the differential operator

$$\partial f := f(\partial/\partial x_1, \ldots \partial/\partial x_n)$$

Then $\mathbb{Q}[\mathbf{x}_n]$ acts on itself by:

$$f \odot g := (\partial f)(g)$$

We also define an inner product of $\mathbb{Q}[\mathbf{x}_n]$:

$$\langle f, g \rangle := \text{constant term of } f \odot g$$

Defining the harmonic space

Let $I \subset \mathbb{Q}[\mathbf{x}_n]$ be a homogeneous ideal. Its harmonic space V is defined as:

$$V := I^{\perp} = \{g \in \mathbb{Q}[\mathbf{x}_n] \, : \, \langle f, g \rangle = 0 \text{ for all } f \in I\}$$

A basis of V is called a *harmonic basis*.

Fact: If I is $\mathfrak{S}_n$-invariant, then $\mathbb{Q}[\mathbf{x}_n]/I \cong V$ as graded $\mathfrak{S}_n$-modules.

Now, let V_n be the harmonic space associated to R_n.

Motivating V_n

Why we want to study V_n, instead of R_n?

Answer: It is hard to determine whether $f + I_n = 0$ for a given $f \in \mathbb{Q}[\mathbf{x}_n]$. We can avoid this challenge by studying V_n. Elements of V_n are polynomials, not cosets.

Describe V_n

Fact: V_n is the smallest space that contains δ_n and is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Here, δ_n is the *Vandermonde determinant*:

$$\delta_n := \prod_{1 \le i < j \le n} (x_i - x_j).$$

Fact: The following is a basis of V_n.

$$\{(x_1^{c_1} \cdots x_n^{c_n}) \odot \delta_n \ : \ 0 \le c_i \le n - i\}.$$

From R_n to $R_{n,\lambda}$

Sean Griffin generalized R_n to $R_{n,\lambda}$. Let $k \leq n$ be nonnegative integers and let λ be a partition of k with s parts. Then let $I_{n,\lambda} \subseteq \mathbb{Q}[\mathbf{x}_n]$ be the ideal generated by $x_1^s, \ldots, x_n^s$ and $e_d(S)$, where the range of S and d will be illustrated in the next example.

Let $R_{n,\lambda} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$ be the associated quotient ring. Let $V_{n,\lambda}$ be the harmonic space.

An example of $I_{n,\lambda}$

Assume $n = 9, k = 7, s = 4$, and $\lambda = (3, 2, 2, 0)$.
$I_{9,(3,2,2,0)}$ is generated by $x_1^4, \ldots, x_9^4$ together with: $e_d(S)$, where possible d, S are:

9	8	7
6	5	4
3		

$|S| = 9$

·	·	·
8	7	6
5		

$|S| = 8$

·	·	·
·	·	·
7		

$|S| = 7$

Some special cases of $R_{n,\lambda}$

1. When $k = s = n$ and $\lambda = (1^n)$, then $R_{n,\lambda} = R_n$.
2. When $k = n$ and λ has no 0s, the ring $R_{n,\lambda}$ is the *Tanisaki quotient* studied by Tanisaki and Garsia-Procesi.
3. When $\lambda = (1^k, 0^{s-k})$, the ring $R_{n,\lambda}$ was introduced by Haglund, Rhoades and Shimozono to give a representation-theoretic model for the Haglund-Remmel-Wilson Delta Conjecture

Injective tableaux

Let λ be a partition. Let $\mathrm{Inj}(\lambda; \leq n)$ be the family of tableaux of shape λ' such that:

1. No two entries are the same.
2. Each entry is at most n.

$\mathrm{Inj}((4,2,1,0,0); \leq 9)$ contains

2	6	5
4	1	
3		
9		

Generalizing Vandermonde

For any subset $S \subseteq [n]$, define

$$\delta_S := \prod_{\substack{i,j \in S \\ i<j}} (x_i - x_j)$$

Take $T \in \mathrm{Inj}(\lambda; \leq n)$, where λ has s parts. Let R_i be the set of numbers in row i of T. Then

$$\delta_T := \delta_{R_1} \cdots \delta_{R_{\lambda_1}} \times \prod x_i^{s-1}$$

where the final product is over all $i \in [n]$ which do not appear in T.

δ_T example

Let T be the following element in $\mathrm{Inj}((4,2,1,0,0); \leq 9)$:

2	6	5
4	1	
3		
9		

Then $R_1 = \{2,5,6\}$, and

$$\delta_{R_1} = (x_2 - x_5)(x_2 - x_6)(x_5 - x_6)$$

Then we have

$$\begin{aligned} \delta_T &= \delta_{\{2,5,6\}} \times \delta_{\{1,4\}} \times \delta_{\{3\}} \times \delta_{\{9\}} \times x_7^4 x_8^4 \\ &= (x_2 - x_5)(x_2 - x_6)(x_5 - x_6) \times (x_1 - x_4) \times 1 \times 1 \times x_7^4 x_8^4. \end{aligned}$$

Describing $V_{n,\lambda}$

Theorem ([Rhoades-Y-Zhao])

Let $k \leq n$ and λ be a partition of k. The harmonic space $V_{n,\lambda}$ is the smallest subspace of $\mathbb{Q}[\mathbf{x}_n]$ which

- *contains δ_T for any $T \in \mathrm{Inj}(\lambda, \leq n)$, and*
- *is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$.*

For Tanisaki quotients, this statement was proved by N.Bergeron and Garsia.

A spanning set of $V_{n,\lambda}$

Goal: construct a basis of $V_{n,\lambda}$.

Fact: The following is a spanning set of $V_{n,\lambda}$:

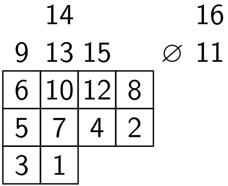
$$\{(x_1^{b_1} \cdots x_n^{b_n}) \odot \delta_T \ : \ T \in \mathrm{Inj}(\lambda; \leq n), \ b_i \geq 0\}$$

Strategy: Extract a basis from this spanning set.

Ordered set partition

Given $k \leq n$ and a partition λ of k with s parts, let $\mathcal{OP}_{n,\lambda}$ be the family of sequences $\sigma = (B_1, \ldots, B_s)$ of subsets of $[n]$ such that $[n] = B_1 \sqcup \cdots \sqcup B_s$ and $|B_i| \geq \lambda_i$ for all i.

For example, if $n = 16$ and $\lambda = (3, 3, 2, 2, 0, 0)$, then $\mathcal{OP}_{n,\lambda}$ contains the following:



Inversions

Assume i is in a box. An *inversion* of i is a number j such that

1. $j > i$.
2. j is on the left of i in the same row.
3. The number below j does not exists or is less than i.

	14				16
9	13	15		$\varnothing$	11
6	10	12	8		
5	7	4	2		
3	1				

Inversions

Assume i is not in a box. An *inversion* of i is a column such that

1. The column is on the right of i.
2. The column has no boxes, or its highest number in box is less than i.

	14				16
9	13	15		$\varnothing$	11
6	10	12	8		
5	7	4	2		
3	1				

Generalizing Lehmer code

Assign a sequence of n numbers to each $\sigma \in \mathcal{OP}_{n,\lambda}$.
The i^{th} entry is the number of inversions of i.

	14				16
9	13	15		$\varnothing$	11
6	10	12	8		
5	7	4	2		
3	1				

$$\mathrm{code}(\sigma) = (1, 1, 0, 2, 0, 0, 0, 2, 3, 0, 0, 0, 4, 4, 3, 0).$$

δ_σ

Let $T(\sigma)$ be the element in $\mathrm{Inj}(\lambda; \leq n)$ obtained by removing all numbers outside of boxes.

	14				16
9	13	15		$\varnothing$	11
6	10	12	8		
5	7	4	2		
3	1				

6	10	12	8
5	7	4	2
3	1		

Define δ_σ by the rule

$$\delta_\sigma := (x_1^{c_1} \cdots x_n^{c_n}) \odot \delta_{T(\sigma)}$$

where $\mathrm{code}(\sigma) = (c_1, \ldots, c_n)$.

Harmonic Basis

Theorem ([Rhoades-Y-Zhao])

Let $k \leq n$ be positive integers and let λ be a partition of k with s parts. The set

$$\{\delta_\sigma \,:\, \sigma \in \mathcal{OP}_{n,\lambda}\}$$

is a harmonic basis of $R_{n,\lambda}$.

This result implies a combinatorial formula for the Hilbert series of $R_{n,\lambda}$:

$$\mathrm{rev}(\mathrm{Hilb}(R_{n,\lambda}; q)) = \sum_{\sigma \in \mathcal{OP}_{n,\lambda}} q^{sum(\mathtt{code}(\sigma))}.$$

A future direction

We can introduce a new set of variables $y_1, \ldots, y_n$ to $V_{n,\lambda}$. Define $DV_{n,\lambda}$ to be the smallest space such that:

1. It contains δ_T for any $T \in \mathrm{Inj}(\lambda, \leq n)$
2. It is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and $\partial/\partial y_1, \ldots, \partial/\partial y_n$
3. It is closed under $y_1(\partial/\partial x_1) + \cdots + y_n(\partial/\partial x_n)$

Question: What is its Bigraded Frobenius image?

Haiman solved the special case: $\lambda = (1^n)$.

Thanks for listening!!

- B. Rhoades, T. Yu, and Z. Zhao. Harmonic bases for generalized coinvariant algebras. *Electronic Journal of Combinatorics*, **4 (4)** (2020))