A BIJECTION BETWEEN K-KOHNERT DIAGRAMS AND REVERSE SET-VALUED TABLEAUX

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Abstract. Lascoux polynomials are K-theoretic analogues of the key polynomials. They both have combinatorial formulas involving tableaux: reverse set-valued tableaux (RSVT) rule for Lascoux polynomials and reverse semistandard Young tableaux (RSSYT) rule for key polynomials. Furthermore, key polynomials have a simple algorithmic model in terms of Kohnert diagrams, which are in bijection with RSSYT. Ross and Yong introduced K-Kohnert diagrams, which are analogues of Kohnert diagrams. They conjectured a K-Kohnert diagram rule for Lascoux polynomials. We establish this conjecture by constructing a weight-preserving bijection between RSVT and K-Kohnert diagrams.

1. Introduction

Fix a positive integer n throughout this paper. A *weak composition of length n* is a sequence of n non-negative integers. If α is a weak composition, we use α_i to denote its i^{th} entry.

Key polynomials κ_α are homogeneous polynomials labeled by weak compositions. They were first introduced by Demazure [Dem74] as the characters of the Demazure modules. Further studies [Koh90, LS90, LS89, RS95, Kas93, Las13, Lit95, AS18, Ass22] provided several combinatorial formulas.

Lascoux polynomials $\mathfrak{L}_\alpha^{(\beta)}$ are K-theoretic generalizations of key polynomials [Las01]. They are inhomegeneous polynomials with an extra variable β. Setting $\beta = 0$ in $\mathfrak{L}_\alpha^{(\beta)}$ yields κ_α. There are several existing combinatorial formulas for $\mathfrak{L}_\alpha^{(\beta)}$ involving set-valued skyline fillings and set-valued tableaux [BSW20, Yu21]. In this paper, we will define Lascoux polynomials by a combinatorial formula involving reverse set-valued tableaux (RSVT). It first appeared implicitly in [BSW20] and was rediscovered by Shimozono and the second author [SY21]. Specifically, for each weak composition α, there is a set $\mathsf{RSVT}(\alpha)$, which consists of certain RSVT satisfying a left key condition (see subsection 2.1). Then $\mathfrak{L}_\alpha^{(\beta)}$ can be written as a sum over $\mathsf{RSVT}(\alpha)$:

$$\mathfrak{L}_\alpha^{(\beta)} := \sum_{T \in \mathsf{RSVT}(\alpha)} \beta^{\mathsf{ex}(T)} x^{\mathsf{wt}(T)}.$$

Ross and Yong [RY15] defined a generalization of Kohnert's move on diagrams [Koh90]. We call them K-Kohnert moves. Repeatedly applying K-Kohnert moves on the key diagram of α yields a set of diagrams, which is denoted as $\mathsf{KKD}(\alpha)$ (see subsection 2.3).

Conjecture 1.1. [RY15] *The Lascoux polynomials indexed by α, is given by*

$$\mathfrak{L}_\alpha^{(\beta)} = \sum_{D \in \mathsf{KKD}(\alpha)} \beta^{\mathsf{ex}(D)} \boldsymbol{x}^{\mathsf{wt}(D)}.$$

Pechenik and Scrimshaw [PS20] proved a special case of this conjecture where all positive numbers in α are the same. This paper will prove the conjecture for all α.

Theorem 1.2. *Conjecture 1.1 is true.*

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To prove this theorem, we define two maps: Ψ_α on $\mathsf{KKD}(\alpha)$ (see subsection 3.1) and Φ_α on $\mathsf{RSVT}(\alpha)$ (see subsection 3.2). We will show Ψ_α (resp. Φ_α) is a well-defined map to $\mathsf{RSVT}(\alpha)$ (resp. $\mathsf{KKD}(\alpha)$). Finally, we establish the following.

Theorem 1.3. *The maps $\Psi_\alpha : \mathsf{KKD}(\alpha) \to \mathsf{RSVT}(\alpha)$ and $\Phi_\alpha : \mathsf{RSVT}(\alpha) \to \mathsf{KKD}(\alpha)$ are mutually inverses of each other. Moreover, they preserve $\mathsf{wt}(\cdot)$ and $\mathsf{ex}(\cdot)$.*

The paper is organized as follows. In section 2, we review related combinatorial rules for κ_α and $\mathfrak{L}_\alpha^{(\beta)}$. In section 3, we define two maps Ψ_α and Φ_α on $\mathsf{KKD}(\alpha)$ and $\mathsf{RSVT}(\alpha)$ respectively. The following sections will prove Theorem 1.3. In section 4, we introduce a partial order on all weak compositions. We call it the Bruhat order and show it is equivalent to the left swap order in [AS18]. In section 5, we describe the sets $\mathsf{KKD}(\alpha)$ and $\mathsf{RSVT}(\alpha)$ recursively using the Bruhat order. In section 6, we introduce two auxiliary operators $\sharp_g$ and $\flat_e$ on $\mathsf{KD}(\alpha)$ and discuss their properties. In section 7, we give recursive descriptions of maps Ψ_α and Φ_α in terms of $\sharp_g$ and $\flat_e$. Then we show Ψ_α (resp. Φ_α) is a well-defined map to $\mathsf{RSVT}(\alpha)$ (resp. $\mathsf{KKD}(\alpha)$) using the recursive definitions developed in section 5. Finally we prove Theorem 1.3.

2. Background

2.1. $\mathsf{RSSYT}(\alpha)$ **and** $\mathsf{RSVT}(\alpha)$**.** Given a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell \geqslant 0)$, a *Young diagram* of shape λ is a finite collection of boxes, aligned at the left, in which the i^{th} row has λ_i boxes. We use English convention for our Young diagrams and tableaux, so the first row is the highest row.

A *reverse semistandard Young tableau* of shape λ is a filling of the Young diagram λ with positive number such that

(i) each box contains exactly one number,
(ii) the entries in each row weakly decrease from left to right, and
(iii) the entries of each column strictly decrease from top to bottom.

Let RSSYT_λ be the set of all the reverse semistandard Young tableaux of shape λ.

Following [SB02], we introduce another set of tableaux where a box might have more than one number. A *reverse set-valued tableau* of shape λ is a filling of the Young diagram λ with positive numbers such that

(i) each box contains a finite and non-empty set of positive integers,
(ii) if a set A is to the left of a set B in the same row, then $\min(A) \geqslant \max(B)$, and
(iii) if a set C is below a set A in the same column, then $\min(A) > \max(C)$.

Let RSVT_λ be all the reverse set-valued tableaux of shape λ.

Let the *weight vector* for T be the weak composition whose i^{th} component is the the total number of appearance of i in T, denoted by $\mathsf{wt}(T)$. Given any weak composition α, let $|\alpha| = \sum_{i\geqslant 1} \alpha_i$. Given $T \in \mathsf{RSVT}_\lambda$, define $\mathsf{L}(T)$ to be an element in RSSYT_λ constructed by only keeping the largest number in each box of T. We call these numbers the *leading numbers* of T. Any number in T that is not a leading number is called a *extra number*. Let the *excess* of T be the number of extra numbers in T, so we can denote it by $\mathsf{ex}(T) = |\mathsf{wt}(T)| - |\lambda|$.

Next we give the definition of *left key* of T, denoted by $\mathsf{K}_-(T)$, where T is a RSSYT. It was first given in [Wil13, Section 5]. We give the description as in [SY21, Definition 3.11].

Definition 2.1. *Let C_1, C_2 be two adjacent columns from a* RSSYT *with C_1 on the left. We may view C_1 and C_2 as sets. We define $C_1 \lhd C_2$ as follows. Assume $C_2 = \{a_1 < a_2 < \cdots < a_m\}$. Start by finding the smallest $b_1 \in C_1$ such that $b_1 \geqslant a_1$. Then find the smallest $b_2 \in C_2$ such that $b_2 \geqslant a_2$ and $b_2 > b_1$. Continue until we find all $b_1, b_2, \ldots, b_m$. Then $C_1 \lhd C_2 := b_1 < b_2 < \ldots b_m$. Let $C_1, \ldots C_k$ be k columns in a* RSSYT*, then we can define recursively,*

$$C_1 \lhd C_2 \cdots \lhd C_k := C_1 \lhd (C_2 \lhd \cdots \lhd C_k).$$

Given a reverse semistandard Young tableau T with columns $C_1, C_2, \dots, C_n$. Then its left key $\mathsf{K}_-(T)$ is a RSSYT constructed by taking $C_1 \lhd C_2 \cdots \lhd C_k$ as its k^{th} column.

Given a reverse set-valued tableau T, its left key $K_-(T)$ is defined as $K_-(\mathsf{L}(T))$.

Example 2.2. *Consider the following $T \in \mathsf{RSVT}_{(3,2)}$. We have $\mathsf{wt}(T) = (2,2,2,1,0,1)$ and $\mathsf{ex}(T) = 3$. We can also compute $\mathsf{L}(T)$ and its left key.*

$$T = \begin{array}{|c|c|c|}\hline 64 & 32 & 2 \\ \hline 31 & 1 \\ \cline{1-2}\end{array}, \quad \mathsf{L}(T) = \begin{array}{|c|c|c|}\hline 6 & 3 & 2 \\ \hline 3 & 1 \\ \cline{1-2}\end{array}, \quad \mathsf{K}_-(T) = \mathsf{K}_-(\mathsf{L}(T)) = \begin{array}{|c|c|c|}\hline 6 & 6 & 6 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}.$$

Given a weak composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, let α^+ be the partition obtained from α by sorting the numbers in decreasing order and ignoring the trailing 0's. Define the *key tableau* for α to be the unique element in $\mathsf{RSSYT}_{\alpha^+}$ whose j^{th} column consists of the numbers $\{i | \alpha_i \geqslant j\}$. Denote this tableau by $\mathsf{key}(\alpha)$.

Remark 2.3. *For any reverse set-valued tableau T, $\mathsf{K}_-(T)$ is a key tableau.*

With above concepts, we now define the subsets of $\mathsf{RSSYT}_{\alpha^+}$ and RSVT_{α^+} that will be used to compute κ_α and $\mathfrak{L}^{(\beta)}_\alpha$.

$$\mathsf{RSSYT}(\alpha) := \{T \in \mathsf{RSSYT}_{\alpha^+} : \mathsf{K}_-(T) \leqslant \mathsf{key}(\alpha)\}$$
$$\mathsf{RSVT}(\alpha) := \{T \in \mathsf{RSVT}_{\alpha^+} : \mathsf{K}_-(T) \leqslant \mathsf{key}(\alpha)\}$$

Here the $\leqslant$ relation means entry-by-entry comparison. For example, T in Example 2.2 is in $\mathsf{RSVT}((0,0,2,0,0,3))$ but not in $\mathsf{RSVT}((0,2,0,0,3))$.

We now list the combinatorial formulas in [LS90, LS89, RS95] for *key polynomials*, and in [BSW20, SY21] for *Lascoux polynomials* labeled by a weak composition α:

$$\kappa_\alpha := \sum_{T \in \mathsf{RSSYT}(\alpha)} x^{\mathsf{wt}(T)}, \quad \mathfrak{L}^{(\beta)}_\alpha := \sum_{T \in \mathsf{RSVT}(\alpha)} \beta^{\mathsf{ex}(T)} x^{\mathsf{wt}(T)}.$$

2.2. **Viewing $\mathsf{RSVT}(\alpha)$ as a pair of diagrams.** In this subsection, we introduce another perspective on $\mathsf{RSVT}(\alpha)$. A *diagram* is a finite subset of $\mathbb{N} \times \mathbb{N}$. We may represent a diagram by putting a box at row r and column c for each (c, r) in the diagram. We adopt the convention where columns begin at 1 from the left and rows begin at 1 from the bottom. The *weight* of a diagram D, denoted as $\mathsf{wt}(D)$, is a weak composition whose i^{th} entry is the number of boxes in its i^{th} row.

A *diagram pair* is an ordered pair $D = (D_1, D_2)$ such that D_1 and D_2 are disjoint diagrams. We may represent D by putting a box at (c, r) for each $(c, r) \in D_1$ and putting a box with label X at (c, r) for each $(c, r) \in D_2$. Cells in D_1 are called *Kohnert cells*. Cells in D_2 are called *ghost cells*. The *weight* of D, denoted as $\mathsf{wt}(D)$, is a weak composition whose i^{th} entry is the number of Kohnert cells and ghost cells in its i^{th} row. Let the *excess* of D, denoted by $\mathsf{ex}(D)$, be $|D_2|$.

Now we embed the set of RSVT into the set of diagram pairs. Given an RSVT T, we send it to (L, E). The set L (resp. E) consists of all (r, c) such that r is a leading (resp. extra) number in column c of T. This map is injective. If we know (L, E) is the image of some RSVT T, we can uniquely recover T: First, for each c, build a column that consists of r such that $(c, r) \in L$. The column should be decreasing from top to bottom. Then for each $(c, r) \in E$, put r in the lowest cell whose largest number is larger than r. This will be column c of T. Now we may view each $RSVT$ as a diagram pair. We write $T = (L, E)$ to denote this correspondence. It is clear that this correspondence preserves $\mathsf{wt}(\cdot)$ and $\mathsf{ex}(\cdot)$.

Example 2.4. *Let T be the RSVT in the previous example. It corresponds to the diagram pair* $(\{(1,3),(1,6),(2,1),(2,3),(3,2)\},\{(1,1),(1,4),(2,2)\})$, *which can be presented as*

$$\begin{array}{ccc} \cdot & & \\ & & \\ \mathsf{X} & & \\ \cdot & \cdot & \\ & \mathsf{X} & \cdot \\ \mathsf{X} & \cdot & \end{array}$$

Viewing T as a diagram pair, we have $\mathsf{wt}(T) = (2,2,2,1,0,1)$ and $\mathsf{ex}(T) = 3$, which agrees with the previous example.

We may also view $\mathsf{RSSYT}(\alpha)$ as a subset of $\mathsf{RSVT}(\alpha)$. Thus, $\mathsf{RSSYT}(\alpha)$ is the set of diagram pairs $(L,\emptyset) \in \mathsf{RSVT}(\alpha)$. With this convention, we have the following observation.

Remark 2.5. *If the diagram pair (L,E) is $\mathsf{RSVT}(\alpha)$, then $(L,\emptyset) \in \mathsf{RSSYT}(\alpha)$.*

2.3. $\mathsf{KD}(\alpha)$ **and** $\mathsf{KKD}(\alpha)$**.** We give another combinatorial definition of key polynomials due to Kohnert [Koh90]. A diagram pair is called a *key diagram pair* if its Kohnert cells are left-justified and has no ghost cells. Given a weak composition α, we let D_α the key diagram pair associated to α: On its row i, there are α_i left-justified Kohnert cells and no ghost cells.

Next, we define a *Kohnert move* on a diagram pair with no ghost cells: Select the rightmost box in any row and move it downward to the first position available, possibly jumping over other cells as needed. Let $\mathsf{KD}(\alpha)$ be the closure of $\{D_\alpha\}$ under all possible Kohnert moves.

Theorem 2.6. [Koh90] *The key polynomials indexed by α, is given by*

$$\kappa_\alpha = \sum_{D \in \mathsf{KD}(\alpha)} \boldsymbol{x}^{\mathsf{wt}(D)}$$

Remark 2.7. *There is a natural identification between $\mathsf{KD}(\alpha)$ and $\mathsf{RSSYT}(\alpha)$ which yields $\mathsf{KD}(\alpha) = \mathsf{RSSYT}(\alpha)$. Take $T \in \mathsf{RSSYT}(\alpha)$. By our convention in the previous subsection, T is viewed as a diagram pair $(L,\emptyset)$. This result is well-known to experts. For example, it follows from work done in [AS18]. For completeness, we will recover this result in section 5.*

Example 2.8. *Let $\alpha = (0,2,1)$, then*

$$\mathsf{key}(\alpha) = \begin{array}{cc} 3 & 2 \\ 2 & \end{array}, \quad D(\alpha) = \begin{array}{cc} \cdot & \\ \cdot & \cdot \\ & \end{array}, \text{ thus we obtain:}$$

$$\mathsf{KD}(\alpha) = \left\{ \begin{array}{cc} \cdot & \\ \cdot & \cdot \\ & \end{array}, \begin{array}{cc} & \\ \cdot & \cdot \\ \cdot & \end{array}, \begin{array}{cc} \cdot & \\ \cdot & \\ & \cdot \end{array}, \begin{array}{cc} & \\ \cdot & \\ \cdot & \cdot \end{array}, \begin{array}{cc} \cdot & \\ & \\ \cdot & \cdot \end{array} \right\}, \text{ and}$$

$$\mathsf{RSSYT}(\alpha) = \left\{ \begin{array}{cc} 3 & 2 \\ 2 & \end{array}, \begin{array}{cc} 2 & 2 \\ 1 & \end{array}, \begin{array}{cc} 3 & 1 \\ 2 & \end{array}, \begin{array}{cc} 2 & 1 \\ 1 & \end{array}, \begin{array}{cc} 3 & 1 \\ 1 & \end{array} \right\}.$$

If we view each RSSYT as a diagram pair, it is clear that $\mathsf{RSSYT}(\alpha) = \mathsf{KD}(\alpha)$.

Ross and Yong [RY15, Section 1.2] generalized Kohnert moves. We state their construction below.

A *K-Kohnert move* is an operation on a diagram pair. It selects the rightmost cell in a row. The selected cell cannot be a ghost cell. Then move this cell downward to the first position available. It can jump over other Kohnert cells, but cannot jump over any ghost cells. After the move, it may or may not leave a ghost cell at the original position. When a K-Kohnert move leaves a ghost cell, we also refer it as a *ghost move*. Let $\mathsf{KKD}(\alpha)$ be the closure of $\{D_\alpha\}$ under all possible K-Kohnert moves. We make the following observations.

Remark 2.9. *Let α be a weak composition. We have*

- $\mathsf{KD}(\alpha) \subseteq \mathsf{KKD}(\alpha)$.
- *If* $(K, G) \in \mathsf{KKD}(\alpha)$, *then* $(K, \emptyset) \in \mathsf{KD}(\alpha)$.

Remark 2.10. *Usually, an element of $\mathsf{KD}(\alpha)$ is viewed as a diagram. We defined $\mathsf{KD}(\alpha)$ as a set of diagram pairs so we can work with $\mathsf{KD}(\alpha)$ and $\mathsf{KKD}(\alpha)$ using the same technique. In particular, with our convention, $\mathsf{KD}(\alpha)$ is viewed as a subset of $\mathsf{KKD}(\alpha)$.*

Ross and Yong [RY15] conjectured a formula for Lascoux polynomials involving K-Kohnert diagrams.

Conjecture 2.11. [RY15] *The Lascoux polynomials indexed by α, is given by*

$$\mathfrak{L}_\alpha^{(\beta)} = \sum_{D \in \mathsf{KKD}(\alpha)} \beta^{\mathsf{ex}(D)} \boldsymbol{x}^{\mathsf{wt}(D)}.$$

We prove this conjecture by establishing bijections between $\mathsf{KKD}(\alpha)$ and $\mathsf{RSVT}(\alpha)$ that preserve $\mathsf{wt}(\cdot)$ and $\mathsf{ex}(\cdot)$. Moreover, when restricted to $\mathsf{KD}(\alpha) \subseteq \mathsf{KKD}(\alpha)$ and $\mathsf{RSSYT}(\alpha) \subseteq \mathsf{RSVT}(\alpha)$, our bijections restrict to the identity maps. We will describe our bijections in the next two subsections.

Example 2.12. *Continue Example 2.8 for $\alpha = (0, 2, 1)$, we get*

$$\mathsf{KKD}(\alpha) = \mathsf{KD}(\alpha) \bigcup \left\{ \begin{array}{cc} \mathsf{X} & \\ \cdot & \cdot \\ \cdot & \end{array}, \begin{array}{cc} \cdot & \\ \cdot & \mathsf{X} \\ & \cdot \end{array}, \begin{array}{cc} & \\ \cdot & \mathsf{X} \\ \cdot & \cdot \end{array}, \begin{array}{cc} \mathsf{X} & \\ \cdot & \\ \cdot & \cdot \end{array}, \begin{array}{cc} \cdot & \\ \mathsf{X} & \\ \cdot & \cdot \end{array}, \begin{array}{cc} \mathsf{X} & \\ \cdot & \mathsf{X} \\ \cdot & \cdot \end{array} \right\}, \text{ and}$$

$$\mathsf{RSVT}(\alpha) = \mathsf{RSSYT}(\alpha) \bigcup \left\{ \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 21 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 3 & 21 \\ \hline 2 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 2 & 21 \\ \hline 1 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 32 & 1 \\ \hline 1 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 21 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 3 & 21 \\ \hline 21 & \\ \cline{1-1} \end{array} \right\}.$$

Viewing the 6 elements in $\mathsf{RSVT}(\alpha)$ with at least one extra number as diagram pairs, we obtain the following. Note they are different from the elements in $\mathsf{KKD}(\alpha)$ with at least one ghost cell.

$$\left\{ \begin{array}{cc} \cdot & \\ \cdot & \cdot \\ \mathsf{X} & \end{array}, \begin{array}{cc} \cdot & \\ \cdot & \cdot \\ & \mathsf{X} \end{array}, \begin{array}{cc} & \\ \cdot & \cdot \\ \cdot & \mathsf{X} \end{array}, \begin{array}{cc} \cdot & \\ \mathsf{X} & \\ \cdot & \cdot \end{array}, \begin{array}{cc} \cdot & \\ \cdot & \\ \mathsf{X} & \cdot \end{array}, \begin{array}{cc} \cdot & \\ \cdot & \cdot \\ \mathsf{X} & \mathsf{X} \end{array} \right\}$$

Thus, Conjecture 1.1 is correct when $\alpha = (0, 2, 1)$ since

$$\sum_{D \in \mathsf{KKD}(\alpha)} \beta^{\mathsf{ex}(D)} \boldsymbol{x}^{\mathsf{wt}(D)} = \sum_{T \in \mathsf{RSVT}(\alpha)} \beta^{\mathsf{ex}(T)} \boldsymbol{x}^{\mathsf{wt}(T)}.$$

2.4. Kohnert Tableaux.

Assaf and Searles introduced Kohnert tableaux in [AS18]. We will use Kohnert tableaux to prove the correctness of our bijections.

Definition 2.13. [AS18, Definition 2.3] *Let α be a weak composition. A Kohnert tableau* with *content α is a Young diagram filled by numbers such that:*

(1) Column c of the tableau consists of numbers $\{i : \alpha_i \geqslant c\}$, with each number appearing exactly once.

(2) If a number i appears in row r, then $i \geqslant r$.

(3) If a number i appears in (c,r) and $(c+1,r')$, then $r \geqslant r'$.

(4) Let i,j appear in column c with $j > i$ and j is lower than i. Then there is an i in column $c+1$ that is strictly above the j in column c.

Let $\mathsf{KT}(\alpha)$ be the set of all Kohnert tableaux with content α.

Assaf and Searles constructed bijections between $\mathsf{KT}(\alpha)$ and $\mathsf{KD}(\alpha)$ in [AS18]. For each $T \in \mathsf{KT}(\alpha)$, we may ignore its numbers and view each cell as a Kohnert cell. By [AS18, Lemma 2.4], the resulting diagram pair is in $\mathsf{KD}(\alpha)$.

The inverse of the map above is called *Kohnert Labeling with respect to* α, denoted as $\mathsf{Label}_\alpha(\cdot)$. We may describe it as the following algorithm on certain diagram pairs.

Let D be an arbitrary diagram pair such that column c of D has $|\{i : \alpha_i \geqslant c\}|$ Kohnert cells and no ghost cells. Initialize sets $S_1, S_2, \dots$ as $S_c = \{i : \alpha_i \geqslant c\}$. Iterate through boxes of D from right to left, and from bottom to top within each column. For the box (c,r), find the smallest $i \in S_c$ such that i does not appear at $(c+1,r')$ for all $r' > r$. We remove i from S_c and fill i in (c,r) of D. If no such i exists or $i < r$, terminate the algorithm. After all boxes are filled, output the final tableau.

By [AS18, Lemma 2.6, Lemma 2.7, Theorem 2.8], the labeling algorithm on D produces an output if and only if $D \in \mathsf{KD}(\alpha)$. Moreover, if we restrict the algorithm on $\mathsf{KD}(\alpha)$, then this is a bijection from $\mathsf{KD}(\alpha)$ to $\mathsf{KT}(\alpha)$ whose inverse is described above.

Example 2.14. *Let $\alpha = (0,2,1)$, we have*

$$\mathsf{KT}(\alpha) = \left\{ \begin{ytableau} 3 \\ 2 & 2 \\ \none \end{ytableau},\ \begin{ytableau} \none \\ 2 & 2 \\ 3 \end{ytableau},\ \begin{ytableau} 3 \\ 2 \\ \none & 2 \end{ytableau},\ \begin{ytableau} \none \\ 3 \\ 2 & 2 \end{ytableau},\ \begin{ytableau} 3 \\ \none \\ 2 & 2 \end{ytableau} \right\},$$

where the relative order in the sets corresponds to $\mathsf{KD}(\alpha)$ from Example 2.8 under the above labeling algorithm.

3. Describing the maps

For each composition α, we have introduced two sets of diagram pairs: $\mathsf{KKD}(\alpha)$ and $\mathsf{RSVT}(\alpha)$. We will define two maps: Ψ_α on $\mathsf{KKD}(\alpha)$ and Φ_α on $\mathsf{RSVT}(\alpha)$. In section 7, we will show the image of Ψ_α (resp. Φ_α) lies in $\mathsf{RSVT}(\alpha)$ (resp. $\mathsf{KKD}(\alpha)$).

3.1. **An informal description of Ψ_α.** In this section, we describe a map Ψ_α from $\mathsf{KKD}(\alpha)$ to the set of all diagram pairs. First, we describe an operator on $\mathsf{KD}(\alpha)$. Let G be a diagram. Then $\sharp_G(\cdot)$ acts on $\mathsf{KD}(\alpha)$ in the following way: Take $D \in \mathsf{KD}(\alpha)$. Iterate through cells of G from right to left. Within each column, go from bottom to top. For $(c,r) \in G$, search for the largest $r' \leqslant r$ such that (c,r') is a Kohnert cell in D. Moreover, if we raise the cell (c,r') to (c,r), the resulting diagram is still in $\mathsf{KD}(\alpha)$. After finding such r', we move cell (c,r') to (c,r). After iterating over all cells in G, we denote the final Kohnert diagram by $\sharp_G(D)$. If we cannot find such an r' during an iteration, then $\sharp_G(D)$ is undefined.

Example 3.1. *Let D be the fourth Kohnert diagram in Example 2.8. Let G be the diagram $\{(1,3),(2,2)\}$. We may compute $\sharp_G(D)$ as follows. We label (c,r) and (c,r') involved in each step above and below the arrows.*

$$\begin{ytableau} \none \\ \cdot \\ \cdot & \cdot \end{ytableau} \xrightarrow[(2,1)]{(2,2)} \begin{ytableau} \none \\ \cdot & \cdot \\ \cdot \end{ytableau} \xrightarrow[(1,1)]{(1,3)} \begin{ytableau} \cdot \\ \cdot & \cdot \\ \none \end{ytableau}$$

Now, we may describe the map Ψ_α. Take $D = (K, G) \in \mathsf{KKD}(\alpha)$. Compute $(L, \emptyset) = \sharp_G((K, \emptyset)) \in \mathsf{KD}(\alpha)$. Then $\Psi_\alpha(D) := (L, (K \sqcup G) - L)$.

Example 3.2. *Let $D = (K, G)$ be the last diagram pair in Example 2.12. The previous example shows $\sharp_G(K) = (\{(1,2),(1,3),(2,2)\}, \emptyset)$. Thus, D is sent to the diagram pair $(\{(1,2),(1,3),(2,2)\}, \{(1,1),(2,1)\})$. Notice that this diagram pair corresponds to the following* RSVT*:*

$$\begin{array}{|c|c}\hline \cdot & \multicolumn{1}{c}{} \\ \hline \cdot & \multicolumn{1}{c|}{\cdot} \\ \hline \mathsf{X} & \multicolumn{1}{c|}{\mathsf{X}} \\ \hline \end{array} = \begin{array}{|c|c|}\hline 3 & 21 \\ \hline 21 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

We say this is an informal description of Ψ_α since the map is not obviously well-defined. It seems possible that $\sharp_G((K, \emptyset))$ is undefined. In section 7, we will provide an alternative description of the map Ψ_α and check the following.

Lemma 3.3. *The map Ψ_α is a well-defined map from* $\mathsf{KKD}(\alpha)$ *to* $\mathsf{RSVT}(\alpha)$*.*

3.2. **An informal description of Φ_α.** The map Φ_α can be described similarly on $\mathsf{RSVT}(\alpha)$. First, we need an analogue of the $\sharp_G(\cdot)$ operator. Let E be a diagram. Then $\flat_E(\cdot)$ acts on $\mathsf{KD}(\alpha)$ in the following way: Take $D \in \mathsf{KD}(\alpha)$. Iterate through cells of E from left to right. Within each column, go from top to bottom. For $(c, r) \in G$, search for the smallest $r' \geqslant r$ such that (c, r') is a Kohnert cell in D. Moreover, if we drop the cell (c, r') to (c, r), the resulting diagram is still in $\mathsf{KD}(\alpha)$. After finding such r', we move cell (c, r') to (c, r). After iterating over all cells in G, we denote the final Kohnert diagram by $\flat_G(D)$. If we cannot find such an r' during an iteration, then $\flat_G(D)$ is undefined.

Example 3.4. *Let D be the first Kohnert diagram in Example 2.8. Let E be the diagram $\{(1,1),(2,1)\}$. We may compute $\flat_E(D)$ as follows. We label (c, r) and (c, r') involved in each step above and below the arrows.*

$$\begin{array}{|c|c}\hline \cdot & \multicolumn{1}{c}{} \\ \hline \cdot & \multicolumn{1}{c|}{\cdot} \\ \hline \end{array} \xrightarrow[(1,3)]{(1,1)} \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \xrightarrow[(2,2)]{(2,1)} \begin{array}{|c|c}\hline \cdot & \multicolumn{1}{c}{} \\ \hline \cdot & \multicolumn{1}{c|}{\cdot} \\ \hline \end{array}$$

Now, we may describe the map Φ_α. Take $T \in \mathsf{RSVT}(\alpha)$ and we may write T as a diagram pair (L, E). Compute $(K, \emptyset) = \flat_E((L, \emptyset))$. Then $\Phi_\alpha(T) := (K, (L \sqcup E) - K)$.

Example 3.5. *Let T be the last* RSVT *in Example 2.12. We may write $T = (L, E)$ as $(\{(1,2),(1,3),(2,2)\}, \{(1,1),(2,1)\})$. The previous example computes $(K, \emptyset) = \flat_E((L, \emptyset))$. Thus, $(L \sqcup E) - K = \{(1,3),(2,2)\}$ and T is sent to*

$$\begin{array}{|c|c}\hline \mathsf{X} & \multicolumn{1}{c}{} \\ \hline \cdot & \multicolumn{1}{c|}{\mathsf{X}} \\ \hline \cdot & \multicolumn{1}{c|}{\cdot} \\ \hline \end{array}$$

Notice that this is an element of $\mathsf{KKD}(\alpha)$*.*

Again, Φ_α is not obviously well-defined. In section 7, we will provide an alternative description of the map Φ_α and check the following.

Lemma 3.6. *The map Φ_α is a well-defined map from* $\mathsf{RSVT}(\alpha)$ *to* $\mathsf{KKD}(\alpha)$*.*

Now we restate our main result. The proof is also in section 7.

Theorem 3.7. *The maps $\Psi_\alpha : \mathsf{KKD}(\alpha) \to \mathsf{RSVT}(\alpha)$ and $\Phi_\alpha : \mathsf{RSVT}(\alpha) \to \mathsf{KKD}(\alpha)$ are mutually inverses of each other. Moreover, they preserve* $\mathsf{wt}(\cdot)$ *and* $\mathsf{ex}(\cdot)$*.*

4. Bruhat Order on Weak Compositions

Partial order on weak compositions has been studied in [AS18, AQ19, FG21, FGPS20]. In this section, we give a definition via the key tableau associated to the weak composition. In subsection 4.1, we also study $m(\alpha, S)$ (resp. $M(\alpha, S)$) which is the unique minimum (resp. maximum) weak composition in a certain set of weak compositions. In subsection 4.2, we used properties of $M(\alpha, S)$ to show that our Bruhat order is equivalent to the left swap order defined in [AS18, AQ19], which implies that the Bruhat order is equivalent to the inclusion order on $\mathsf{KD}(\alpha), \mathsf{KKD}(\alpha), \mathsf{RSSYT}(\alpha)$ and $\mathsf{RSVT}(\alpha)$.

4.1. **Bruhat order.**

We may define a partial order on all weak compositions.

Definition 4.1. *Let α, γ be two weak compositions. We define $\alpha \leqslant \gamma$ if $\mathsf{key}(\alpha)$ and $\mathsf{key}(\gamma)$ have the same shape and $\mathsf{key}(\alpha) \leqslant \mathsf{key}(\gamma)$ entry-wise. This order is called the Bruhat order.*

Definition 4.2. *For $S \subseteq [n]$, let $\mathbb{1}_S$ be the weak composition whose i^{th} entry is 1 if $i \in S$ and 0 otherwise. For a weak composition α, the support of α is the set $\{i | \alpha_i > 0\}$, denoted as $\mathsf{supp}(\alpha)$.*

Weak compositions with only 0s and 1s are in natural bijection with subsets of $[n]$. The bijections are $S \mapsto \mathbb{1}_S$ and $\alpha \mapsto \mathsf{supp}(\alpha)$. With the Bruhat order above, we may define a partial order on subsets of $[n]$.

Definition 4.3. *For $S, S' \subseteq [n]$, we say $S \leqslant S'$ if $\mathbb{1}_S \leqslant \mathbb{1}_{S'}$.*

We have other alternative descriptions of this order.

Lemma 4.4. *Take $S, S' \subseteq [n]$. The following are equivalent:*

1. *$S \leqslant S'$*
2. *$|S| = |S'|$ and for each $j \in [|S|]$, the j^{th} largest number in S is at most the j^{th} largest number in S'.*
3. *$|S| = |S'|$ and for each $s \in S$, $|[s, n] \cap S| \leqslant |[s, n] \cap S'|$.*

Proof. We first prove statement 1 and 2 are equivalent. By definition $S \leqslant S'$ if and only if $\mathsf{key}(\mathbb{1}_S) \leqslant \mathsf{key}(\mathbb{1}_{S'})$. The number on row j column 1 of $\mathsf{key}(\mathbb{1}_S)$ (resp. $\mathsf{key}(\mathbb{1}_{S'})$) is the j^{th} largest number in S (resp. S'). Thus, $\mathsf{key}(\mathbb{1}_S) \leqslant \mathsf{key}(\mathbb{1}_{S'})$ is equivalent to statement 2.

Next, we show the statement 2 and 3 are equivalent. Assume the statement 2 is true. Let $s \in S$ be the j^{th} largest number in S. Then $|[s, n] \cap S| = j$. Since the j^{th} largest number in S' is at least s, $|[s, n] \cap S'| \geqslant j$. Now assume statement 3 is true. Let s be the j^{th} largest number in S. We know there are at least j numbers in $[s, n] \cap S'$, so the j^{th} largest number in S' is at least s. □

Take $S \subseteq [n]$ and a weak composition α. Consider the set $\{\gamma : \gamma \geqslant \alpha, \mathsf{supp}(\gamma) \subseteq S\}$. Let $m(\alpha, S)$ be the unique minimum element in the set, if it exists. Later, we will show $m(\alpha, S)$ exists as long as the set is non-empty. First, we introduce an algorithm to compute $m(\alpha, S)$ or assert it does not exist. Initialize *list* to be an empty list and initialize σ to be the weak composition with all 0s. Iterate over $i = 1, \ldots, n$. Perform the following two processes in each iteration:

- (Adding process): If $\alpha_i > 0$, then add α_i to *list*.
- (Removing process): If $i \in S$ and *list* is non-empty, then remove $\max(\mathit{list})$ from *list* and assign it to σ_i.

After all iterations, if *list* is empty, then $m(\alpha, S)$ is σ. Otherwise, such $m(\alpha, S)$ does not exist.

Example 4.5. *Let $n = 7$, $\alpha = (1, 3, 0, 2, 0, 0, 2)$ and $S = \{3, 4, 5, 6, 7\}$. Then we trace σ and list during the algorithm:*

- *Before iteration 1: $\sigma = (0, 0, 0, 0, 0, 0, 0)$; list is empty.*
- *After iteration 1: $\sigma = (0, 0, 0, 0, 0, 0, 0)$; list contains 1.*
- *After iteration 2: $\sigma = (0, 0, 0, 0, 0, 0, 0)$; list contains 1, 3.*

- *After iteration 3:* $\sigma = (0,0,3,0,0,0,0)$*; list contains* 1.
- *After iteration 4:* $\sigma = (0,0,3,2,0,0,0)$*; list contains* 1.
- *After iteration 5:* $\sigma = (0,0,3,2,1,0,0)$*; list is empty.*
- *After iteration 6:* $\sigma = (0,0,3,2,1,0,0)$*; list is empty.*
- *After iteration 7:* $\sigma = (0,0,3,2,1,0,2)$*; list is empty.*

Since the list is empty after the iterations, the algorithm outputs $m(\alpha, S) = (0,0,3,2,1,0,2)$.

We take steps to show this algorithm is correct. We start with the following observation, which connects this algorithm with the $\lhd$ operator in Definition 2.1.

Lemma 4.6. *Assume the algorithm outputs $m(\alpha, S) = \sigma$. Let T_c (resp. A_c) be the set consisting of numbers in column c of $\mathsf{key}(\sigma)$ (resp. $\mathsf{key}(\alpha)$). Then $T_c = S \lhd A_c$.*

Proof. We know $i \in A_c$ if and only if during the adding process of the iteration i, a number at least c is added to *list*. Similarly, $i \in T_c$ if and only if during the removing process of the iteration i, a number at least c is removed from *list*.

Assume $T_c = \{t_1 < t_2 < \cdots < t_s\}$ and $A_c = \{a_1 < a_2 < \cdots < a_s\}$. During the adding process of iteration a_1, the algorithm puts α_{a_1} into *list*. This is the first time that *list* gains a number at least c. Thus, $t_1 \geqslant a_1$. Moreover, assume there exists $t \in S$ such that $a_1 \leqslant t < t_1$. During the removing process of iteration t, *list* has a number at least c. The algorithm will remove a number at least c from *list*, contradicting to $t \notin T_c$. Thus, t_1 is the smallest in S with $t_1 \geqslant a_1$.

Now consider t_j with $j > 1$. During the removing process of t_j, we remove a number at least c for the j^{th} time. Thus, we have added at least j such numbers to *list*, so $a_j \leqslant t_j$. Now assume there is $t < t_j$ such that $t \in S$, $t \geqslant a_j$, and $t > t_{j-1}$. During the removing process of iteration t, there is a number at least c in *list*, so such a number will be removed. We have a contradiction since $t \notin T_c$. Thus, t_j is the smallest in S such that $t_j \geqslant a_j$ and $t_j > t_{j-1}$. $\square$

Next, we investigate the condition for the algorithm to assert $m(\alpha, S)$ does not exist.

Lemma 4.7. *The following are equivalent:*

(1) $|S| \geqslant |\mathsf{supp}(\alpha)|$*. In addition, if we let $S' \subseteq S$ consists of the largest numbers in S with $|S'| = |\mathsf{supp}(\alpha)|$, then $S' \geqslant \mathsf{supp}(\alpha)$.*
(2) The algorithm asserts $m(\alpha, S)$ exists. (i.e. The list is empty when the algorithm ends.)
(3) There is a weak composition γ such that $\mathsf{supp}(\gamma) \subseteq S$ and $\gamma \geqslant \alpha$.

Proof. First, assume $|S| < |\mathsf{supp}(\alpha)|$. We check the last two statements do not hold.

(2) Consider the i^{th} iteration. First, the size of *list* is increased by one if $i \in \mathsf{supp}(\alpha)$. Next, the size of *list* is fixed or decreased by one if $i \in S$. Throughout this algorithm, *list* gains $|\mathsf{supp}(\alpha)|$ numbers and loses at most $|S|$ numbers. It is not empty when the algorithm ends.
(3) Assume such γ exists. We know $|\mathsf{supp}(\gamma)| = |\mathsf{supp}(\alpha)| > |S|$, contradicting to $\mathsf{supp}(\gamma) \subseteq S$.

Now assume $|S| \geqslant |\mathsf{supp}(\alpha)|$ and define S' as above. Suppose $S' \not\geqslant \mathsf{supp}(\alpha)$. We check the last two statements in the lemma do not hold.

(2) By Lemma 4.4, there exists $j \in \mathsf{supp}(\alpha)$ such that
$$|[j,n] \cap \mathsf{supp}(\alpha)| > |[j,n] \cap S'| = |[j,n] \cap S|.$$
Between the j^{th} iteration and the last iteration inclusively, *list* gains $|[j,n] \cap \mathsf{supp}(\alpha)|$ numbers and loses at most $|[j,n] \cap S|$ numbers. It is not empty when the algorithm ends.
(3) Assume such γ exists. By $\mathsf{supp}(\gamma) \subseteq S$, we have $S' \geqslant \mathsf{supp}(\gamma) \geqslant \mathsf{supp}(\alpha)$. Contradiction.

Finally, assume $S' \geqslant \mathsf{supp}(\alpha)$. We check the last two statements in the lemma are true.

(2) Assume when the algorithm ends, *list* is not empty. Find the largest j such that *list* is not empty since the j^{th} iteration. First, we know *list* is empty right before the j^{th} iteration. Second, we know a number is added to *list* during the j^{th} iteration, so $j \in \mathsf{supp}(\alpha)$. By

Lemma 4.4, $|[j,n] \cap \mathsf{supp}(\alpha)| \leqslant |[j,n] \cap S'| = |[j,n] \cap S|$. Between the j^{th} iteration and the last iteration inclusively, *list* gains $|[j,n] \cap \mathsf{supp}(\alpha)|$ numbers. Since *list* is not empty since iteration j, *list* loses $|[j,n] \cap S|$ numbers. Thus, *list* is empty after the last iteration.

(3) From the previous statement, we know the algorithm will produce a weak composition σ. Just need to check $\mathsf{supp}(\sigma) \in S$ and $\sigma \geqslant \alpha$. It is apparent that $\mathsf{supp}(\sigma) \subseteq S$. Since all positive numbers in α are assigned into σ, $\mathsf{key}(\sigma)$ and $\mathsf{key}(\alpha)$ are of the same shape. Next we show that $\sigma \geqslant \alpha$ by comparing $\mathsf{key}(\sigma)$ and $\mathsf{key}(\alpha)$ column-by-column: Let T_c (resp. A_c) be the set consisting of numbers in column c of $\mathsf{key}(\sigma)$ (resp. $\mathsf{key}(\alpha)$). By Lemma 4.6, $T_c = S \lhd A_c$. Thus, $T_c \geqslant A_c$.

□

Now we can prove the correctness of our algorithm.

Lemma 4.8. *The algorithm correctly computes $m(\alpha, S)$.*

In other words, consider the set $\{\gamma : \gamma \geqslant \alpha, \mathsf{supp}(\gamma) \subseteq S\}$.

- *If list is empty after the iterations, then the output σ is the unique minimum in the set.*
- *Otherwise, the set is empty.*

Moreover, the second case happens only when the set is empty.

Proof. If *list* is not empty when the algorithm ends, $\{\gamma : \gamma \geqslant \alpha, \mathsf{supp}(\gamma) \subseteq S\} = \emptyset$ by the previous lemma.

Now assume *list* is empty when the algorithm ends. Then the set is non-empty: By the proof of the previous lemma, the output σ is in this set. We check σ is the least element. Assume there is a γ in the set with $\gamma \not\geqslant \sigma$. Let T'_c consists of numbers in column c of $\mathsf{key}(\gamma)$. Define T_c and A_c similarly for $\mathsf{key}(\sigma)$ and $\mathsf{key}(\alpha)$ respectively. Then we can find c such that $T'_c \not\geqslant T_c$. Let t'_j be the j^{th} smallest number in T'_c. Define t_j and a_j similarly for T_c and A_c. Then we can find smallest j such that $t'_j < t_j$. Notice that $t'_j \geqslant a'_j$ and $t'_j \in S$. Moreover, $t'_j > t'_{j-1} \geqslant t_{j-1}$ if $j > 1$. We have a contradiction to the fact $T_c = S \lhd A_c$ from Lemma 4.6. □

Corollary 4.9. *Let α be a weak composition and $S \subseteq [n]$. $m(\alpha, S)$ exists if and only if*

- $|S| \geqslant |\mathsf{supp}(\alpha)|$, *and*
- *Let $S' \subseteq S$ consists of the largest numbers in S with $|S'| = |\mathsf{supp}(\alpha)|$. Then $S' \geqslant \mathsf{supp}(\alpha)$.*

Proof. Follows from the previous two lemmas. □

Analogously, we may also look at the set $\{\gamma : \gamma \leqslant \alpha, \mathsf{supp}(\gamma) \subseteq S\}$. Similarly, if it is non-empty, it will contain a unique maximum element. Let $M(\alpha, S)$ be this element. To compute it, we only need to slightly change our algorithm above: Let i goes from n to 1, instead of 1 to n. Similar to Corollary 4.9, we have the following for $M(\alpha, S)$.

Corollary 4.10. *Let α be a weak composition and $S \subseteq [n]$. $M(\alpha, S)$ exists if and only if*

- $|S| \geqslant |\mathsf{supp}(\alpha)|$, *and*
- *Let $S' \subseteq S$ consists of the smallest numbers in S with $|S'| = |\mathsf{supp}(\alpha)|$. Then $S' \leqslant \mathsf{supp}(\alpha)$.*

Proof. The proof is similar to the proof of Corollary 4.9. □

We end this subsection by a property that connects $m(\gamma, S)$ and $M(\alpha, S)$. Let γ be a weak composition. We use $\overline{\gamma}$ to denote the weak composition obtained by decreasing each positive entry of γ by 1.

Lemma 4.11. *Let α, γ be two weak compositions. Take $S \subseteq [n]$ with $|S| = |\mathsf{supp}(\alpha)|$. Then the following are equivalent:*

- *$m(\gamma, S)$ exists and $\alpha \geqslant \mathbb{1}_S + m(\gamma, S)$.*
- *$M(\alpha, S)$ exists and $\overline{M(\alpha, S)} \geqslant \gamma$.*

Proof. Assume the first statement is true. Notice $\mathsf{supp}(\mathbb{1}_S + m(\gamma, S)) = S$, so $M(\alpha, S)$ exists and

$$M(\alpha, S) \geqslant \mathbb{1}_S + m(\gamma, S).$$

Decrease each positive entry by 1 on both sides and get

$$\overline{M(\alpha, S)} \geqslant m(\gamma, S).$$

Then we get the second statement since $m(\gamma, S) \geqslant \gamma$.

Now assume the second statement is true. Notice $\mathsf{supp}(\overline{M(\alpha, S)}) \subseteq S$, so $m(\gamma, S)$ exists and

$$\overline{M(\alpha, S)} \geqslant m(\gamma, S).$$

By $|S| = |\mathsf{supp}(\alpha)|$, $\mathsf{supp}(M(\alpha, S)) = S$, so

$$M(\alpha, S) \geqslant \mathbb{1}_S + m(\gamma, S).$$

Then we get the first statement since $\alpha \geqslant M(\alpha, S)$. ☐

4.2. **Left swap order.** Assaf and Searles [AS18] also defined a partial order on weak compositions called the *left swap order*. In this subsection, we introduce this order and show that it is equivalent to the Bruhat order.

Definition 4.12. [AS18, AQ19, Definition 2.3.4] *A* left swap *on a weak composition α exchanges two parts $\alpha_i < \alpha_j$ with $i < j$. The left swap order on weak compositions is the transitive closure of the relation $\gamma \preceq \alpha$ whenever γ is a left swap of α.*

When γ is obtain from α by exchanging the i^{th} and j^{th} parts of α, we write $\gamma = (i\ j)\alpha$.

Proposition 4.13. [AQ19, Prop. 2.3.9] *Given weak compositions α, γ, we have $\gamma \preceq \alpha$ if and only if the key diagram pair of γ is in $\mathsf{KD}(\alpha)$.*

To show the equivalence between the left swap order and the Bruhat order, we need a few lemmas. We start with the following, which summarizes how $M(\alpha, S)$ is changed when we changes S in a nice way.

Lemma 4.14. *Let α be a weak composition and $S \subseteq [n]$ with $S \leqslant \mathsf{supp}(\alpha)$. Take $g \notin S$ such that $|[g, n] \cap \mathsf{supp}(\alpha)| > |[g, n] \cap S|$. Then there exists $s \in S$ such that $s < g$ and*

$$M(\alpha, S') = (s\ \ g)M(\alpha, S),$$

where $S' = (S \sqcup \{g\}) - \{s\}$. In particular, $M(\alpha, S)$is a left swap of $M(\alpha, S')$.

Proof. Run the algorithm that computes $M(\alpha, S)$. After initialization, the algorithm iterates from $i = n$ to $i = 1$. Right after the iteration with $i = g$, the algorithm has put $|[g, n] \cap \mathsf{supp}(\alpha)|$ numbers to *list* and has removed at most $|[g, n] \cap S|$ numbers from *list*. Since $|[g, n] \cap \mathsf{supp}(\alpha)| > |[g, n] \cap S|$, the list is non-empty. Let x be the largest number in the current list. Then this x will be picked sometime in the future. Let $s \in S \cap [1, g)$ be the largest such that in the iteration $i = s$, the algorithm assigns x to σ_i. We may run the algorithm to compute $M(\alpha, (S \sqcup \{g\}) - \{s\})$. It behaves the same as on $M(\alpha, S)$, except it assigns x to σ_g and keeps $\sigma_s = 0$. Thus, $M(\alpha, (S \sqcup \{g\}) - \{s\}) = (g\ s)M(\alpha, S)$. ☐

We know from definition that $M(\alpha, S) \leqslant \alpha$. The next lemma will describe their relationship in the left swap order.

Lemma 4.15. *Let α be a weak composition. Let S be a set such that $S \leqslant \mathsf{supp}(\alpha)$. Then $M(\alpha, S) \preceq \alpha$.*

Proof. Find the smallest $g \geqslant 1$ such that $[g,n] \cap S = [g,n] \cap \mathsf{supp}(\alpha)$. Prove this lemma by induction on g. For the base case, we assume $g = 1$. Then $\mathsf{supp}(\alpha) = S$ which implies $M(\alpha, S) = \alpha$.

Next, assume $g > 1$. Since $[g-1,n] \cap S \neq [g-1,n] \cap \mathsf{supp}(\alpha)$ and $S \leqslant \mathsf{supp}(\alpha)$, we know $g - 1 \in \mathsf{supp}(\alpha) - S$. Thus, $|[g-1,n] \cap S| < |[g-1,n] \cap \mathsf{supp}(\alpha)|$. By Lemma 4.14, there exists $s \in S$ such that if we let $S' = (S - \{s\}) \sqcup \{g-1\}$, we have $M(\alpha, S) \preceq M(\alpha, S')$. Notice that $[g-1,n] \cap S' = [g-1,n] \cap \mathsf{supp}(\alpha)$. We may apply our inductive hypothesis and get $M(\alpha, S) \preceq M(\alpha, S') \preceq \alpha$. ◻

Finally, we need the following intuitive lemma, which says both partial orders are preserved by the operator $\alpha \mapsto \overline{\alpha}$.

Lemma 4.16. *Given weak compositions α and γ with $\mathsf{supp}(\alpha) = \mathsf{supp}(\gamma)$. Then we have*

- *$\gamma \leqslant \alpha$ if and only if $\overline{\gamma} \leqslant \overline{\alpha}$.*
- *$\gamma \preceq \alpha$ if and only if $\overline{\gamma} \preceq \overline{\alpha}$.*

Proof. Immediate from definitions. ◻

Now we are ready to prove the equivalence of these two partial orders.

Proposition 4.17. *Given weak compositions α and γ, we have $\gamma \preceq \alpha$ if and only if $\gamma \leqslant \alpha$.*

Proof. First we show if $\gamma \preceq \alpha$, then $\gamma \leqslant \alpha$. It suffices to show when γ is a left swap of α. Say $\gamma = (i\, j)\alpha$ where $i < j$ and $\alpha_i < \alpha_j$. Let T_c (resp. T'_c) consists of numbers in column c of $\mathsf{key}(\alpha)$ (resp. $\mathsf{key}(\gamma)$). When $c \leqslant \alpha_i$ or $c > \alpha_j$, we have $T_c = T'_c$. When $\alpha_i < c \leqslant \alpha_j$, T'_c is obtained from T_c by replacing j with i. Therefore $\mathsf{key}(\gamma) \leqslant \mathsf{key}(\alpha)$ and $\gamma \leqslant \alpha$.

Next we assume $\gamma \leqslant \alpha$ and show $\gamma \preceq \alpha$. We prove by induction on $\max(\alpha)$. If $\max(\alpha) = 0$, then α, γ only contain 0s. Our claim is immediate. Now assume $\max(\alpha) > 0$. We consider two cases.

- If $\mathsf{supp}(\gamma) = \mathsf{supp}(\alpha)$, then $\overline{\gamma} \leqslant \overline{\alpha}$ by Lemma 4.16. By our inductive hypothesis, $\overline{\gamma} \preceq \overline{\alpha}$. By Lemma 4.16 again, $\gamma \preceq \alpha$.
- Assume $\mathsf{supp}(\gamma) \neq \mathsf{supp}(\alpha)$. Let $S = \mathsf{supp}(\gamma)$. First, notice that $\gamma \leqslant M(\alpha, S)$ and these two weak compositions have the same support. By the previous case, $\gamma \preceq M(\alpha, S)$. It remains to check $M(\alpha, S) \preceq \alpha$, which follows from $S \leqslant \mathsf{supp}(\alpha)$ and Lemma 4.15.

◻

Consequently, we know several statements are equivalent to $\gamma \leqslant \alpha$.

Corollary 4.18. *Given weak compositions α, γ, the following are equivalent:*

(1) $\gamma \leqslant \alpha$;
(2) $\mathsf{KD}(\gamma) \subseteq \mathsf{KD}(\alpha)$;
(3) $\mathsf{KKD}(\gamma) \subseteq \mathsf{KKD}(\alpha)$;
(4) $\mathsf{RSSYT}(\gamma) \subseteq \mathsf{RSSYT}(\alpha)$;
(5) $\mathsf{RSVT}(\gamma) \subseteq \mathsf{RSVT}(\alpha)$.

Proof. We show the following directions.

- (1) $\iff$ (2) This follows from Propositions 4.13 and 4.17.
- (1) $\iff$ (4) This is true by definition.
- (1) $\iff$ (5) This is true by definition.
- (2) $\implies$ (3) This is true by definition.
- (3) $\implies$ (2) Since $\mathsf{KKD}(\gamma) \subseteq \mathsf{KKD}(\alpha)$ and $\mathsf{KD}(\gamma) \subseteq \mathsf{KKD}(\gamma)$, we have $\mathsf{KD}(\gamma) \subseteq \mathsf{KKD}(\alpha)$. Since there is no ghost cells in elements of $\mathsf{KD}(\gamma)$, we have $\mathsf{KD}(\gamma) \subseteq \mathsf{KD}(\alpha)$.

◻

5. Recursive descriptions of $\mathsf{KKD}(\alpha)$ and $\mathsf{RSVT}(\alpha)$

To check our maps Φ_α and Ψ_α are well-defined, we need to study their domains. In this section, we give necessary and sufficient criteria on when a diagram pair is in $\mathsf{KKD}(\alpha)$ (resp. $\mathsf{RSVT}(\alpha)$). Our criteria will be recursive.

For a diagram pair $D = (K, G)$, we can send it to a triple (K_1, G_1, d) where K_1, G_1 are disjoint subsets of $[n]$ and d is a diagram pair. They are defined as follows:

- $K_1 := \{r : (1, r) \in K\}$
- $G_1 := \{r : (1, r) \in G\}$
- d is the diagram pair with a Kohnert cell (resp. ghost cell) at (c, r) if D has a Kohnert cell (resp. ghost cell) at $(c + 1, r)$ with $c \geqslant 1$.

The map $D \to (K_1, G_1, d)$ is invertible: given disjoint $K_1, G_1 \subseteq [n]$ and a diagram pair d, we can uniquely recover D. Thus, we may identify a diagram pair with its image and write $D = (K_1, G_1, d)$.

Fix a weak composition α in this section. An element of $\mathsf{KKD}(\alpha)$ or $\mathsf{RSVT}(\alpha)$ can be written as (K_1, G_1, d). We will find conditions on this triple to determine when $(K_1, G_1, d) \in \mathsf{KKD}(\alpha)$ and $\mathsf{RSVT}(\alpha)$.

5.1. Describing $\mathsf{KKD}(\alpha)$.

First, we describe the condition for a diagram pair (K_1, G_1, d) to live in $\mathsf{KKD}(\alpha)$.

Theorem 5.1. *The diagram pair (K_1, G_1, d) is in $\mathsf{KKD}(\alpha)$ if and only if it satisfies:*

(1) K_1 and G_1 are disjoint subsets of $[n]$.
(2) $K_1 \leqslant \mathsf{supp}(\alpha)$.
(3) For each $g \in G_1$, $|[g, n] \cap \mathsf{supp}(\alpha)| > |[g, n] \cap K_1|$.
(4) $d \in \mathsf{KKD}(\overline{M(\alpha, K_1)})$

The rest of this subsection proves it. First, we want to show if a diagram pair D satisfies these conditions, then $D \in \mathsf{KKD}(\alpha)$. This can be implied from the following lemma:

Lemma 5.2. *If $D = (K_1, G_1, d)$ satisfies conditions (1)-(4) in Theorem 5.1, then we can find $\gamma \leqslant \alpha$ such that $D \in \mathsf{KKD}(\gamma)$. Moreover, $\gamma_i = 0$ if $i \notin K_1 \sqcup G_1$.*

Notice $D \in \mathsf{KKD}(\gamma) \subseteq \mathsf{KKD}(\alpha)$ by Corollary 4.18. Thus, this lemma implies the reverse direction of Theorem 5.1.

Proof. We describe an algorithm that turns D into a key diagram pair of some weak composition γ via reverse K-Kohnert moves. First, we consider d. Since $d \in \mathsf{KKD}(\overline{M(\alpha, K_1)})$, we may do reverse K-Kohnert moves on d to obtain the key diagram pair of $\overline{M(\alpha, K_1)}$.

Now, Kohnert cells in D form the key diagram pair of $M(\alpha, K_1)$. If G_1 is empty, we are done. Otherwise, let $g = \min(G_1)$. By Lemma 4.14, we can find $k \in K_1 \cap [1, g)$ such that $M(\alpha, (K_1 \sqcup \{g\}) - \{k\}) = (g\ k)M(\alpha, K_1)$. We perform reverse K-Kohnert moves to lift the entire row k of D to row g, and remove the ghost at $(1, g)$. Next we assign $(K_1 \sqcup \{g\}) - \{k\}$ to K_1 and assign $G_1 - \{g\}$ to G_1. Now, Kohnert cells in D still form the key diagram pair of $M(\alpha, K_1)$. We may repeat the steps above until G_1 is empty. The resulting diagram pair is a key diagram pair for some weak composition γ. Clearly, $\gamma_i = 0$ if $(1, i)$ was not a cell in D. ☐

Proof of Theorem 5.1. The reverse direction is already shown. For each $D = (K_1, G_1, d) \in KKD(\alpha)$, we need to show the four conditions are satisfied. We prove by induction on $\max(\alpha)$. When $\max(\alpha) = 0$, the claim is immediate.

Assume the statement is true for any weak composition whose maximum entry is less than $\max(\alpha)$. For any $D \in \mathsf{KKD}(\alpha)$, we want to check it satisfies the four conditions. Condition (1) is immediate. We prove the other conditions by an induction on K-Kohnert moves. If D is the key diagram pair of α, then the last three conditions are immediate. Now assume $(K_1, G_1, d) \in \mathsf{KKD}(\alpha)$

satisfies the last three conditions. Perform one K-Kohnert move and obtain (K_1', G_1', d'). We want to show (K_1', G_1', d') also satisfies the last three conditions. If the K-Kohnert move is not on column 1, then $K_1 = K_1'$ and $G_1 = G_1'$, which gives us the second and the third condition. Notice that d' is obtained from d by one K-Kohnert move, so the last condition is also satisfied.

Now suppose the K-Kohnert move is on column 1. $K_1' = (K_1 - \{i\}) \sqcup \{j\}$ with $i > j$, and G_1' is either G_1 or $G_1 \sqcup \{i\}$. We check the last three conditions below.

(2) $\mathsf{supp}(\alpha) \geqslant K_1 \geqslant K_1'$.

(3) We check $|[g, n] \cap \mathsf{supp}(\alpha)| > |[g, n] \cap K_1'|$, for each $g \in G_1 \sqcup \{i\}$.
If $g \in G_1$, we have
$$|[g, n] \cap \mathsf{supp}(\alpha)| > |[g, n] \cap K_1| \geqslant |[g, n] \cap K_1'|.$$
For $g = i$, we have
$$|[i, n] \cap \mathsf{supp}(\alpha)| \geqslant |[i, n] \cap K_1| > |[i, n] \cap K_1'|.$$

(4) We have $d' = d \in \mathsf{KKD}(\overline{M(\alpha, K_1)})$. By our inductive hypothesis, d satisfies all four conditions of $\mathsf{KKD}(\overline{M(\alpha, K_1)})$. Using Lemma 5.2, we may perform reverse K-Kohnert moves on d and get the key diagram pair of $\gamma \leqslant \overline{M(\alpha, K_1)}$. Since we can make a K-Kohnert move on $(1, i)$, d has no cells on row i. Thus, we know $\gamma_i = 0$. To show $d \in \mathsf{KKD}(\overline{M(\alpha, K_1')})$, we show $\gamma \leqslant \overline{M(\alpha, K_1')}$.

Let $T = \mathsf{key}(M(\alpha, K_1))$. For each column of T that contains i, we replace i by the largest i' such that $i' < i$ and i' is not in this column. Then we sort the column into strictly decreasing order. Let T' be the resulting tableau. It is clear that T' is a key. Let $\sigma = \mathsf{wt}(T')$. We make two observations about T' and σ:

- Column 1 of T' consists of K_1', so $\mathsf{supp}(\sigma) = K_1'$.
- The tableau T' is entry-wise less than T. Thus, $\sigma \leqslant M(\alpha, K_1) \leqslant \alpha$.

By definition, these two observations yield $\sigma \leqslant M(\alpha, K_1')$.

It remains to check $\gamma \leqslant \overline{\sigma}$. Let $g_1 > \cdots > g_s$ be the numbers in column c of $\mathsf{key}(\gamma)$. By $\gamma_i = 0$, none of these numbers is i. Let $t_1 > \cdots > t_s$ (resp. $t_1' > \cdots > t_s'$) be numbers in column $c + 1$ of T (resp. T'). By $\gamma \leqslant \overline{M(\alpha, K_1)}$, we know $g_k \leqslant t_k$ for $1 \leqslant k \leqslant s$. If i is not in column $c + 1$ of T, then we are done. Otherwise, assume $t_a = i$. We know $t_1', \ldots, t_s'$ are obtained from $t_1, \ldots, t_s$ by changing $t_a, \cdots, t_b$ into $t_a - 1, \ldots, t_b - 1$. Since $g_a \leqslant i$ and $g_a \neq i$, we have $g_a \leqslant i - 1 = t_a'$. Then $g_{a+1} \leqslant i - 2 = t_{a+1}'$. Following this argument, we have $g_k \leqslant t_k'$ for $1 \leqslant k \leqslant s$. Thus, $\gamma \leqslant \overline{\sigma} \leqslant M(\alpha, K_1')$.

□

5.2. **Describing $\mathsf{RSVT}(\alpha)$.** Take $T \in \mathsf{RSVT}(\alpha)$ and write T as (L_1, E_1, t). First, we notice the following relation between $K_-(T)$, $K_-(t)$ and L_1:

Lemma 5.3. *Assume $T = (L_1, E_1, t) \in \mathsf{RSVT}(\alpha)$. Assume $K_-(t) = \mathsf{key}(\gamma)$. Then $\mathsf{wt}(K_-(T)) = \mathbb{1}_{L_1} + m(\gamma, L_1)$.*

Proof. Let $S = \mathsf{key}(\mathbb{1}_{L_1} + m(\gamma, L_1))$. View S and T as tableaux. Let T_c be the set consisting of leading numbers in column c of T, so $T_1 = L_1$. It suffices to show that $K_-(T) = S$. We compare these two tableaux column by column. Apparently, column 1 of $K_-(T)$ and column 1 of S both consist of L_1.

Consider the column c of $K_-(T)$ and S for $c > 1$. Column c of S agrees with column $c - 1$ of $\mathsf{key}(m(\gamma, L_1))$. Let $\mathsf{key}(\gamma)_{c-1}$ consists of numbers in column $c-1$ of $\mathsf{key}(\gamma)$. By Lemma 4.6, column $c - 1$ of $\mathsf{key}(m(\gamma, L_1))$ consists of $L_1 \lhd \mathsf{key}(\gamma)_{c-1}$. Since $\mathsf{key}(\gamma) = K_-(t)$, we have $\mathsf{key}(\gamma)_{c-1} = T_2 \lhd \cdots \lhd T_c$. Thus, column c of S consists of $L_1 \lhd T_2 \lhd \cdots \lhd T_c = T_1 \lhd T_2 \lhd \cdots \lhd T_c$, which agrees with column c of $K_-(T)$. □

Theorem 5.4. *The diagram pair (L_1, E_1, t) is in $\mathsf{RSVT}(\alpha)$ if and only if it satisfies:*

(1) L_1 *and* E_1 *are disjoint subsets of* $[n]$*.*
(2) $L_1 \leqslant \mathsf{supp}(\alpha)$*.*
(3) Let L_1' *be the set of row indices of Kohnert cells in column 1 of* t*. For each* $e \in E_1$*,* $|(e,n] \cap L_1| > |(e,n] \cap L_1'|$*.*
(4) $t \in \mathsf{RSVT}(\overline{M(\alpha, L_1)})$

Proof. First, we show that if $T = (L_1, E_1, t) \in \mathsf{RSVT}(\alpha)$, then the four conditions are satisfied.

(1) L_1 and E_1 are clearly disjoint.
(2) Notice that column 1 of $K_-(T)$ consists of numbers in L_1, while column 1 of $\mathsf{key}(\alpha)$ consists of numbers in $\mathsf{supp}(\alpha)$. By $K_-(T) \leqslant \mathsf{key}(\alpha)$, $L_1 \leqslant \mathsf{supp}(\alpha)$.
(3) View T as a tableau. For $e \in E_1$, assume it is on row j column 1 of T. Then $|L_1 \cap (e,n]| = j$. The set L_1' consists of leading numbers in column 2 of T. The j^{th} largest number in L_1', if exists, is the leading number at row j column 2 of T. Thus, it is weakly less than e, so $|(e,n] \cap L_1'| < j$.
(4) By $T \in \mathsf{RSVT}(\alpha)$, $K_-(T) \leqslant \mathsf{key}(\alpha)$. Let $\gamma = \mathsf{wt}(K_-(t))$. By Lemma 5.3, $\mathbb{1}_{L_1} + m(\gamma, L_1) \leqslant \alpha$. By Lemma 4.11,
$$\mathsf{wt}(K_-(t)) = \gamma \leqslant \overline{M(\alpha, L_1)}.$$
Thus $t \in \mathsf{RSVT}(\overline{M(\alpha, L_1)})$.

Now, we check if we have $T = (L_1, E_1, t)$ satisfying these four conditions, then $T \in \mathsf{RSVT}(\alpha)$. We may construct T as a tableau. First, we build column 1 of T. We arrange numbers in L_1 into a strictly decreasing column. For each $e \in E_1$, by condition 2, $|(e,n] \cap L_1| > 0$. Then we put e on row $|(e,n] \cap L_1|$. Clearly, this column is an RSVT with leading numbers from L_1 and extra numbers from E_1.

Next, we may build the tableau corresponding to t recursively. Let L_1' be the set of leading numbers in column 1 of t. Put the column we just constructed on the left of t. Only need to check all numbers in row j of our new column are weakly larger than the leading number in row j column 1 of t, which is the j^{th} largest number in L_1'.

- Let e be an extra number on row j of our new column, By condition 3, $|[e,n] \cap L_1'| < j$. Thus, the j^{th} largest number in L_1' is at most e.
- By condition 4, if we let $\gamma = \mathsf{wt}(K_-(t))$, then $\gamma \leqslant \overline{M(\alpha, L_1)}$. By Lemma 4.11, $m(\gamma, L_1)$ exists. Then by Corollary 4.9, the j^{th} largest number in $\mathsf{supp}(\gamma) = L_1'$ is weakly less than the j^{th} largest number in L_1, which is the leading number in row j of our new column.

Now we have constructed a tableau T which if viewed as a diagram pair, corresponds to (L_1, E_1, t). It remains to check $K_-(T) \leqslant key(\alpha)$. Notice that $K_-(T) = \mathsf{key}(\mathbb{1}_{L_1} + m(\gamma, L_1))$. By Lemma 4.11 and $\gamma \leqslant \overline{M(\alpha, L_1)}$, we have $K_-(T) \leqslant key(\alpha)$. Thus, $(L_1, E_1, t) \in \mathsf{RSVT}(\alpha)$. □

The recursive descriptions of $\mathsf{KKD}(\alpha)$ and $\mathsf{RSVT}(\alpha)$ share many similarities. They only differ at the third condition, which is the condition on the positions of ghost cells. From this observation, we get the following result which was discussed in Remark 2.7.

Corollary 5.5. *We have* $\mathsf{RSSYT}(\alpha) = \mathsf{KD}(\alpha)$*.*

Proof. We know $\mathsf{KD}(\alpha)$ is the subset of $\mathsf{KKD}(\alpha)$ containing all diagram pairs with no ghost cells. By Theorem 5.1, $\mathsf{KD}(\alpha)$ consists of $(K_1, \emptyset, d)$ such that

- $K_1 \leqslant \mathsf{supp}(\alpha)$.
- $d \in \mathsf{KD}(\overline{M(\alpha, K_1)})$.

On the other hand, $\mathsf{RSSYT}(\alpha)$ is the subset of $\mathsf{RSVT}(\alpha)$ containing all diagram pairs with no ghost cells. By Theorem 5.4, $\mathsf{RSSYT}(\alpha)$ consists of $(L_1, \emptyset, t)$ such that

- $L_1 \leqslant \mathsf{supp}(\alpha)$.
- $t \in \mathsf{RSSYT}(\overline{M(\alpha, L_1)})$.

An induction on $\max(\alpha)$ yields $\mathsf{KKD}(\alpha) = \mathsf{RSSYT}(\alpha)$. $\square$

This recursive description of $\mathsf{RSVT}(\alpha)$ leads to the following lemma.

Lemma 5.6. *Let $T = (L, E) = (L_1, E_1, t)$ be an element in $\mathsf{RSVT}(\alpha)$. If $(L, \emptyset) \in KD(\gamma)$ for another weak composition γ, then $T \in \mathsf{RSVT}(\gamma)$.*

Proof. Prove by induction on $\max(\gamma)$. Notice that t is in $\mathsf{RSVT}(\overline{M(\alpha, K_1)})$. If we ignore ghost cells of t, it is in $\mathsf{KD}(\overline{M(\gamma, K_1)})$. By our inductive hypothesis, $t \in \mathsf{RSVT}(\overline{M(\gamma, K_1)})$.

Now we check $T = (L_1, E_1, t)$ satisfies the four conditions of $\mathsf{RSVT}(\gamma)$. Condition 1 and 3 are implied by $T \in \mathsf{RSVT}(\alpha)$. Condition 2 follows from $(L, \emptyset) \in \mathsf{KD}(\gamma)$. The last condition is checked in the previous paragraph. $\square$

Remark 5.7. *Alternatively, we may prove Lemma 5.6 while viewing T as a tableau. Let T' be the tableau we get after keeping the smallest number in each cell of T. By $(L, \emptyset) \in \mathsf{KD}(\gamma)$, we know $T' \in \mathsf{RSSYT}(\gamma)$. Consequently, $K_-(T) = K_-(T') \leqslant \gamma$, so $T \in \mathsf{RSVT}(\gamma)$.*

6. Two operators on Kohnert diagrams

In order to prove the well-definedness and bijectivity of Ψ_α and Φ_α defined in Section 3, we introduce two auxiliary operators $\sharp_g$ and $\flat_e$ on $\mathsf{KD}(\alpha)$ and study their properties. Later in Section 7, we will use these two operators to give alternative descriptions of Ψ_α and Φ_α.

6.1. **Introducing the $\sharp_g$ operator.** We define an operator $\sharp_g$ on $\mathsf{KD}(\alpha)$ for each $g \in [n]$.

Definition 6.1. *For each $g \in [n]$, define $\sharp_g : \mathsf{KD}(\alpha) \to \mathsf{KD}(\alpha) \times [n]$. Take $D = (K_1, \emptyset, d) \in \mathsf{KD}(\alpha)$. Find the largest $k \in K_1 \cap [1, g]$ such that $D' = (K_1 - \{k\} \sqcup \{g\}, \emptyset, d)$ is still in $\mathsf{KD}(\alpha)$. If such k exists, then $\sharp_g(D) := (D', k)$. Otherwise, $\sharp_g(D)$ is undefined.*

We would like to determine when $\sharp_g(D)$ is well-defined. This is partially answered by the following lemma:

Lemma 6.2. *Assume $D = (K_1, \emptyset, d) \in \mathsf{KD}(\alpha)$. If $\sharp_g(D)$ is defined, then $|[g, n] \cap \mathsf{supp}(\alpha)| > |(g, n] \cap K_1|$.*

The proof involves Kohnert tableaux from subsection 2.4.

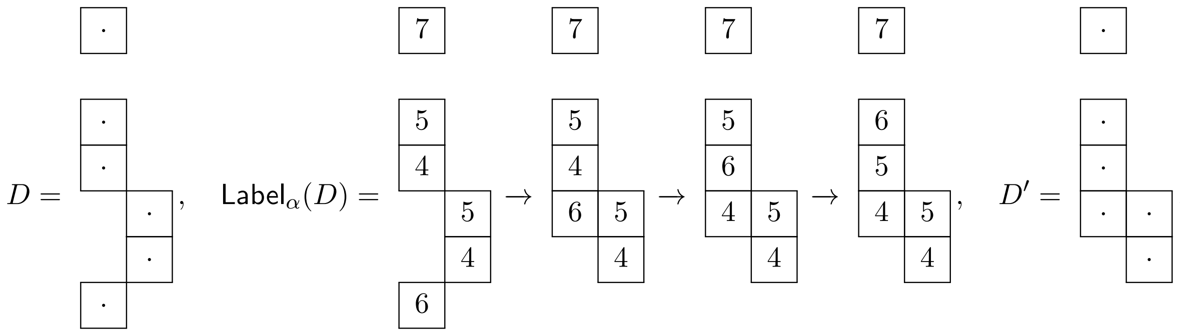
Proof. Assume $\sharp_g(D) = (D', k)$. Then in column 1 of D', there are $|(g, n] \cap K_1| + 1$ cells weakly above row g. In $\mathsf{Label}_\alpha(D')$, these cells are filled by distinct number in $[g, n] \cap \mathsf{supp}(\alpha)$. Thus, $|[g, n] \cap \mathsf{supp}(\alpha)| > |(g, n] \cap K_1|$. $\square$

Next, we will show the converse of this lemma. First, we introduce an algorithm called *sharp algorithm*. Its input would be a number g and $D = (K_1, \emptyset, d) \in \mathsf{KD}(\alpha)$ such that $|[g, n] \cap \mathsf{supp}(\alpha)| > |(g, n] \cap K_1|$. It will output a diagram pair D' with only Kohnert cells. It will also output a filling of D'. Later, we will check the filling is a Kohnert tableau with content α, which implies $D' \in \mathsf{KD}(\alpha)$. Finally, we will check D' is the first component of $\sharp_g(D)$.

The sharp algorithm consists of five steps:

- Step 1: Compute $\mathsf{Label}_\alpha(D)$.
- Step 2: Since $|[g, n] \cap \mathsf{supp}(\alpha)| > |(g, n] \cap K_1|$, there is a number m such that $m \geqslant g$ but m is weakly below row g in column 1. Find the highest such m. Let k be the row index of this m.
- Step 3: Let $D' = ((K_1 - \{k\}) \sqcup \{g\}, \emptyset, d)$. This is the first output.
- Step 4: To compute the filling, we start from $\mathsf{Label}_\alpha(D)$ and move the m from $(1, k)$ to $(1, g)$. The resulting filling satisfies the first three conditions from Definition 2.13.
- Step 5: If there is an $u < m$ such that u, m violates condition four in column 1, we find the smallest such u and swap it with m. Repeat this step until no such u exists. The final filling will be the second output.

Example 6.3. *Consider $\alpha = (0,0,0,2,2,1,1)$, and $D \in \mathsf{KD}(\alpha)$ as shown below. Let $g = 3$. The sharp algorithm gives $m = 6$ and $k = 1$. The output D' is obtained by moving the Kohnert cell $(1,k)$ to $(1,g)$ in D. To obtain the filling, we need to first move m to row g in $\mathsf{Label}_\alpha(D)$. Next, swap m with 4 and then 5.*



Lemma 6.4. *The filling produced by the sharp algorithm is a Kohnert tableau with content α. Consequently, the filling is $\mathsf{Label}_\alpha(D')$ and $D' \in \mathsf{KD}(\alpha)$.*

Proof. We claim that after Step 4 and after each iteration of Step 5, the filling satisfies the first three conditions of Definition 2.13. Moreover, if $i < j$ violates the last condition in column c, then $j = m$ and $c = 1$.

After Step 4, the filling clearly satisfies the first three conditions. Now assume $i < j$ violates the last condition in column c. Clearly, $c = 1$ and m is i or j. Assume m is i, then j is below row g in column 1. If j were below row k, then m, j would have violated condition 4 before this move. On the other hand, if j were above row k, then we would have picked j instead of m. In either case, we reach a contradiction. Thus, j must be m.

If there is a u such that $u < m$ violates condition 4, we pick the smallest such u. Assume our claim holds now. We need to show our claim is still true after we swap u and m. We check the first three conditions:

(1) Condition 1 clearly holds.
(2) Only need to check condition 2 for m. Recall $u < m$. Since u satisfies condition 2 before the move, so does m after the move.
(3) Only need to check condition 3 for u. Since u, m violate condition 4 before this swap, and that u satisfies condition 3 before moving m to $(1, g)$, there is no u in column 2 strictly above the m in column 1. Thus, after the move, u satisfies condition 3.

Now assume $i < j$ violates condition 4 in column c. Clearly, $c = 1$ and one of i, j is u or m. We just need to check u cannot be i or j and m cannot be i:

- Assume $i = u$. Then u, j would have violated condition 4 before the move, contradicting to our claim.
- Assume $j = u$. Then $i < u < m$. Before the move, i, m violates condition 4. Then we would have picked i and swapped it with m, rather than u. Contradiction.
- Assume $i = m$. If j were below m before the move, then m, j would have violated condition 4 before the move. Now assume j were between u and m before the move. Then u, j would have violated condition 4 before the move.

Now our claim holds after each move. When the sharp algorithm terminates, there is no violation of condition 4 with the form i, m. Thus, the filling satisfies condition 4, so it is in $\mathsf{KT}(\alpha)$. ☐

Lemma 6.5. *The D' yielded by the sharp algorithm is the first component of $\sharp_g(D)$.*

Proof. The lemma is trivial if $g \in K_1$. Thus, we may assume $g \notin K_1$. We already showed $D' \in \mathsf{KD}(\alpha)$ by constructing $\mathsf{Label}_\alpha(D')$. Take $r \in K_1$ with $k < r < g$. It suffices to show $(K_1 - \{r\} \sqcup \{g\}, \emptyset, d) \notin \mathsf{KD}(\alpha)$.

In column 1 of $\mathsf{Label}_\alpha(D)$, assume $s_1, \ldots, s_p$ are the numbers below row g and weakly above row r. By how we picked m, we know $s_1, \ldots, s_p < g$. We may run the labeling algorithm on $((K_1 - \{r\}) \sqcup \{g\}, \emptyset, d)$. It behaves the same as on D on cells prior to $(1, r)$. After filling all these cells before $(1, r)$, we know $s_1, \ldots, s_p$ are still in the set S_1. However, there remains only $p - 1$ empty cells below row g. Thus, at least one number of $s_1, \ldots, s_p$ will be placed weakly above row g. Since this number is less than g, the labeling algorithm will terminate and produce no output. Thus, this diagram is not in $\mathsf{KD}(\alpha)$. □

Thus, we know the sharp algorithm outputs the first component of $\sharp_g(D)$, together with its Kohnert Labeling. Now we can tell when $\sharp_g(D)$ is well-defined.

Lemma 6.6. *Assume $D = (K_1, \emptyset, d) \in \mathsf{KD}(\alpha)$. Then $\sharp_g(D)$ is well-defined if and only if*

$$|[g, n] \cap \mathsf{supp}(\alpha)| > |(g, n] \cap K_1|.$$

Proof. The forward direction is given by Lemma 6.2. The other direction follows from the sharp algorithm. □

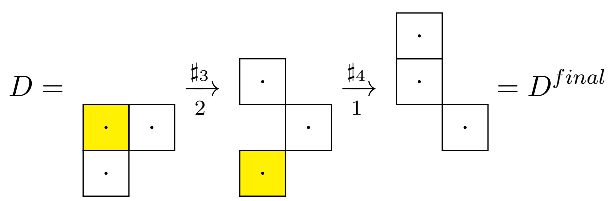
Corollary 6.7. *Take $D \in \mathsf{KD}(\alpha)$. Assume $\sharp_g(D) = (D', k)$. Assume $(1, k)$ is filled by m in $\mathsf{Label}_\alpha(D)$. Then $\mathsf{Label}_\alpha(D)$ and $\mathsf{Label}_\alpha(D')$ agree at the cell (c, r), if (c, r) satisfies one of the following:*

- *$c > 1$;*
- *$r < g$ and $r \neq k$;*
- *$r > m$.*

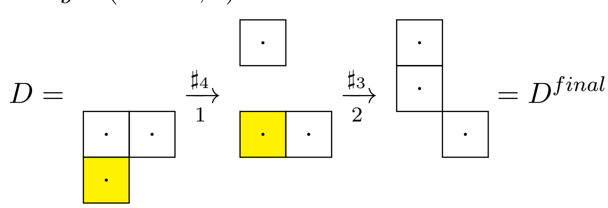
Proof. By the behavior of the sharp algorithm, $\mathsf{Label}_\alpha(D')$ is obtained from $\mathsf{Label}_\alpha(D)$ by moving m from $(1, k)$ to $(1, g)$ and repeatedly swapping m with a number above it. The number m will not go above row m, so only $(1, k)$ and cells between row g and row m in column 1 are affected. □

6.2. **Commutativity of $\sharp_g$ operators.** Next, we observe that two $\sharp_g$ operators might "commute" under certain conditions. Consider the following example:

Example 6.8. *Let $\alpha = (0, 0, 2, 1)$. Let D be the following element in $\mathsf{KD}(\alpha)$. We first apply $\sharp_3$ and get $(D^1, 2)$. Then apply $\sharp_4$ on D^1 and get $(D^{final}, 1)$.*



We can try to swap the order of these two operators. We first apply $\sharp_4$ on D and get $(D^2, 1)$. Then we apply $\sharp_3$ on D^2 and get $(D^{final}, 2)$.



Observe that changing the order of these two operators will not affect the final Kohnert diagram.

This phenomenon is captured by the following two lemmas.

Lemma 6.9. *Take $D \in \mathsf{KD}(\alpha)$. Take $g_1, g_2 \in [n]$ with $g_1 < g_2$. Assume $\sharp_{g_1}(D) = (D^1, k_1)$ and $\sharp_{g_2}(D^1) = (D^{final}, k_2)$. If $k_1 > k_2$, then the two operators "commute". That is:*

- *$\sharp_{g_2}(D) = (D^2, k_2)$ for some $D^2 \in \mathsf{KD}(\alpha)$, and*

- $\sharp_{g_1}(D^2) = (D^{final}, k_1)$.

Proof. Let C be the first column of $\mathsf{Label}_\alpha(D)$. Define C^1 and C^{final} similarly. In C, let m_1 (resp. m_2) be the number at row k_1 (resp. k_2).

First, we claim m_1 is below row g_2 in C^1. If not, then we may find a number u that is weakly above row g_2 in C but below row g_2 in C^1. By Corollary 6.7, the u is above row g_1 in C^1, so u is higher than m_2. By $u \geqslant g_2$, we should pick u rather than m_2 when computing $\sharp_{g_2}(D^1)$. Contradiction.

Now consider C^1. By Corollary 6.7, m_2 is still at row k_2. All numbers between m_2 and row g_2 will be less than g_2. Thus, $m_1 < g_2$. Since C and C^1 only differ between row g_1 and row m_1, all numbers between m_2 and row g_2 will be less than g_2 in C. Thus, $\sharp_{g_2}$ will also pick m_2 when acting on D. $\sharp_{g_2}(D) = (D^2, k_2)$. Column 1 of $\mathsf{Label}_\alpha(D^2)$ agrees with C between row k_1 and g_1. Thus, $\sharp_{g_1}$ will pick m_1 when acting on D^2. □

Lemma 6.10. *Take $D \in \mathsf{KD}(\alpha)$. Take $g_1, g_2 \in [n]$ with $g_1 < g_2$. Assume $\sharp_{g_2}(D) = (D^2, k_2)$ and $\sharp_{g_1}(D_2) = (D^{final}, k_1)$. If $k_1 > k_2$, then the two operators "commute". That is:*

- *$\sharp_{g_1}(D) = (D^1, k_1)$ for some $D^1 \in \mathsf{KD}(\alpha)$, and*
- *$\sharp_{g_2}(D^1) = (D^{final}, k_2)$.*

Proof. Let C be the first column of $\mathsf{Label}_\alpha(D)$. Define C^2 and C^{final} similarly. In C, let m_1 (resp. m_2) be the number at row k_1 (resp. k_2).

First, C and C^2 agree between row k_2 and row g_2. Thus, $\sharp_{g_1}$ would also pick m_1 when acting on D, so $\sharp_{g_1}(D) = (D^1, g_2)$.

Since $\sharp_{g_2}$ picks m_2 when acting on D, all numbers between row k_2 and row g_2 in C are less than g_2. In particular, $m_1 < g_2$. We know column 1 of D^1 is obtained from C by changing cells between row k_1 and row m_1. Thus, in column 1 of D^1, all numbers between row k_2 and row g_2 are still less than g_2. When acting on D^1, $\sharp_{g_2}$ would still pick m_2. □

6.3. **Introducing the $\flat_k$ operator.** Next, we define the operator $\flat_k$, which can be viewed as the (partial) inverse of $\sharp_g$.

Definition 6.11. *For each $k \in [n]$, define $\flat_k : \mathsf{KD}(\alpha) \to \mathsf{KD}(\alpha) \times [n]$. Take $D = (K_1, \emptyset, d) \in \mathsf{KD}(\alpha)$. Find the smallest $g \in K_1 \cap [k, n]$ such that $D' = ((K_1 - \{g\}) \sqcup \{k\}, \emptyset, d)$ is still in $\mathsf{KD}(\alpha)$. If such g exists, then $\flat_k(D) := (D', k)$. Otherwise, $\flat_k(D)$ is undefined.*

Lemma 6.12. *Take $D \in \mathsf{KD}(\alpha)$.*

- *Take $g \in [n]$. If $\sharp_g(D) = (D', k)$, then $\flat_k(D') = (D, g)$.*
- *Take $k \in [n]$. If $\flat_k(D) = (D', g)$, then $\sharp_g(D') = (D, k)$.*

In other words, $\flat_k$ and $\sharp_g$ are (partial) inverses of each other.

Proof. Assume $D = (K_1, \emptyset, d)$. Consider the first statement. If $g \in K_1$, then $k = g$ and $\flat_k(D') = (D, g)$ trivially. Now assume $g \notin K_1$. Then $((K_1 - \{r\}) \sqcup \{g\}, \emptyset, d)$ is not in $\mathsf{KD}(\alpha)$ for all $r \in K_1$ with $k < r < g$. Thus, $\flat_k(D') = (D, g)$. The second statement can be proved similarly. □

We would like to determine when $\flat_k(D)$ is well-defined. This is answered by the following lemma:

Lemma 6.13. *Assume $D = (K_1, \emptyset, d) \in \mathsf{KD}(\alpha)$. Let K_2 be the set of row indices for cells in column 2 of D. Then $\flat_k(D)$ is well-defined if and only if $k \in K_1$ or $|K_1 \cap (k, n]| > |K_2 \cap (k, n]|$.*

Proof. First, assume the condition fails. We show $\flat_k(D)$ is undefined. Assume by contradiction that $\flat_k(D) = (D', g)$. Define K_1' and K_2' similarly for D'. Then $K_1' = (K_1 - \{g\}) \sqcup \{k\}$ and $K_2' = K_2$. Thus,

$$|K_1' \cap (k, n]| < |K_1 \cap (k, n]| \leqslant |K_2 \cap (k, n]| = |K_2' \cap (k, n]|.$$

Then consider $\mathsf{Label}_\alpha(D')$. There are $|K_2' \cap (k,n]|$ distinct numbers above row k in column 2. They all must appear above row k in column 1, but there are not enough cells for them. Contradiction.

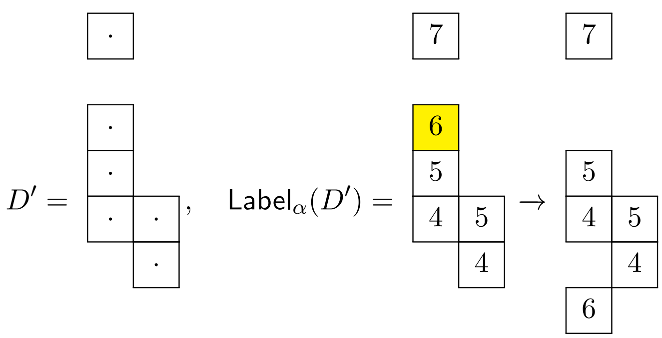
Now assume the condition holds, we show $\flat_k(D)$ is well-defined. Clearly, we are done if $k \in K_1$. Now assume $|K_1 \cap (k,n]| > |K_2 \cap (k,n]|$ and consider $\mathsf{Label}_\alpha(D)$. By our assumption, we can find m above row k in column 1 such that there is no m above row k in column 2. Pick the lowest such m and move it to $(1,k)$. Then the resulting filling is in $\mathsf{KT}(\alpha)$:

(1) Condition 1 of $\mathsf{KT}(\alpha)$ is clear.
(2) Since we moved a cell down, condition 2 is clear.
(3) Condition 3 holds for m since there is no m above row k in column 2.
(4) Only need to check there is no violations of condition 4 in column 1. Let $i < j$ be a violation. Then m must be i or j. If $i = m$, then m, j would have violated condition 4 before the move. Now assume $j = m$. If i were above m in $\mathsf{Label}_\alpha(D)$, then i, m would have violated condition 4 before the move. On the other hand, if i were below m in $\mathsf{Label}_\alpha(D)$, then there is an i above row k in column 2, so i, m cannot be a violation.

Thus, after moving one cell down to $(1,k)$ in D, the resulting diagram is still a Kohnert diagram, so $\flat_k(D)$ is well-defined. □

Remark 6.14. *In the previous proof of well-definedness, we choose the cell containing m and move it down to row k. The resulting diagram is still in $\mathsf{KD}(\alpha)$. Notice that this might not be the lowest cell that can do this job. See the following example.*

Example 6.15. *Following Example 6.3. We would like to compute $\flat_1(D')$. In D', there are 4 cells in column 1 above row 1 and there are 2 cells in column 2 above row 1. Thus, the condition in Lemma 6.13. is satisfied. We want to check $\flat_1(D')$ is well-defined. The proof of well-definedness gives $m = 6$. After moving the 6 to row 1, the resulting filling is in $\mathsf{KT}(\alpha)$, which implies the underlying diagram is in $\mathsf{KD}(\alpha)$.*



However, moving the cell $(1,3)$ to $(1,1)$ in D' will also make the resulting diagram in $\mathsf{KD}(\alpha)$.

Similar to $\sharp_g$, the $\flat_k$ operator can commute under certain condition:

Lemma 6.16. *Take $D \in \mathsf{KD}(\alpha)$. Take $k_1, k_2 \in [n]$ with $k_1 > k_2$. Assume $\flat_{k_1}(D) = (D^1, g_1)$ and $\flat_{k_2}(D^1) = (D^{final}, g_2)$. If $g_1 < g_2$, then the two operators "commute". That is:*

- *$\flat_{k_2}(D) = (D^2, g_2)$ for some $D^2 \in \mathsf{KD}(\alpha)$, and*
- *$\flat_{k_1}(D^2) = (D^{final}, g_1)$.*

Proof. It follows directly from Lemma 6.10 and Lemma 6.12. □

6.4. **Relations between $\sharp_g$ and $\flat_e$.** In this section, we investigate the relationship between the two operators introduced above. We already know the effect of $\sharp_g$ can be reversed by the $\flat_e$ operator, and vice versa. Next, we show that a sequence of $\sharp_g$ can also be reversed by a sequence of $\flat_e$, and vice versa.

Lemma 6.17. *Let $D^0 = (K_1, \emptyset, d) \in \mathsf{KD}(\alpha)$, and $1 < g_1 < g_2 < \cdots < g_m \leqslant n$ with $g_i \notin K_1$. For $i = 1, 2, \dots, m$, compute $\sharp_{g_i}(D^{i-1}) = (D^i, k_i)$. Assume $D^1, \dots, D^m$ are all well-defined.*

Find the permutation σ such that $k_{\sigma(1)} < \cdots < k_{\sigma(m)}$ and define $T^m = D^m$. For $i = m, m-1, \dots, 1$, compute $\flat_{k_{\sigma(i)}}(T^i) = (T^{i-1}, g_i')$. Then $T^0 = D^0$ and $\{g_1, \dots, g_m\} = \{g_1', \dots, g_m'\}$.

Proof. We may represent $D^0, \dots, D^m$ using the following diagram:

$$D^0 \xrightarrow[k_1]{\sharp_{g_1}} D^1 \xrightarrow[k_2]{\sharp_{g_2}} D^2 \xrightarrow[k_3]{\sharp_{g_3}} \dots \xrightarrow[k_m]{\sharp_{g_m}} D^m .$$

We put the operator above the arrow and put the second output under the arrow.

Suppose we find $k_i > k_{i+1}$. By Lemma 6.9, we can swap the order of $\sharp_{g_i}$ and $\sharp_{g_{i+1}}$, not affecting the last diagram D^m. Thus, after sorting the output numbers into increasing order, we have

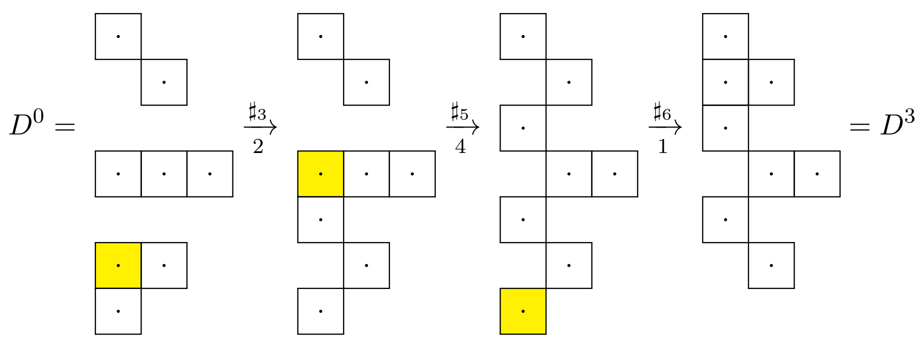
$$D^0 \xrightarrow[k_{\sigma(1)}]{\sharp_{g_{\sigma(1)}}} \tilde{D}^1 \xrightarrow[k_{\sigma(2)}]{\sharp_{g_{\sigma(2)}}} \tilde{D}^2 \xrightarrow[k_{\sigma(3)}]{\sharp_{g_{\sigma(3)}}} \dots \xrightarrow[k_{\sigma(m)}]{\sharp_{g_{\sigma(m)}}} D^m ,$$

where $\tilde{D}^i$ are some diagrams in $\mathsf{KD}(\alpha)$. Finally, we have

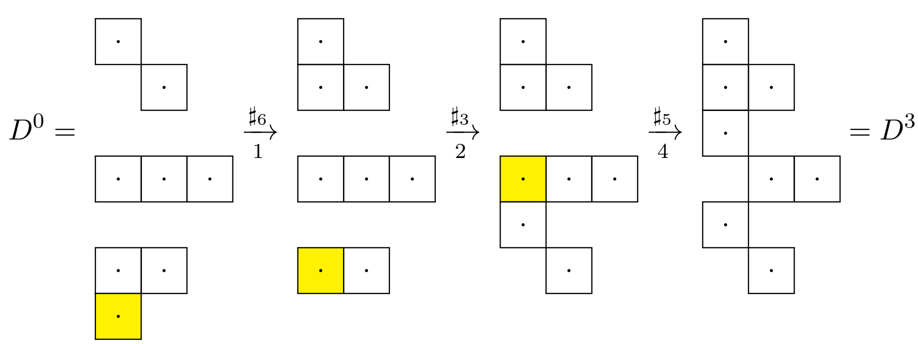
$$D^0 \xleftarrow[g_{\sigma(1)}]{\flat_{k_{\sigma(1)}}} \tilde{D}^1 \xleftarrow[g_{\sigma(2)}]{\flat_{k_{\sigma(2)}}} \tilde{D}^2 \xleftarrow[g_{\sigma(3)}]{\flat_{k_{\sigma(3)}}} \dots \xleftarrow[g_{\sigma(m)}]{\flat_{k_{\sigma(m)}}} D^m .$$

By $T^m = D^m$, we have $D^0 = T^0$. □

Example 6.18. *Consider $\alpha = (0, 0, 2, 0, 3, 1, 2)$, $g_1 = 3, g_2 = 5, g_3 = 6$ and $D^0 \in \mathsf{KD}(\alpha)$. Starting with D^0, we compute $\sharp_{g_1}, \sharp_{g_2}$ and then $\sharp_{g_3}$ to obtain D^3.*



We obtain $k_1 = 2, k_2 = 4$ and $k_3 = 1$, which are highlighted in the above figure. We can pick the permutation σ with one-line notation 312 and obtain $k_{\sigma(1)} = 1, k_{\sigma(2)} = 2$ and $k_{\sigma(3)} = 4$. Then we have $g_{\sigma(1)} = 6, g_{\sigma(2)} = 3$ and $g_{\sigma(3)} = 5$, which yields the following sequence of operation with the same D^3 as the final output.



Applying $\flat_4, \flat_2$ then $\flat_1$ on D^3, we will recover D^0.

Lemma 6.19. *Let $D^0 = (K_1, \emptyset, d) \in \mathsf{KD}(\alpha)$, and $n > e_1 > e_2 > \cdots > e_m \geqslant 1$, with $e_i \notin K_1$. For $i = 1, 2, \ldots, m$, compute $\flat_{e_i}(D^{i-1}) = (D^i, k_i)$. Assume $D^1, \ldots, D^m$ are all well-defined.*

Find the permutation σ such that $k_{\sigma(1)} > \cdots > k_{\sigma(m)}$. Now define $T^m = D^m$. For $i = m, m-1, \ldots, 1$, compute $\sharp_{k_{\sigma(i)}}(T^i) = (T^{i-1}, e'_i)$. Then $T^0 = D^0$ and $\{e_1, \ldots, e_m\} = \{e'_1, \ldots, e'_m\}$.

Proof. The proof is the same as the previous proof, using Lemma 6.16 instead of Lemma 6.9. □

7. Recursive descriptions of the maps

We have described our maps Ψ_α and Φ_α via $\sharp_G(K)$ and $\flat_E(L)$ in section 3. These descriptions are simple to state but hard to work with. Now, we will describe the Ψ_α and Φ_α recursively, involving definitions from section 6 and section 7. Using the new alternative descriptions, we can establish Lemma 3.3, Lemma 3.6 and Theorem 1.3.

7.1. **Recursive description of Ψ_α.** Let G be an arbitrary diagram. First, we can recursively describe the operator $\sharp_G(\cdot)$. If D is empty, then $\sharp_G(D)$ is also empty if $G = \emptyset$, or undefined otherwise. If D is not empty, write D as $(K_1, \emptyset, d)$. Let $G_{\geqslant 2}$ be the diagram $\{(c-1, r) : (c, r) \in G, c \geqslant 2\}$. View d as an element of $\mathsf{KD}(\overline{M(\alpha, K_1)})$ and find $d' = \sharp_{G_{\geqslant 2}}(d)$ recursively. Let G_1 be the set $\{r : (1, r) \in G\}$ and assume $G_1 = \{g_1 < \cdots < g_{|G_1|}\}$. Let D^0 be the Kohnert diagram $(K_1, \emptyset, d')$. Then compute $\sharp_{g_i}(D^{i-1}) = (D^i, k_i)$ for $1 \leqslant i \leqslant |G_1|$. The final output is $D^{|G_1|}$.

Lemma 7.1. *The description is equivalent to the description of $\sharp_G(K)$ in subsection 3.1.*

Proof. Recall that $\sharp_G(K)$ iterates over cells of G from right to left. Within each column, it goes from bottom to top. For a cell $(c, r) \in G$, it picks the highest cell weakly below (c, r) such that once this cell is raised to (c, r), the diagram is still in $\mathsf{KD}(\alpha)$. Then it moves the chosen cell to (c, r).

Let $D = (K_1, \emptyset, d)$ be the Kohnert diagram at the beginning of the iteration of $(c, r) \in G$. Assume $c \geqslant 2$. By the recursive description of $\mathsf{KD}(\alpha)$, the following two statements are equivalent:

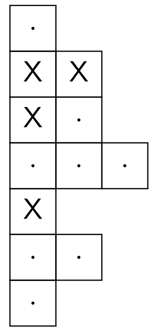
- (c, r') is a cell in D such that if we move it to (c, r), the diagram is still in $\mathsf{KD}(\alpha)$.
- $(c-1, r')$ is a cell in d such that if we move it to $(c-1, r)$, the diagram is still in $\mathsf{KD}(\overline{M(\alpha, K_1)})$.

Thus, iterations of $(c, r) \in G$ with $c \geqslant 2$ will behave the same as if $\sharp_{G_{\geqslant 2}}$ acts on $d \in \mathsf{KD}(\overline{M(\alpha, K_1)})$. Then iterations of $(1, g) \in G$ can be characterized by the $\sharp_g$ operator. □

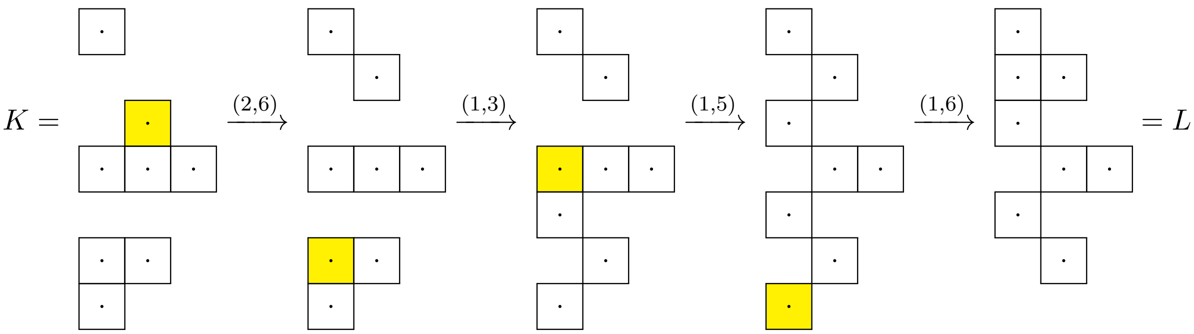
Now we can recursively describe the map Ψ_α. To make our description concise, we extend $\sharp_g(\cdot)$ to diagram pairs (K, G) such that $(K, \emptyset) \in \mathsf{KD}(\alpha)$ and G has no cells in column 1. The operator $\sharp_g(\cdot)$ acts as if acting on $(K, \emptyset)$.

Now take $D = (K_1, G_1, d) \in \mathsf{KKD}(\alpha)$. If D is the empty pair, we have $\Psi_\alpha(D) = D$. Otherwise, let $t = \Psi_\gamma(d)$, where $\gamma = \overline{M(\alpha, K_1)}$. Assume $G_1 = \{g_1 < \cdots < g_{|G_1|}\}$. Let D^0 be the diagram pair $(K_1, \emptyset, t)$. Then compute $\sharp_{g_i}(D^{i-1}) = (D^i, k_i)$ and write D^i as $(K_1^i, \emptyset, t)$. Finally, $\Psi_\alpha(D)$ is $(K_1^{|G_1|}, \{k_1, \ldots, k_{|G_1|}\}, t)$.

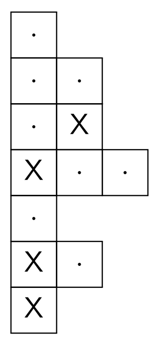
Example 7.2. *Consider $\alpha = (0, 0, 2, 0, 3, 1, 2)$. Let $D = (K, G)$ be the following element in $\mathsf{KKD}(\alpha)$.*

$D =$ 

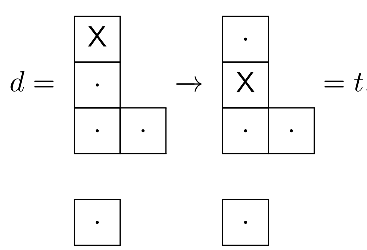
If we compute $\Psi_\alpha(D)$ using the description in subsection 3.1, we would go through the following iterations.



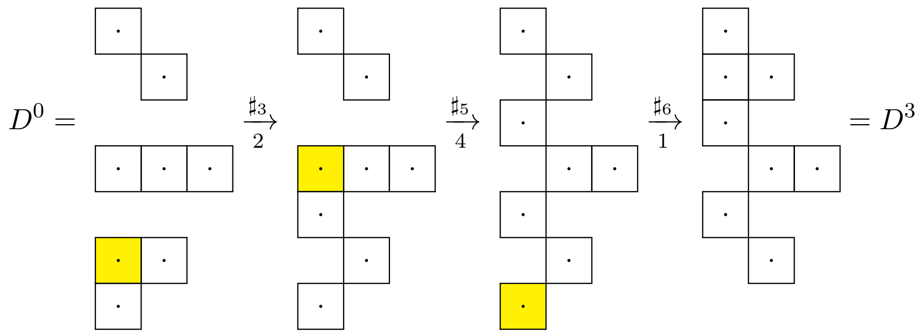
Thus, we have

$\Psi_\alpha(D) = (L, (K \sqcup G) - L) =$ 

Now we try our new recursive description. We may write D as (K_1, G_1, d), where $K_1 = \{1, 2, 4, 7\}, G_1 = \{3, 5, 6\}$ and d is illustrated below. Our new description would first view d as an element of $\mathsf{KKD}(\overline{M(\alpha, K_1)}) = \mathsf{KKD}((0, 1, 0, 2, 0, 0, 1))$ *and send it to t:*



It remains to perform $\sharp_3$, $\sharp_5$ and $\sharp_6$.



Finally, the image is just $(\{3,5,6,7\},\{1,2,4\},t)$*, which agrees with the computation above.*

It is clear that this recursive description agrees with the original description of Ψ_α. To prove Lemma 3.3, we need to show $t, D^0, \dots, D^{|G_1|}$ exist and satisfy our assumptions. Besides, we need to check the final output is a diagram pair in $\mathsf{RSVT}(\alpha)$.

Proof of Lemma 3.3. Prove by induction on $\max(\alpha)$. We may assume Ψ_γ is a well-defined map from $\mathsf{KKD}(\gamma)$ to $\mathsf{RSVT}(\gamma)$, where $\gamma = \overline{M(\alpha, K_1)}$ Thus, we know $t \in \mathsf{RSVT}(\gamma)$.

Then clearly if we ignore ghost cells in D^0, it is in $\mathsf{KD}(\alpha)$. Moreover, D^0 has no ghost cells in column 1. Next, we need to show the diagram pairs D^i are well-defined. By Theorem 5.1, we know for each g_i, $|[g_i, n] \cap \mathsf{supp}(\alpha)| > |[g_i, n] \cap K_1|$. Notice that the first $i-1$ iterations will not move any cells above row g_{i-1}. Thus, $[g_i, n] \cap K_1 = [g_i, n] \cap K_1^{i-1}$. By Lemma 6.6, D^i exists.

Next, we need to check the image is in $\mathsf{RSVT}(\alpha)$. In other words, $T = (K^{|G_1|}, \{k_1, \dots, k_{|G_1|}\}, t)$ should satisfy all four conditions in Theorem 5.4. Let L_1' be the set of row indices of Kohnert cells in column 1 of t.

(1) The first condition is immediate.
(2) Since Kohnert cells of $D^{|G_1|} = (K_1^{|G_1|}, \emptyset, t)$ is in $\mathsf{KD}(\alpha)$, we have $K_1^{|G_1|} \leqslant \mathsf{supp}(\alpha)$.
(3) For each k_i, we show $|(k_i, n] \cap K_1^{|G_1|}| > |(k_i, n] \cap L_1'|$. Since Kohnert cells of $D^{i-1} = (K_1^{i-1}, \emptyset, t)$ is in $\mathsf{KD}(\alpha)$, we know the Kohnert cells of t is in $\mathsf{KD}(\overline{M(\alpha, K_1^{i-1})})$. Thus, $L_1' \leqslant \mathsf{supp}(\overline{M(\alpha, K_1^{i-1})}) \subseteq K_1^{i-1}$. We have
$$|(k_i, n] \cap K_1^{i-1}| \geqslant |(k_i, n] \cap L_1'|.$$
Since K_1^i is obtained from K_1^{i-1} by replacing k_i with a larger number, we have $|(k_i, n] \cap K_1^i| = |(k_i, n] \cap K_1^{i-1}| + 1 > |(k_i, n] \cap L_1'|$. To obtain $K_1^{|G_1|}$, we replace each of $k_{i+1}, \dots, k_{|G_1|}$ in K_1^i with a larger number. Therefore, $|(k_i, n] \cap K_1^{|G_1|}| > |(k_i, n] \cap L_1'|$.
(4) Kohnert cells of t is in $\mathsf{KD}(\overline{M(\alpha, K_1^{|G_1|})})$ and $t \in \mathsf{RSVT}(\overline{M(\alpha, K_1)})$. By Lemma 5.6, $t \in \mathsf{RSVT}(\overline{M(\alpha, K_1^{|G_1|})})$.

□

7.2. **Recursive description of Φ_α.** Let E be an arbitrary diagram. We can recursively describe the operator $\flat_E(\cdot)$. If D is empty, then $\flat_E(D)$ is also empty if $E = \emptyset$, or undefined otherwise. If D is not empty, write D as $(L_1, \emptyset, t)$. Let $E_{\geqslant 2}$ be the diagram $\{(c-1, r) : (c, r) \in E, c \geqslant 2\}$. Let E_1 be the set $\{r : (1, r) \in E\}$. Assume $E_1 = \{e_1 > \cdots > e_{|E_1|}\}$. Let D^0 be the Kohnert diagram $(L_1, \emptyset, t)$. Then compute $\flat_{e_i}(D^{i-1}) = (D^i, g_i)$ for $1 \leqslant i \leqslant |E_1|$ and write D^i as $(L_1^i, \emptyset, t)$. View t as an element of $\mathsf{KD}(\overline{M(\alpha, L_1^{|E_1|})})$ and find $t' = \flat_{E_{\geqslant 2}}(t)$ recursively. Finally, $\flat_E(D) = (L_1^{|E_1|}, \emptyset, t')$.

Lemma 7.3. *The description is equivalent to the description of* $\flat_E(L)$ *in subsection 3.2.*

Proof. Recall that $\flat_E(L)$ iterates over cells of E from left to right. Within each column, it goes from top to bottom. For a cell $(c, r) \in E$, it picks the lowest cell weakly above (c, r) such that once this cell is lowered to (c, r), the diagram is still in $\mathsf{KD}(\alpha)$. Then it moves the chosen cell to (c, r).

Let $D = (L_1, \emptyset, t)$ be the Kohnert diagram at the beginning of the iteration of $(c, r) \in E$. The iterations of $(1, e) \in E$ can be characterized by the $\flat_e$ operator. Assume $c \geqslant 2$. By the recursive description of $\mathsf{KD}(\alpha)$, the following two statements are equivalent for any $r' < r$:

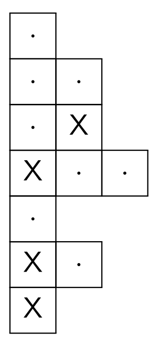
- (c, r') is a cell in D such that if we move it to (c, r), the diagram is still in $\mathsf{KD}(\alpha)$.
- $(c-1, r')$ is a cell in t such that if we move it to $(c-1, r)$, the diagram is still in $\mathsf{KD}(\overline{M(\alpha, L_1^{|E_1|})})$.

Thus, iterations of $(c, r) \in E$ with $c \geqslant 2$ will behave the same as if $\flat_{E_{\geqslant 2}}$ acts on $t \in \mathsf{KD}(\overline{M(\alpha, L_1^{|E_1|})})$. $\square$

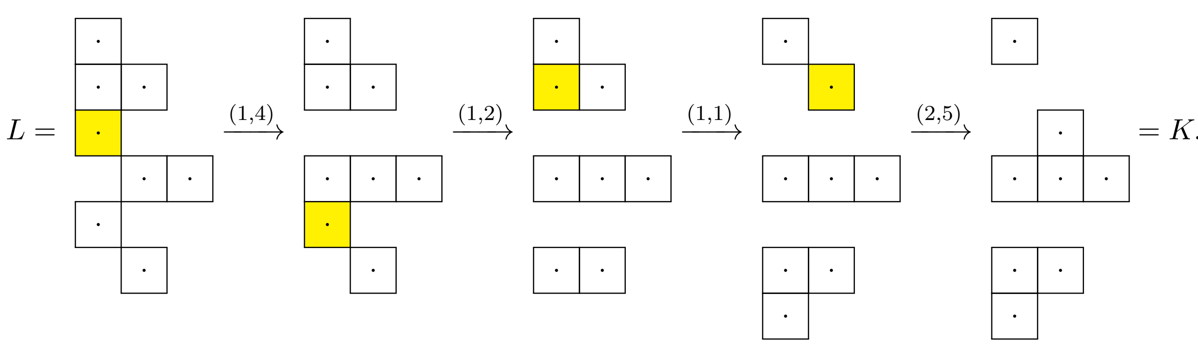
Now we can recursively describe the map Φ_α. To make our description concise, we extend $\flat_e(\cdot)$ to diagram pairs (K, G) such that $(K, \emptyset) \in \mathsf{KD}(\alpha)$ and G has no cells in column 1. The operator $\flat_e(\cdot)$ acts as if acting on $(K, \emptyset)$.

Now take $D = (L_1, E_1, t) \in \mathsf{RSVT}(\alpha)$. If D is the empty diagram pair, we have $\Phi_\alpha(D) = D$. Otherwise, assume $E_1 = \{e_1 > \cdots > e_{|E_1|}\}$. Let D^0 be the diagram pair $(L_1, \emptyset, t)$. Then compute $\flat_{e_i}(D^{i-1}) = (D^i, g_i)$ and write D^i as $(L_1^i, \emptyset, t)$. Notice that $t \in \mathsf{RSVT}(\overline{M(\alpha, L_1)})$ and its Kohnert cells is in $\mathsf{KD}(\overline{M(\alpha, L_1^{|E_1|})})$. Thus, by Lemma 5.6, we may view t as an element of $\mathsf{RSVT}(\gamma)$, where $\gamma = \overline{M(\alpha, L_1^{|E_1|})}$. Let $d = \Phi_\gamma(t)$. Finally, $\Phi_\gamma(T) = (L_1^{|E_1|}, \{g_1, \ldots, g_{|E_1|}\}, d)$.

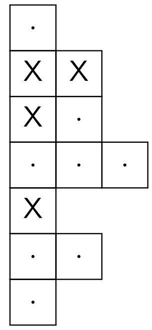
Example 7.4. *Consider $\alpha = (0, 0, 2, 0, 3, 1, 2)$. Let $D = (L, E)$ be the following element in $\mathsf{RSVT}(\alpha)$.*

$D =$ 

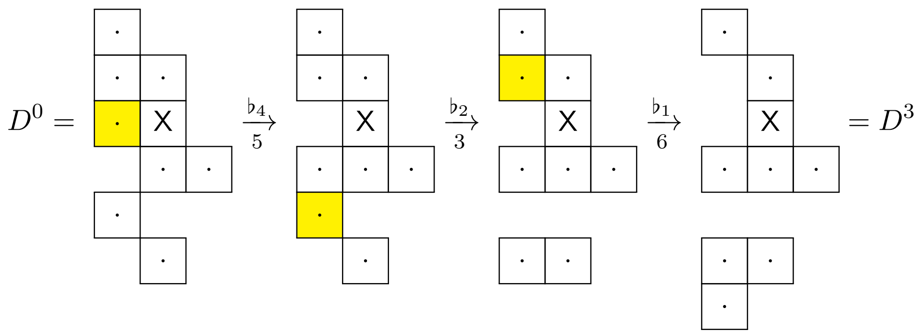
If we want to compute $\Phi_\alpha(D)$ using the description in subsection 3.2, we would go through the following iterations.



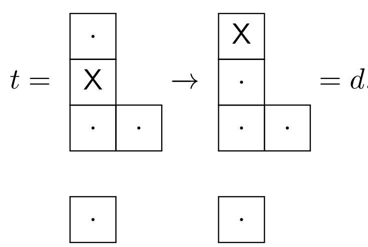
Thus, $\Phi_\alpha(D)$ *is:*

$(K, (L \sqcup E) - K) =$ 

Now we try our new recursive description. We may write D *as* (L_1, E_1, t)*, where* $L_1 = \{3, 5, 6, 7\}, E_1 = \{1, 2, 4\}$ *and* t *is illustrated below. Our new description would first perform* $\flat_4, \flat_2$ *and* $\flat_1$ *on* D^0*:*



Now, view t *as an element of* $\mathsf{KKD}(\overline{M(\alpha, L_1^3)}) = \mathsf{KKD}((0, 1, 0, 2, 0, 0, 1))$ *and send it to* d*:*



Finally, $\Phi_\alpha(D) = (\{1, 2, 4, 7\}, \{3, 5, 6\}, d)$*, which agrees with the computation above.*

It is clear that this recursive description agrees with the original description of Φ_α. To prove Lemma 3.6, we need to show $d, D^0, \ldots, D^{|E_1|}$ exist and satisfy our assumptions. Moreover, we need to check the final output is a diagram pair in $\mathsf{KKD}(\alpha)$.

Proof of Lemma 3.6. Prove by induction on $\max(\alpha)$. We may assume Φ_γ a well-defined map from $\mathsf{RSVT}(\gamma)$ to $\mathsf{KKD}(\gamma)$ for any γ with $\max(\gamma) < \max(\alpha)$.

Clearly D^0 is a diagram pair whose Kohnert cells form a diagram in $\mathsf{KD}(\alpha)$ and has no ghost cells in column 1. Next, we need to show D^i is well-defined for an arbitrary $1 \leqslant i \leqslant |E_1|$. Notice that the first $i-1$ iterations will not move any cells weakly below row e_{i-1}. Let L_1' consists of row indices of Kohnert cells in column 1 of t. Thus,

$$|(e_{i-1}, n] \cap L_1^{i-1}| = |(e_{i-1}, n] \cap L_1| > |(e_{i-1}, n] \cap L_1'|,$$

where the inequality follows from Theorem 5.4. By Lemma 6.13, D^i exists. Finally, by the inductive hypothesis, $d \in \mathsf{KKD}(\overline{M(\alpha, L_1^{|E_1|})})$.

Next, we need to check the final image is in $\mathsf{KKD}(\alpha)$. In other words, $(L^{|E_1|}, \{g_1, \dots, g_{|E_1|}\}, d)$ should satisfy all four conditions in Theorem 5.1.

(1) The first condition is immediate.
(2) Since Kohnert cells of $D^{|E_1|} = (L_1^{|E_1|}, \emptyset, t)$ is in $\mathsf{KD}(\alpha)$, we have $L_1^{|E_1|} \leqslant \mathsf{supp}(\alpha)$.
(3) For each g_i, we show $|[g_i, n] \cap \mathsf{supp}(\alpha)| > |[g_i, n] \cap L_1^{|E_1|}|$. Since $(L_1, E_1, t) \in \mathsf{RSVT}(\alpha)$, we have $L_1 \leqslant \mathsf{supp}(\alpha)$. Since L_1^j is obtained from L_1^{j-1} by replacing g_j with a smaller number, $L_1^{|E_1|} \leqslant \dots \leqslant L_1^0 \leqslant \mathsf{supp}(\alpha)$. By Lemma 4.4,
$$|[g_i, n] \cap L_1^{|E_1|}| \leqslant \dots \leqslant |[g_i, n] \cap L_1^0| \leqslant |[g_i, n] \cap \mathsf{supp}(\alpha)|.$$
Notice that $|[g_i, n] \cap L_1^i| = |[g_i, n] \cap L_1^{i-1}| - 1$. Thus, $|[g_i, n] \cap L_1^{|E_1|}| < |[g_i, n] \cap \mathsf{supp}(\alpha)|$.
(4) This is checked above.

□

7.3. **Proof of Theorem 1.3.** In this subsection, we prove Theorem 1.3.

Proof of Theorem 1.3. The maps clearly preserve $\mathsf{wt}(\cdot)$ and $\mathsf{ex}(\cdot)$. To show they are mutually inverses, we only need to check the following two statements.

(1) Take $D \in \mathsf{KKD}(\alpha)$. Let $T = \Psi_\alpha(D) \in \mathsf{RSVT}(\alpha)$. Then $\Phi_\alpha(T) = D$.
(2) Take $T \in \mathsf{RSVT}(\alpha)$. Let $D = \Phi_\alpha(D) \in \mathsf{KKD}(\alpha)$. Then $\Psi_\alpha(D) = T$.

We only establish the first statement using Lemma 6.17. The second statement can be proved similarly using Lemma 6.19 instead.

We prove by induction on $\max(\alpha)$. When $\max(\alpha) = 0$, $D = (\emptyset, \emptyset)$ and our claim is immediate.

Now assume $\max(\alpha) > 0$. Let $D = (K_1, G_1, d)$. First, we compute $\Psi_\alpha(D)$ using our recursive description. Let $t = \Psi_\gamma(d)$ where $\gamma = \overline{M(\alpha, K_1)}$. Assume $G_1 = \{g_1 < g_2 < \dots < g_{|G_1|}\}$. Let $D^0 = (K_1, \emptyset, t)$ and $\sharp_{g_i}(D^{i-1}) = (D^i, k_i) = ((K_1^i, \emptyset, t), k_i)$ for $i = 1, \dots, |G_1|$. Then we know D is sent to $T = (K_1^{|G_1|}, E_1, t\}) \in \mathsf{RSVT}(\alpha)$, where $E_1 = \{k_1, \dots, k_{|G_1|}\}$.

Now we compute $\Phi_\alpha(T)$ using our recursive description. Now write E_1 as $\{e_1 > \dots > e_{|G_1|}\}$. After applying $\flat_{e_1}, \dots, \flat_{e_{|G_1|}}$ on $(K_1^{|G_1|}, \emptyset, t)$, by Lemma 6.17, the resulting diagram pair is $(K_1, \emptyset, t)$ and G_1 consists of the output numbers. Finally, by the inductive hypothesis, $\Phi_\gamma(t) = d$. Thus, $\Phi_\alpha(T) = (K_1, G_1, d) = D$. □

Now we have the desired weight-preserving bijection between $\mathsf{KKD}(\alpha)$ and $\mathsf{RSVT}(\alpha)$. We can claim the Ross-Yong conjecture is correct.

Corollary 7.5. *The Lascoux polynomials indexed by α, has a combinatorial formula with $\mathsf{KKD}(\alpha)$, i.e.,*
$$\mathfrak{L}_\alpha^{(\beta)} = \sum_{D \in \mathsf{KKD}(\alpha)} \beta^{\mathsf{ex}(D)} \boldsymbol{x}^{\mathsf{wt}(D)}.$$

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