Tableaux rules for key polynomials and Lascoux polynomials

Tianyi Yu (UCSD)

March 29, 2022

Outline

1. Key polynomials are characters of the Demazure modules. We introduce two rules to compute a key polynomial using tableaux.

2. Lascoux polynomials are K-theoretic analogues of key polynomials. We introduce two rules to compute a Lascoux polynomial using tableaux.

Operator

Define an operator on $\mathbb{Z}[x_1, x_2, \ldots, x_n]$:

$$\pi_i(f) = (x_i - x_{i+1})^{-1}(x_i f - x_{i+1} s_i f).$$

- $\pi_1(x_1) = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2$
- $\pi_1(x_1^2) = \frac{x_1^3 - x_2^3}{x_1 - x_2} = x_1^2 + x_1 x_2 + x_2^2$

Key polynomials

For weak composition α,

$$\kappa_\alpha := \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition} \\ \pi_i(\kappa_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

- $\kappa_{210} = x_1^2 x_2$.
- $\kappa_{120} = \pi_1(\kappa_{210}) = \frac{x_1^3 x_2 - x_1 x_2^3}{x_1 - x_2} = x_1^2 x_2 + x_1 x_2^2$.
- $\kappa_{102} = \pi_2(\kappa_{120}) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2$.

Schur polynomials

$$\kappa_{012} = \pi_1(\kappa_{102}) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3,$$

which is the Schur polynomial of $(2,1)$ in variables x_1, x_2, x_3.

Fact: When α is a weakly increasing weak composition, κ_α is the Schur polynomial of $\mathrm{rev}(\alpha)$.

Combinatorial formula for Schur polynomials: SSYT

Schur polynomials can be computed by semi-standard Young tableaux (SSYT):

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3)$$
$$= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Combinatorial formula for Schur polynomials: RSSYT

Schur polynomials can also be computed by reversed semi-standard Young tableaux (RSSYT):

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} & s_{(2,1)}(x_1, x_2, x_3) \\ = & \, x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2 x_1 x_2 x_3 \end{aligned}$$

Keys

A *key* is a SSYT where each number in column j is also in column $j-1$.

A *reverse key* is a RSSYT where each number in column j is also in column $j-1$.

Keys, reverse keys, and weak compositions are in bijection with each other:

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \Longleftrightarrow \begin{array}{|c|c|c|}\hline 4 & 4 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \Longleftrightarrow (1,0,3,2)$$

Let $\mathrm{key}(\cdot)$ and $\mathrm{key}^R(\cdot)$ send a weak composition to its corresponding key or reversed key.

Right keys and left keys

Each SSYT is associated with a key.

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad K_+(T) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Each RSSYT is associated with a reverse key.

$$T = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad K_-(T) = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$$

SSYT formula for key polynomials

Theorem (Lascoux, Schützenberger, 1980)

$$\kappa_\alpha = \sum_T x^{\mathrm{wt}(T)}$$

where T is a SSYT such that $K_+(T) \leq \mathrm{key}(\alpha)$.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\kappa_{102} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2.$$

RSSYT formula for key polynomials

Theorem (Lascoux, Schützenberger, 1980)

$$\kappa_\alpha = \sum_T x^{\mathrm{wt}(T)}$$

where T is a RSSYT such that $K_-(T) \leq \mathrm{key}^R(\alpha)$.

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\kappa_{102} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2.$$

Operator

Define an operator on $\mathbb{Z}[\beta][x_1, x_2, \ldots, x_n]$:

$$\pi_i^{(\beta)}(f) = \pi_i(f) + \beta\pi_i(x_{i+1}f).$$

- $\pi_1^{(\beta)}(x_1) = x_1 + x_2 + \beta x_1 x_2$
- $\pi_1^{(\beta)}(x_1^2) = x_1^2 + x_1 x_2 + x_2^2 + \beta(x_1^2 x_2 + x_1 x_2^2).$

Lascoux polynomials

For weak composition α,

$$\mathfrak{L}_{\alpha}^{(\beta)} := \begin{cases} x^{\alpha} & \text{if } \alpha \text{ is a partition} \\ \pi_i^{(\beta)}(\mathfrak{L}_{s_i\alpha}^{(\beta)}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

- $\mathfrak{L}_{210}^{(\beta)} = x_1^2 x_2$.
- $\mathfrak{L}_{120}^{(\beta)} = x_1^2 x_2 + x_1 x_2^2 + \beta x_1^2 x_2^2$.

SVT formula for $\mathfrak{L}_\alpha^{(\beta)}$

Theorem (Buch 2002)

When α is weakly increasing, $\mathfrak{L}_\alpha^{(\beta)}$ is a sum over set-valued tableaux (SVT) with shape $\mathrm{rev}(\alpha)$.

The following contribute to $\mathfrak{L}_{(0,1,2)}^{(\beta)}$

12	3
3	

$\beta x_1 x_2 x_3^2$

1	13
23	

$\beta^2 x_1^2 x_2 x_3^2$

1	2
23	

$\beta x_1 x_2^2 x_3$

....

SVT formula for $\mathfrak{L}_\alpha^{(\beta)}$

Theorem (Y 2021)

$$\mathfrak{L}_\alpha^{(\beta)} = \sum_{T} \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)}$$

where T is a SVT such that no matter how you pick one number from each box, the resulting SSYT contributes to κ_α.

Example $\mathfrak{L}^{(\beta)}_{(1,0,2)}$

$$
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}
$$

$$
\begin{array}{|c|c|}\hline 1 & 12 \\ \hline 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 23 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 13 \\ \hline 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 23 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 13 \\ \hline 3 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 23 \\ \hline 2 \\ \cline{1-1}\end{array}
$$

$$
\begin{array}{|c|c|}\hline 1 & 123 \\ \hline 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 1 & 13 \\ \hline 23 \\ \cline{1-1}\end{array}
$$

Thus, we may write $\mathfrak{L}^{(\beta)}_{(1,0,2)}$ as

$$
\begin{aligned}
&x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 \\
&+\beta(x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1x_2x_3^2 + x_1^2x_3^2 + x_1x_2^2x_3) \\
&+\beta^2(x_1^2x_2^2x_3 + x_1^2x_2x_3^2)
\end{aligned}
$$

RSVT rule for $\mathfrak{L}_\alpha^{(\beta)}$

Theorem (Buciumas, Scrimshaw, Weber 2020; Shimozono, Y 2021)

$$\mathfrak{L}_\alpha^{(\beta)} = \sum_T \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)}$$

where T is a RSVT such that if you pick the largest number from each box, the resulting RSSYT contributes to κ_α.

Example $\mathfrak{L}^{(\beta)}_{(1,0,2)}$

$$
\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}
$$

$$
\begin{array}{|c|c|}\hline 2 & 21 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 21 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 32 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 32 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 31 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 32 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}
$$

$$
\begin{array}{|c|c|}\hline 32 & 21 \\ \hline 1 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 321 \\ \hline 1 \\ \cline{1-1}\end{array}
$$

Thus, we may write $\mathfrak{L}^{(\beta)}_{(1,0,2)}$ as

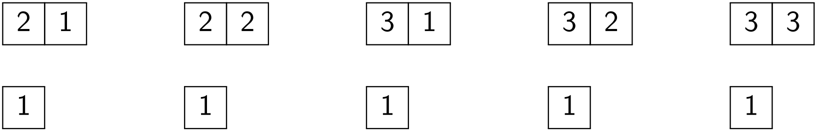
$$
\begin{aligned}
&x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 \\
+&\beta(x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1x_2x_3^2 + x_1^2x_3^2 + x_1x_2^2x_3) \\
+&\beta^2(x_1^2x_2^2x_3 + x_1^2x_2x_3^2)
\end{aligned}
$$

A future direction

Find a weight preserving bijection between SVT and RSVT that can unify the rules above.

Other combinatorial formulas for κ_α and $\mathfrak{L}_\alpha^{(\beta)}$

- Skyline fillings rule for κ_α [Mason 2009].
- Set-valued skyline fillings for $\mathfrak{L}_\alpha^{(\beta)}$. Conjectured [Monical 2017] proved [Buciumas, Scrimshaw, Weber 2020]



Other combinatorial formulas for κ_α and $\mathfrak{L}_\alpha^{(\beta)}$

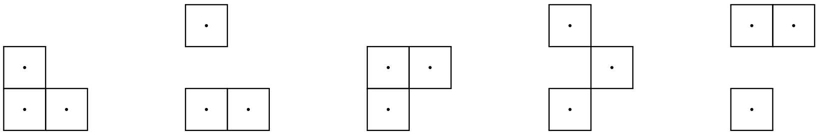
- Reduced compatible sequence rule for κ_α [Reiner, Shimozono 1995].
- Compatible sequence rule for $\mathfrak{L}_\alpha^{(\beta)}$ [Shimozono, Y 2021].

$$\begin{pmatrix}3 & 1 & 4\\ 2 & 1 & 1\end{pmatrix} \begin{pmatrix}3 & 4 & 1\\ 2 & 2 & 1\end{pmatrix} \begin{pmatrix}3 & 1 & 4\\ 3 & 1 & 1\end{pmatrix} \begin{pmatrix}3 & 4 & 1\\ 2 & 3 & 1\end{pmatrix} \begin{pmatrix}3 & 4 & 1\\ 3 & 3 & 1\end{pmatrix}$$

Remark: This rule establishes a conjecture of Reiner and Yong.

Other combinatorial formulas for κ_α and $\mathfrak{L}_\alpha^{(\beta)}$

- Kohnert diagrams rule for κ_α [Kohnert 1991].
- K-Kohnert diagrams rule for $\mathfrak{L}_\alpha^{(\beta)}$. Conjectured [Ross, Yong 2015]. Rectangle case proved [Pechenik, Scrimshaw 2019]. General case proved [Pan, Y 2022].



Thanks for listening!!

- M. Shimozono, and T Yu. Grothendieck to Lascoux expansions. arXiv preprint arXiv:2106.13922 (2021).
- T Yu. Set-valued tableaux rule for Lascoux polynomials. arXiv preprint arXiv:2110.00164 (2021).