TOP-DEGREE COMPONENTS OF GROTHENDIECK AND LASCOUX POLYNOMIALS

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ABSTRACT. We define a filtered algebra $\widehat{V_1} \subset \widehat{V_2} \subset \cdots \subset \widehat{V} \subset \mathbb{Q}[x_1, x_2, \dots]$ which gives an algebraic interpretation of a classical q-analogue of Bell numbers. The space $\widehat{V_n}$ is the span of the Castelnuovo–Mumford polynomials $\widehat{\mathfrak{G}}_w$ with $w \in S_n$. Pechenik, Speyer and Weigandt define $\widehat{\mathfrak{G}}_w$ as the top-degree components of the Grothendieck polynomials and extract a basis of $\widehat{V_n}$. We describe another basis consisting of $\widehat{\mathfrak{L}}_\alpha$, the top-degree components of Lascoux polynomials. Our basis connects the Hilbert series of $\widehat{V_n}$ and $\widehat{V}$ to rook-theoretic results of Garsia and Remmel. To understand $\widehat{\mathfrak{L}}_\alpha$, we introduce a combinatorial construction called a "snow diagram" that augments and decorates any diagram D. When D is the key diagram of α, its snow diagram yields the leading monomial of $\widehat{\mathfrak{L}}_\alpha$. When D is the Rothe diagram of w, its snow diagram yields the leading monomial of $\widehat{\mathfrak{G}}_w$, agreeing with the work of Pechenik, Speyer and Weigandt.

1. INTRODUCTION

In this paper, we introduce and study a new graded subalgebra $\widehat{V}$ of the polynomial ring $\mathbb{Q}[x_1, x_2, ...]$ in infinitely many x-variables. It is equipped with a distinguished filtration $\widehat{V_1} \subset \widehat{V_2} \subset \cdots \subset \widehat{V}$ satisfying $\widehat{V} = \bigcup_n \widehat{V_n}$. Both $\widehat{V_n}$ and $\widehat{V}$ are graded vector spaces under the polynomial degree. We will examine $\widehat{V_n}$ and $\widehat{V}$ using two different bases.

Pechenik, Speyer and Weigandt [PSW21] introduce the *Castelnuovo–Mumford polynomials*, which are the top-degree components of Grothendieck polynomials. For $w \in S_n$, we denote its Castelnuovo–Mumford polynomial as $\widehat{\mathfrak{G}}_w$.[1] We define $\widehat{V_n}$ as the $\mathbb{Q}$-span of $\widehat{\mathfrak{G}}_w$ with $w \in S_n$ and let $\widehat{V} = \bigcup_{n \geqslant 1} \widehat{V_n}$. We check $\widehat{V}$ is a filtered algebra. The work of Pechenik, Speyer and Weigandt [PSW21]. extracts a basis of $\widehat{V_n}$ and $\widehat{V}$ consisting of $\widehat{\mathfrak{G}}_w$.

We introduce another basis of $\widehat{V_n}$ and $\widehat{V}$ consisting of $\widehat{\mathfrak{L}}_\alpha$, the top-degree components of Lascoux polynomials. Our basis of $\widehat{V_n}$ naturally corresponds to *non-attacking rook diagrams* within a staircase. Garsia and Remmel [GR86] use these diagrams to give a combinatorial interpretation of $B_n(q)$, a q-analogue of Bell numbers due to Milne [Mil82]. In Theorem 1.4 and Theorem 1.5, we use our basis to relate the Hilbert series of $\widehat{V_n}$ and $\widehat{V}$ to $B_n(q)$.

1.1. Castelnuovo–Mumford polynomials and their span.

The Castelnuovo–Mumford polynomials that span $\widehat{V}$ are defined based on *Grothendieck polynomials*. For $w \in S_n$, the Grothendieck polynomial $\mathfrak{G}_w$, introduced by Lascoux and Schützenberger [LS82a], are polynomial representatives of the K-classes of structure sheaves of Schubert varieties in flag varieties. They are polynomials in β and $x_1, x_2, \dots$ with positive integer coefficients. We may view $\mathfrak{G}_w$ as a polynomial of β. From this perspective, the coefficient of β^d is a homogeneous polynomial in the x-variables with degree $\mathsf{inv}(w) + d$, where $\mathsf{inv}(w)$ is the number of inversions in w (See §2). If we extract the coefficient of β^0, we get the *Schubert polynomial* $\mathfrak{S}_w$. There exist various combinatorial rules for the monomial expansion of Schubert polynomials [BB93] [BJS93]. Let V_n be the $\mathbb{Q}$-span of $\mathfrak{S}_w$ for all $w \in S_n$.

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[1] Pechenik, Speyer and Weigandt [PSW21] denote it as $\mathfrak{CM}_w$.

It is well-known that V_n is the $\mathbb{Q}$-span of all monomials that divide $x_1^{n-1}x_2^{n-2}\dots x_{n-1}$. The vector space V_n has a basis $\{\mathfrak{S}_w : w \in S_n\}$, so its dimension is $n!$.

Let A be a vector subspace of $\mathbb{Q}[x_1, x_2, \dots]$ with a basis consisting of homogeneous polynomials. The *Hilbert series* of A, denoted as $\mathrm{Hilb}(A; q)$, is $\sum_{n\geqslant 0} m_n q^n$, where m_n is the number of polynomials with degree n in that basis of A. Then $\mathrm{Hilb}(V_n; q) = \sum_{w\in S_n} q^{\mathsf{inv}(w)}$, which is the q-analogue of $n!$.

Let S_+ be the set of permutations of $\{1, 2, \dots\}$. Lascoux and Schützenberger [LS82a] define $\mathfrak{G}_w$ and $\mathfrak{S}_w$ for $w \in S_+$. Let V be the $\mathbb{Q}$-span of $\mathfrak{S}_w$ with $w \in S_+$. Then $V = \bigcup_{n\geqslant 1} V_n$ and V has basis $\{\mathfrak{S}_w : w \in S_+\}$. In fact, V is just $\mathbb{Q}[x_1, x_2, \dots]$, so V is an algebra. Recall the definition of a filtered algebra.

Definition 1.1. *Let A be an algebra over $\mathbb{Q}$. We say A is a filtered algebra over $\mathbb{Q}$ if there exists an increasing sequence of vector spaces $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ such that $A = \bigcup_{n\geqslant 1} A_n$ and $A_m \cdot A_n \subseteq A_{m+n}$ for any $m, n \in \mathbb{Z}_{\geqslant 0}$.*

By the increasing sequence $V_1 \subset V_2 \subset \cdots \subset V$, V has the structure of a filtered algebra.

On the other hand, this paper studies the top-degree component of Grothendieck polynomials. For a polynomial $f \in \mathbb{Q}[\beta][x_1, x_2, \dots]$, we may view it as a polynomial of β. We use $\widehat{f}$ to denote the coefficient of the highest power of β. Following the notaion, the *Castelnuovo–Mumford polynomial* can be defined as $\widehat{\mathfrak{G}}_w$. For instance, for the permutation $w \in S_4$ with one-line notation 1243,

$$\mathfrak{G}_w = x_1 + x_2 + x_3 + \beta(x_1x_2 + x_1x_3 + x_2x_3) + \beta^2 x_1x_2x_3, \quad \text{and } \widehat{\mathfrak{G}}_w = x_1x_2x_3.$$

Pechenik, Speyer and Weigandt [PSW21] first introduce and study $\widehat{\mathfrak{G}}_w$. In particular, they describe the *leading monomial* in $\widehat{\mathfrak{G}}_w$ with respect to the *tail lexicographic order*. That is: x^α is larger than x^γ if there exists k such that $\alpha_k > \gamma_k$ and $\alpha_j = \gamma_j$ for all $j > k$. To study the leading monomial of $\widehat{\mathfrak{G}}_w$, Pechenik, Speyer and Weigandt [PSW21]. define a statistic $\mathsf{rajcode}(\cdot)$ on S_n and focus on a subset of S_n called *inverse fireworks permutations* (see §2). We summarize some of results in [PSW21] on $\widehat{\mathfrak{G}}_w$.

(A) The polynomial $\widehat{\mathfrak{G}}_w$ has leading monomial $x^{\mathsf{rajcode}(w)}$. If w is inverse fireworks, then $x^{\mathsf{rajcode}(w)}$ has coefficient 1 in $\widehat{\mathfrak{G}}_w$.
(B) For $w, w' \in S_n$, $\widehat{\mathfrak{G}}_w$ is a scalar multiple of $\widehat{\mathfrak{G}}_{w'}$ if and only if $\mathsf{rajcode}(w) = \mathsf{rajcode}(w')$.
(C) Any $w \in S_n$ has the same $\mathsf{rajcode}$ as exactly one $w' \in S_n$ that is inverse fireworks.

Dreyer, Mészáros and St. Dizier [DMD22] provide an alternative proof of (A) via the climbing chain model for Grothendieck polynomials introduced by Lenart, Robinson, and Sottile [LRS06]. Hafner [Haf22] provides an alternative proof of (A) for vexillary permutations via bumpless pipedreams. Notice that (A) implies the degree of $\widehat{\mathfrak{G}}_w$ is $\mathsf{raj}(w)$, which is the sum of all entries in $\mathsf{rajcode}(w)$. Thus, the x-degree in $\mathfrak{G}_w$ is $\mathsf{raj}(w)$. A diagrammatic way to compute the x-degree of $\mathfrak{G}_w$ when w is vexillary or 1432-avoiding is given by Rajchgot, Robichaux, and Weigandt [RRW22].

We may view $\widehat{\mathfrak{G}}_w$ as the "top-degree counterparts" of the Schubert polynomials. Then $\widehat{V_n} := \mathrm{span}\{\widehat{\mathfrak{G}}_w : w \in S_n\}$ is an analogue of V_n. By work of Pechenik, Speyer and Weigandt, $\widehat{V_n}$ has basis

$$\{\widehat{\mathfrak{G}}_w : w \in S_n \text{ is inverse fireworks}\}. \tag{1}$$

The reason is as follows:

- By (C), for any $w \in S_n$, $\mathsf{rajcode}(w) = \mathsf{rajcode}(w')$ for some inverse fireworks $w' \in S_n$. By (B), $\widehat{\mathfrak{G}}_w$ is a scalar multiple of $\widehat{\mathfrak{G}}_{w'}$, so (1) spans $\widehat{V_n}$.
- By (C), for two different inverse fireworks permutations in S_n, their $\mathsf{rajcode}$ are different. By (A), their leading monomials are different. Thus, (1) consists of polynomials with distinct leading monomials, so it is linearly independent.

By [Cla01], the number of inverse fireworks permutations in S_n is B_n, the n^{th} Bell number. Thus, $\widehat{V_n}$ has dimension B_n. We also define the infinite dimensional vector space $\widehat{V} := \operatorname{span}\{\widehat{\mathfrak{G}}_w : w \in S_+\}$. Analogous to V, we have $\widehat{V} = \bigcup_{n \geqslant 0} \widehat{V_n}$. We check $\widehat{V}$ is a filtered algebra in Proposition 2.7.

To understand the Hilbert series of $\widehat{V_n}$ and $\widehat{V}$, we develop another base for these spaces.

1.2. **Top-degree components of Lascoux polynomials.** Let α be a weak composition. The Lascoux polynomial of α, denoted as $\mathfrak{L}_\alpha$, is introduced by Lascoux [Las03]. Just like $\mathfrak{G}_w$, Lascoux polynomials involve variables $\beta, x_1, x_2, \ldots$ with positive integer coefficients. Grothendieck polynomials and Lascoux polynomials are closely related. An expansion of Grothendieck polynomials into Lascoux polynomials was conjectured by Reiner and Yong [RY21] and proven by Shimozono and Yu [SY21]. Let C_n be the subset of $\alpha \in C_+$ such that $\alpha_i \leqslant n - i$ for $i \in [n]$ and $\alpha_i = 0$ for $i > n$. Following [SY21], for $w \in S_n$, $\mathfrak{G}_w$ expands into $\mathfrak{L}_\alpha$ with $\alpha \in C_n$, where the coefficients are positive integers multiplied by a non-negative power of β.

If we view $\mathfrak{L}_\alpha$ as a polynomial in β, the coefficient of β^d is a homogeneous polynomial in the x-variables with degree $|\alpha| + d$, where $|\alpha|$ is the sum of all entries in α. The coefficient of β^0 is key polynomial κ_α. The key polynomials, introduced by Demazure [Dem74], are characters of Demazure modules. The Grothendieck-to-Lascoux expansion recovers the Schubert-to-Key expansion: $\mathfrak{S}_w$ with $w \in S_n$ expands positively into κ_α with $\alpha \in C_n$. This expansion was found by Lascoux and Schützenberger [LS89] and proven by Reiner and Shimozono [RS95]. Consequently, $\{\kappa_\alpha : \alpha \in C_n\}$ is a basis of V_n.

Inspired by this basis of V_n with key polynomials, we construct our new basis of $\widehat{V_n}$. The *top Lascoux polynomial* of α is defined as $\widehat{\mathfrak{L}}_\alpha$. By the Grothendieck-to-Lascoux expansion, we know $\widehat{\mathfrak{G}}_w$ with $w \in S_n$ expands positively into $\widehat{\mathfrak{L}}_\alpha$ with $\alpha \in C_n$. Therefore, $\widehat{V_n}$ is a subspace of $\operatorname{span}\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n\}$. To show these two spaces are equal and extract a basis of $\widehat{V_n}$, we need a better understanding of $\widehat{\mathfrak{L}}_\alpha$. We define $\mathsf{rajcode}(\cdot)$ on weak compositions and focus on a subset of weak compositions called snowy weak compositions. We show in §4 that $\widehat{\mathfrak{L}}_\alpha$ enjoy analogous properties of $\widehat{\mathfrak{G}}_w$ that are established by Pechenik, Speyer and Weigandt [PSW21].

Theorem 1.2. *Let α, γ be two weak compositions.*

(a) The polynomial $\widehat{\mathfrak{L}}_\alpha$ has leading monomial $x^{\mathsf{rajcode}(w)}$. If α is snowy, then $x^{\mathsf{rajcode}(w)}$ has coefficient 1 in $\widehat{\mathfrak{L}}_\alpha$.
(b) $\widehat{\mathfrak{L}}_\alpha$ is a scalar multiple of $\widehat{\mathfrak{L}}_\gamma$ if and only if $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$.
(c) Any $\alpha \in C_n$ has the same $\mathsf{rajcode}$ as exactly one $\alpha' \in C_n$ that is snowy.

Therefore, $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n \text{ is snowy}\}$ spans the same space as $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n\}$. This span has a B_n-dimensional subspace $\widehat{V_n}$. In §5, we give another basis of $\widehat{V_n}$.

Theorem 1.3. *The space $\widehat{V_n}$ is also the $\mathbb{Q}$-span of $\widehat{\mathfrak{L}}_\alpha$ with $\alpha \in C_n$. It has basis $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n \text{ is snowy}\}$*

To establish this result, we just need to count the number of snowy weak compositions in C_n. Snowy weak compositions naturally correspond to certain diagrams. A *diagram* is a finite subset of $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$. We represent a diagram D by drawing a box in row i column j for each $(i, j) \in D$. Let Rook_+ be the set of diagrams such that each row or column has at most one cell. These are known as *non-attacking rook diagrams*. Let Rook_n consists $R \in \mathsf{Rook}_+$ such that if R has a cell in row i column j, then $i + j \leqslant n$. It is an exercise to show $|\mathsf{Rook}_n| = B_n$. In Lemma 4.20, we build bijections between Rook_n and snowy weak compositions in C_n.

Define $C_+ := \bigcup_{n \geqslant 1} C_n$, the set of all weak compositions. Correspondingly, $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_+ \text{ is snowy}\}$ is another basis of $\widehat{V}$. Based on the two basis of $\widehat{V_n}$ and $\widehat{V}$, we have two expressions for $\operatorname{Hilb}(\widehat{V_n}; q)$

and $\mathrm{Hilb}(\widehat{V};q)$:

$$\mathrm{Hilb}(\widehat{V_n};q) = \sum_{\substack{w\in S_n,\\ w \text{ is inverse}\\ \text{fireworks}}} q^{\mathsf{raj}(w)} = \sum_{\substack{\alpha\in C_n,\\ \alpha \text{ is snowy}}} q^{\mathsf{raj}(\alpha)},$$

$$\mathrm{Hilb}(\widehat{V};q) = \sum_{\substack{w\in S_+,\\ w \text{ is inverse}\\ \text{fireworks}}} q^{\mathsf{raj}(w)} = \sum_{\substack{\alpha\in C_+,\\ \alpha \text{ is snowy}}} q^{\mathsf{raj}(\alpha)}.$$

We can connect the rightmost sums in equations above to rook theoretic results. Garsia and Remmel [GR86] define a statistic on Rook_+ which we denote as $\mathsf{GR}(\cdot)$. They used this statistic to compute $S_{n,k}(q)$, a q-analogue of the Stirling numbers of the second kind. Define $B_n(q) := \sum_{k=0}^{n} S_{n,k}(q)$, a q-analogue of Bell numbers. The work of Garsia and Remmel implies that

$$B_n(q) = \sum_{R\in\mathsf{Rook}_n} q^{\mathsf{GR}(R)},$$

which is a polynomial of q with degree $\binom{n}{2}$. We connect their work to the space $\widehat{V_n}$ and establish the following:

Theorem 1.4. *We have*

$$\mathrm{Hilb}(\widehat{V_n};q) = q^{\binom{n}{2}} B_n(q^{-1}) = \mathsf{rev}(B_n(q)),$$

where $\mathsf{rev}(\cdot)$ is the operator that reverse the coefficients of a polynomial. In other words, it sends a polynomial $f(q)$ of degree d to $q^d f(q^{-1})$.

For $\widehat{V}$, we show the coefficients of $\mathrm{Hilb}(\widehat{V};q)$ form the sequence A126348 in OEIS. Using the combinatorics we developed, we give an explicit expression of $\mathrm{Hilb}(\widehat{V};q)$.

Theorem 1.5. *We have*

$$\mathrm{Hilb}(\widehat{V};q) = \sum_{\alpha \text{ is snowy}} q^{\mathsf{raj}(\alpha)} = \prod_{m>0}\left(1+\frac{q^m}{1-q}\right)$$

Our definition of $\mathsf{rajcode}$ and raj on weak compositions comes from a more general construction. Given a diagram D, we define a procedure to construct its *snow diagram*, denoted as $\mathsf{snow}(D)$ (see §3). It is a diagram where some cells are labeled by $\bullet$ and $\circledast$. We define $\mathsf{rajcode}(D)$ to be the weak composition whose i^{th} entry is the number of cells in row i of $\mathsf{snow}(D)$. Then $\mathsf{raj}(D)$ is defined as the total number of cells in $\mathsf{snow}(D)$. Each weak composition α is naturally associated with a diagram $D(\alpha)$, which is a left justified diagram with α_i cells in row i. We define $\mathsf{rajcode}(\alpha)$ as $\mathsf{rajcode}(D(\alpha))$ and define $\mathsf{raj}(\alpha)$ as $\mathsf{raj}(D(\alpha))$.

Finally, recall that each permutation w is also associated with a diagram called the Rothe diagram, denoted as $RD(w)$. By studying $\mathsf{rajcode}(RD(w))$ and $\mathsf{raj}(RD(w))$, we show they are the same as $\mathsf{rajcode}(w)$ and $\mathsf{raj}(w)$ defined in [PSW21]. In other words, our work unifies the rule to compute the degree and the leading monomial of $\widehat{\mathfrak{G}}_w$ and $\widehat{\mathfrak{L}}_\alpha$.

The rest of the paper is organized as follows. In §2, we provide necessary background information and notation. In §3, we construct a snow diagram from any diagram and define two statistics $\mathsf{rajcode}(\cdot), \mathsf{raj}(\cdot)$ on it. In §4, we study combinatorial properties of the snow diagram of a key diagram, which we use to prove a result on $\widehat{\mathfrak{L}}_\alpha$ (Theorem 1.2). In §5, we derive the Hilbert series of $\widehat{V_n}$ and $\widehat{V}$. In §6, we show the statistics $\mathsf{rajcode}(\cdot)$ and $\mathsf{raj}(\cdot)$ on the snow diagram of a Rothe diagram is equivalent to that defined in [PSW21], and relate it to the Schensted insertion and the shadow diagram. In §7, we present several open problems and future directions.

2. Background

2.1. **Polynomials.** In this subsection, we provide necessary backgrounds for Grothendieck polynomials and Lascoux polynomials. Then we introduce $\widehat{\mathfrak{G}}_w$ and $\widehat{\mathfrak{L}}_\alpha$ which span the spaces $\widehat{V_n}$ and $\widehat{V}$.

The *Grothendieck polynomials* $\mathfrak{G}_w \in \mathbb{Z}_{\geqslant 0}[\beta][x_1, x_2, \dots]$ were recursively defined by Lascoux and Schützenberger [LS82a]. Let $\partial_i(\cdot)$ be the divided difference operators acting on the polynomial ring. For each i, $\partial_i(f) := \dfrac{f - s_i f}{x_i - x_{i+1}}$, where s_i is the operator that swaps x_i and x_{i+1}. Then for $w \in S_n$,

$$\mathfrak{G}_w := \begin{cases} x_1^{n-1}x_2^{n-2}\cdots x_{n-1} & \text{if } w \text{ is } [n, n-1, \dots, 1] \text{ in one-line notation,} \\ \partial_i((1+\beta x_{i+1})\mathfrak{G}_{ws_i}) & \text{if } w(i) < w(i+1). \end{cases}$$

Let S_+ be the set of permutations of $\{1, 2, \dots\}$ such that only finitely many numbers are permuted. Take $w \in S_+$ and assume w only permutes numbers in $[n]$. Let $w' \in S_n$ be the restriction of w to $[n]$ and define $\mathfrak{G}_w$ as $\mathfrak{G}_{w'}$. It is shown in [LS82a] that $\mathfrak{G}_w$ is well-defined.

A *weak composition* is an infinite sequence of non-negative integers with finitely many positive entries. Let C_+ be the set of all weak compositions. For $\alpha \in C_+$, we use α_i to denote its i^{th} entry, and write $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where α_n is the last positive entry. We use x^α to denote the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots$ and $|\alpha| = \sum_{i \geqslant 1}\alpha_i$. The *Lascoux polynomials* $\mathfrak{L}_\alpha$, indexed by weak compositions, are in $\mathbb{Z}_{\geqslant 0}[\beta][x_1, x_2, \dots]$. By [Las03], they are defined recursively

$$\mathfrak{L}_\alpha = \begin{cases} x^\alpha & \text{if } \alpha_1 \geqslant \alpha_2 \geqslant \cdots \\ \pi_i((1+\beta x_{i+1})\mathfrak{L}_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}, \end{cases}$$

where π_i is the operator $\pi_i(f) = \partial_i(x_i f)$.

We say a pair (i, j) is an *inversion* of $w \in S_n$ if $i < j$ and $w(i) > w(j)$. Let $\mathsf{Inv}(w)$ be the set of all inversions in w and let $\mathsf{inv}(w) = |\mathsf{Inv}(w)|$. Then we may view $\mathfrak{G}_w$ as a polynomial of β. The coefficient of β^d is a homogeneous polynomial in the x-variables with degree $\mathsf{inv}(w) + d$. The coefficient of β^0 in $\mathfrak{G}_w$ is the *Schubert polynomial* $\mathfrak{S}_w$. We may also view $\mathfrak{L}_w$ as a polynomial of β. The coefficient of β^d is a homogeneous polynomial in the x-variables with degree $|\alpha| + d$. The coefficient of β^0 in $\mathfrak{L}_w$ is the *key polynomial* κ_α. The Schubert polynomials and the key polynomials are well-studied [BJS93, Dem74, LS].

In this paper, we are interested in the top-degree components of $\mathfrak{G}_w$ and $\mathfrak{L}_\alpha$. For a polynomial $f \in \mathbb{Q}[\beta][x_1, x_2, \dots]$, let $\widehat{f}$ denote the coefficient of the highest power of β. The *Castelnuovo–Mumford polynomial* of $w \in S_n$ or S_+ is defined as $\widehat{\mathfrak{G}}_w$. The *top Lascoux polynomial* of $\alpha \in C_+$ is defined as $\widehat{\mathfrak{L}}_\alpha$.

Pechenik, Speyer and Weigandt [PSW21] first study $\widehat{\mathfrak{G}}_w$. To the best of the authors knowledge, $\widehat{\mathfrak{L}}_\alpha$ has not been defined and studied previously.

Now consider the *tail lexicographic order* on monomials in the x-variables. We say a monomial x^α is larger than x^γ if there exists k such that $\alpha_k > \gamma_k$ and $\alpha_j = \gamma_j$ for all $j > k$. The *leading monomial* of $f \in \mathbb{Q}[x_1, x_2, \dots]$ is the largest monomial that appears in f. Among the four homogeneous polynomials in $\mathbb{Q}[x_1, x_2, \dots]$ above, three of them already have nice descriptions for the leading monomial.

- [BJS93] The leading monomial of $\mathfrak{S}_w$ with $w \in S_n$ is $x^{\mathsf{invcode}(w)}$, where
$$\mathsf{invcode}(w)_i = |\{j : (i, j) \in \mathsf{Inv}(w)\}|.$$
- [LS89] The leading monomial of κ_α is x^α.
- [PSW21] The leading monomial of $\widehat{\mathfrak{G}}_w$ is $x^{\mathsf{rajcode}}(w)$ defined as follows.

Definition 2.1. [PSW21] *Let* $\mathrm{LIS}^w(q)$ *be the length of the longest increasing subsequence of* w *that starts with* q. *The* $\mathsf{rajcode}(w)$ *for* $w \in S_n$ *is a weak composition where* $\mathsf{rajcode}(w)_r := n + 1 - r - \mathrm{LIS}^w(w(r))$ *for* $r \in [n]$ *and* 0 *elsewhere. Then* $\mathsf{raj}(w) := |\mathsf{rajcode}(w)|$.

Example 2.2. *Consider* $w = 3721564 \in S_7$. *We have* $\mathrm{LIS}^w(2) = 3$, *so* $\mathsf{rajcode}(w)_3 = 7+1-3-3 = 2$. *All together, we get* $\mathsf{rajcode}(w) = (4, 5, 2, 1, 1, 1)$ *and* $\mathsf{raj}(w) = 14$.

We will define $\mathsf{rajcode}(\cdot)$ on C_+ and show the leading monomial of $\mathfrak{L}_\alpha$ is $x^{\mathsf{rajcode}}(w)$ in §4.

A connection between $\mathfrak{G}_w$ and $\mathfrak{L}_\alpha$ is established by Shimozono and Yu [SY21]. To describe this connection, we need the following notion.

Definition 2.3. *Let* $f, f_1, f_2, \dots$ *be polynomials in* $\mathbb{Z}_{\geqslant 0}[x_1, x_2, \dots]$. *We say* f *expands positively into* $\{f_1, f_2, \dots\}$ *if there exist* $c_1, c_2, \dots \in \mathbb{Z}_{\geqslant 0}$ *such that* $f = c_1 f_1 + c_2 f_2 + \cdots$.

Now assume $f, f_1, f_2, \dots$ *are polynomials in* $\mathbb{Z}_{\geqslant 0}[\beta][x_1, x_2, \dots]$. *We say* f *expands positively into* $\{f_1, f_2, \dots\}$ *if there exist* $g_1, g_2, \dots \in \mathbb{Z}_{\geqslant 0}[\beta]$ *such that* $f = g_1 f_1 + g_2 f_2 + \cdots$.

Theorem 2.4 ([SY21]). *For* $w \in S_+$, $\mathfrak{G}_w$ *expands positively into* $\{\mathfrak{L}_\alpha : \alpha \in C_+\}$.

Notice that [SY21] gives two explicit combinatorial rules to compute the coefficients. We omit these descriptions since the exact values of these coefficients are not relevant to this paper. We would like to show $\widehat{\mathfrak{G}}_w$ also expands positively into $\widehat{\mathfrak{L}}_\alpha$. We need the following Lemma.

Lemma 2.5. *Take* $f, f_1, f_2, \dots$ *in* $\mathbb{Z}_{\geqslant 0}[\beta][x_1, x_2, \dots]$. *Assume* f *expands positively into* $\{f_1, f_2, \dots\}$. *Then* $\widehat{f}$ *expands positively into* $\widehat{f_1}, \widehat{f_2}, \dots$.

Proof. First, find $g_1, g_2, \dots \in \mathbb{Z}_{\geqslant 0}[\beta]$ such that $f = g_1 f_1 + g_2 f_2 + \cdots$. Let m be the degree of β in f. We extract the coefficient of β^m on both sides. The left hand side gives $\widehat{f}$. For each $g_i f_i$ on the right-hand side, its degree of β is at most m. Thus, the coefficient of β^m is 0 or $c_i \widehat{f_i}$, where c_i is the coefficient of the highest power of β in g_i. Our claim follows. □

Lemma 2.5 together with Theorem 2.4 implies the following.

Corollary 2.6. *For* $w \in S_+$, $\widehat{\mathfrak{G}}_w$ *expands positively into* $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_+\}$.

We end this section by deducing $\widehat{V}$ is a filtered algebra. By the work of Lascoux, Schützenberger [LS82b] and Brion[Bri02], the product $\mathfrak{G}_u \mathfrak{G}_v$ with $u \in S_m$ and $v \in S_n$ expands positively into $\mathfrak{G}_w$ with $w \in S_{m+n}$. By Lemma 2.5, $\widehat{\mathfrak{G}}_u \widehat{\mathfrak{G}}_v$ with $u \in S_m$ and $v \in S_n$ expands positively into $\widehat{\mathfrak{G}}_w$ with $w \in S_{m+n}$. Finally, we conclude the following.

Proposition 2.7. *The space* $\widehat{V}$ *is a filtered algebra with respect to the filtration* $\widehat{V_1} \subset \widehat{V_2} \subset \cdots \subset \widehat{V}$.

2.2. **Diagrams.** A *diagram* is a finite subset of $\mathbb{N} \times \mathbb{N}$. We may represent a diagram by putting a cell at row r and column c for each (r, c) in the diagram. We adopt the convention where columns begin at 1 from the left and rows begin at 1 from the top. The *weight* of a diagram D, denoted as $\mathsf{wt}(D)$, is a weak composition whose i^{th} entry is the number of boxes in its i^{th} row. Now we recall two classical families of diagrams.

Each weak composition α is associated with a diagram called the *key diagram*, denoted as $D(\alpha)$. It is the unique left-justified diagram with weight α. One interesting key diagram is $D(\rho^{(n)})$ where $\rho^{(n)} := (n-1, n-2, \dots, 1)$. Since it looks like a staircase, we also denote $D(\rho^{(n)})$ by Stair_n.

Example 2.8. *The following are two examples of key diagrams. For clarity, we put an "i" on the left of the* i^{th} *row and put a small dot in each cell.*

$$D(0,2,1) = \begin{array}{c|c|c|} 1 & \multicolumn{2}{c}{} \\ \cline{2-3} 2 & \cdot & \cdot \\ \cline{2-3} 3 & \cdot & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} \quad , \qquad \mathsf{Stair}_4 = \begin{array}{c|c|c|c|} \cline{2-4} 1 & \cdot & \cdot & \cdot \\ \cline{2-4} 2 & \cdot & \cdot & \multicolumn{1}{|c}{} \\ \cline{2-3} 3 & \cdot & \multicolumn{2}{|c}{} \\ \cline{2-2} \end{array}$$

Each permutation $w \in S_n$ or S_+ is associated with a diagram called the *Rothe diagram*, denoted as $RD(w)$. It is the diagram: $\{(r, w(r')) : (r, r') \in \mathsf{Inv}(w)\}$.

Example 2.9. *Let* $w = 41532 \in S_5$*. Then* $\mathsf{Inv}(w) = \{(1,2),(1,4),(1,5),(3,4),(3,5),(4,5)\}$*. The Rothe diagram is depicted as follows.*

$$RD(w) = \begin{array}{cccc} 1 & \cdot & \cdot & \cdot \\ 2 & & & \\ 3 & & \cdot & \cdot \\ 4 & & \cdot & \\ 5 & & & \end{array}$$

2.3. K**-Kohnert diagrams.** In this subsection, we introduce a combinatorial formula for Lascoux polynomials. To simplify our description, we introduce the following definition.

Definition 2.10. *A* labeled diagram *is a diagram where each cell can be labeled by a symbol. The* underlying diagram *of a labeled diagram is the diagram obtained by ignoring all labels. The weight of a labeled diagram* D*, denoted as* $\mathsf{wt}(D)$*, is just the weight of its underlying diagram.*

Then a *ghost diagram* is a labeled diagram where cells can be labeled by X. We call cells labeled by X as "*ghosts*". For a ghost diagram D, its *excess*, denoted as $\mathsf{ex}(D)$, is the number of ghosts in D. Next, we define a move on ghost diagrams.

Definition 2.11 ([RY15])**.** *A* K-Kohnert move *is defined on a ghost diagram.*
It selects a cell and moves it up, subject to the following requirements.

- *The selected cell mush be the rightmost cell in its row.*
- *The selected cell is not a ghost.*
- *The cell is moved to the lowest empty spot above it.*
- *The cell may jump over other cells, but cannot jump over any ghosts.*

After the move, it may or may not leave a ghost at the original position. When a K*-Kohnert move leaves a ghost, we also refer to it as a* ghost move*.*

For a weak composition α, a ghost diagram is called a *K-Kohnert diagram* of α if it can be obtained from $D(\alpha)$ by K-Kohnert moves. Let $\mathsf{KKD}(\alpha)$ be the set of all K-Kohnert diagrams of α. As proved in [PY22], K-Kohnert diagrams give a formula for Lascoux polynomials. This rule was first conjectured by Ross and Yong [RY15]. Notice that our convention is different from [PY22] in the sense that row 1 is the top most row in this paper while it is the bottom most row in [PY22].

Theorem 2.12 ([PY22])**.** *Let* α *be a weak composition. Then we have*

$$\mathfrak{L}_\alpha = \sum_{D \in \mathsf{KKD}(\alpha)} x^{\mathsf{wt}(D)} \beta^{\mathsf{ex}(D)}.$$

Example 2.13. *Let* $\alpha = (0,2,1)$*, then* $KKD(\alpha)$ *consists of the following:*

$$\begin{array}{ccc} 1 & & \\ 2 & \cdot & \cdot \\ 3 & \cdot & \end{array}, \quad \begin{array}{ccc} 1 & & \cdot \\ 2 & \cdot & \\ 3 & \cdot & \end{array}, \quad \begin{array}{ccc} 1 & \cdot & \cdot \\ 2 & & \\ 3 & \cdot & \end{array}, \quad \begin{array}{ccc} 1 & \cdot & \\ 2 & \cdot & \cdot \\ 3 & & \end{array}, \quad \begin{array}{ccc} 1 & \cdot & \cdot \\ 2 & \cdot & \\ 3 & & \end{array},$$

$$\begin{array}{ccc} 1 & \cdot & \\ 2 & \cdot & \cdot \\ 3 & \mathsf{X} & \end{array}, \quad \begin{array}{ccc} 1 & & \cdot \\ 2 & \cdot & \mathsf{X} \\ 3 & \cdot & \end{array}, \quad \begin{array}{ccc} 1 & \cdot & \cdot \\ 2 & \mathsf{X} & \\ 3 & \cdot & \end{array}, \quad \begin{array}{ccc} 1 & \cdot & \cdot \\ 2 & \cdot & \\ 3 & \mathsf{X} & \end{array}, \quad \begin{array}{ccc} 1 & \cdot & \cdot \\ 2 & \cdot & \mathsf{X} \\ 3 & & \end{array},$$

$$\begin{array}{c|c|c|}
\cline{2-3}
1 & \cdot & \cdot \\
\cline{2-3}
2 & \cdot & \mathsf{X} \\
\cline{2-3}
3 & \mathsf{X} & \multicolumn{1}{c}{} \\
\cline{2-2}
\end{array}.$$

By the rule above, we have

$$\begin{aligned}
\mathfrak{L}_\alpha &= x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2 \\
&\quad + \beta(x_1x_2^2x_3 + x_1x_2^2x_3 + x_1^2x_2x_3 + x_1^2x_2x_3 + x_1^2x_2^2) \\
&\quad + \beta^2x_1^2x_2^2x_3.
\end{aligned}$$

2.4. **The Schensted Insertion.** If a diagram is top-justified and left-justified, we say it is a *Young diagram*. A filling of a Young diagram with positive integers is called a *tableau*. A tableau is called *partial* if it contains distinct numbers and each row (resp. column) is decreasing from left to right (resp. top to bottom). Notice that usually in literature, columns and rows are increasing. We reverse the convention to make our results easier to state.

The Schensted insertion [Sch61] is an algorithm defined on a partial tableau T and a positive number x that is not in T. It finds the largest x' in the first row of T such that $x > x'$.

- If such x' does not exist, it appends x at the end of row one and terminates.
- Otherwise, it replaces x' by x and insert x' to the next row in the same way.

When the algorithm terminates, the resulting partial tableau is the output.

For $w \in S_n$, we insert $w(n), w(n-1), \ldots, w(1)$ to the empty tableau via the Schensted insertion and denote the result by $P(w)$.

Example 2.14. *We perform the Schensted insertion on $w \in S_7$ with one-line notation* 3721564*.*

$$\begin{array}{|c|}\hline 4\\\hline\end{array} \to \begin{array}{|c|}\hline 6\\\hline 4\\\hline\end{array} \to \begin{array}{|c|c|}\hline 6&5\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 6&5&1\\\hline 4&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 6&5&2\\\hline 4&1&\multicolumn{1}{c}{}\\\cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 7&5&2\\\hline 6&1&\multicolumn{1}{c}{}\\\cline{1-2} 4&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 7&5&3\\\hline 6&2&\multicolumn{1}{c}{}\\\cline{1-2} 4&1&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}$$

One classical application of the Schensted insertion is to study increasing subsequences in a permutation. Recall $\mathrm{LIS}^w(q)$ is the length of the longest increasing subsequence of w that starts with q. It is related to the Schensted insertion as follows.

Lemma 2.15. [Sag01, Lemma 3.3.3] *Take $w \in S_n$ and perform the Schensted insertion on w. For any $r \in [n]$, when $w(r)$ is inserted, it goes to column $\mathrm{LIS}^w(w(r))$ in row one.*

Example 2.16. *Consider the $w \in S_7$ in Example 2.14. Notice that $\mathrm{LIS}^w(w(4)) = 3$. When $w(4) = 1$ is inserted to row one, it indeed goes to column 3.*

2.5. **Stirling numbers, Bell numbers and their q-analogues.** Let n, k be non-negative integers throughout this subsection. Let $S_{n,k}$ be the *Stirling number of the second kind*, defined by the recurrence relation

$$S_{n+1,k} = S_{n,k-1} + kS_{n,k},$$

together with $S_{0,0} = 1$ and $S_{0,k} = 0$ if $k > 0$. Let $B_n := \sum_{j=0}^n S_{n,j}$ be the Bell number which satisfies the following recurrence relation

$$B_{n+1} = \sum_{j=0}^n \binom{n}{j} B_j.$$

There are famous combinatorial models for these numbers. For instance, $S_{n,k}$ counts set partitions of $[n]$ into k parts while B_n counts all set partitions of $[n]$. For the purpose of this paper, we use another combinatorial model.

Definition 2.17. *A diagram is a non-attacking rook diagram if it has at most one cell in each row or column. Let* Rook_n *be the set of all non-attacking rook diagrams contained in* Stair_n*. Define* $\mathsf{Rook}_+ := \bigcup_{n \geqslant 1} \mathsf{Rook}_n$*, the set of all non-attacking rook diagrams. Let* $\mathsf{Rook}_{n,k}$ *be the set of diagrams in* Rook_n *with size* k*.*

It is an exercise to show $S_{n,k} = |\mathsf{Rook}_{n,n-k}|$ and thus $B_n = |\mathsf{Rook}_n|$. In [BCHR], Butler, Can, Haglund, and Remmel built an explicit bijection between $\mathsf{Rook}_{n,n-k}$ and set partitions of $[n]$ with size k.

Now consider the polynomial ring $\mathbb{Q}[q]$. Define $[n]_q := 1 + q + \cdots + q^{n-1}$. Define a *$q$-analogue of $S_{n,k}$* recursively by:

$$S_{n+1,k}(q) = q^{k-1} S_{n,k-1}(q) + [k]_q S_{n,k}(q),$$

with base cases $S_{0,k}(q) = S_{0,k}$. Similarly, define a *q-analogue of B_n* by $B_n(q) := \sum_{j=0}^{n} S_{n,j}(q)$. The coefficients in $B_n(q)$ are given in OEIS A126347. By [Wag04], $B_n(q)$ satisfies the recurrence relation

$$B_{n+1}(q) = \sum_{j=0}^{n} q^j \binom{n}{j} B_j(q).$$

Milne [Mil82] first gave a combinatorial model for $S_{n,k}(q)$ using set partitions. We are going to use the combinatorial model developed by Garsia and Remmel [GR86]. They defined a statistic on Rook_n called "inversion". We rename it as GR_n to distinguish it from the inversion on permutations.

Definition 2.18 ([GR86])**.** *Assume* $R \in \mathsf{Rook}_n$*. For each* $(r, c) \in R$*, mark all cells* (r', c) *with* $r' \in [r]$ *in* Stair_n*. Also, mark all cells* (r, c') *with* $c' \in [c]$ *in* Stair_n*. The number* $\mathsf{GR}_n(R)$ *counts cells in* Stair_n *that are not marked.*

Garsia and Remmel prove that

$$S_{n,k}(q) = \sum_{D \in \mathsf{Rook}_{n,n-k}} q^{\mathsf{GR}_n(D)}, \tag{2}$$

which implies

$$B_n(q) = \sum_{D \in \mathsf{Rook}_n} q^{\mathsf{GR}_n(D)}. \tag{3}$$

From this formula, we know $B_n(q)$ has degree $\binom{n}{2}$ since $\mathsf{GR}_n(\varnothing) = \binom{n}{2}$.

3. Snow diagrams

In this section, we associate each diagram with a labeled diagram called the *snow diagram* which allows us to define two statistics on diagrams.

For each diagram D, we describe the following algorithm that outputs $\mathsf{snow}(D)$. Cells in $\mathsf{snow}(D)$ can be labeled by $\bullet$ or $\circledast$. We call cells labeled by $\bullet$ as "dark clouds" and cells labeled by $\circledast$ as "snowflakes".

- Iterate through rows of D from bottom to top.
- In each row of D, find the rightmost cell with no $\bullet$ in its column. If such a cell exists, label it by $\bullet$ and put a cell labeled by $\circledast$ in all empty positions above it.

Example 3.1. *The following is a diagram together with its snow diagram.*

$$D = \begin{array}{c|c|c|c|} 1 & & & \cdot \\ 2 & \cdot & \cdot & \\ 3 & & & \cdot \\ 4 & & & \\ 5 & \cdot & \cdot & \end{array}, \quad \mathsf{snow}(D) = \begin{array}{c|c|c|c|} 1 & \ast & \ast & \cdot \\ 2 & \bullet & \cdot & \ast \\ 3 & & \ast & \bullet \\ 4 & & \ast & \\ 5 & \cdot & \bullet & \end{array}.$$

The positions of dark clouds will be important, so we make the following definition.

Definition 3.2. *The dark cloud diagram of a diagram D, $\mathsf{dark}(D)$, is the set of cells (r, c) that are dark clouds in $\mathsf{snow}(D)$.*

Example 3.3. *In Example 3.1, $\mathsf{dark}(D) = \{(2, 1), (3, 3), (5, 2)\}$.*

Remark 3.4. *We make the following observations about $\mathsf{dark}(D)$.*

- *The diagram $\mathsf{dark}(D)$ has at most one cell in each row and in each column. In other words, $\mathsf{dark}(D) \in \mathsf{Rook}_+$.*
- *Take $(r, c) \in D$. If there are no $r' > r$ with $(r', c) \in \mathsf{dark}(D)$ and there are no $c' > c$ with $(r, c') \in \mathsf{dark}(D)$, then $(r, c) \in \mathsf{dark}(D)$.*

Finally, we associate a weak composition and a number to each diagram via its snow diagram.

Definition 3.5. *Let D be a diagram. The rajcode of D, $\mathsf{rajcode}(D)$, is the weak composition $\mathsf{wt}(\mathsf{snow}(D))$. The number $\mathsf{raj}(D) := |\mathsf{rajcode}(D)|$ is the total number of cells in $\mathsf{snow}(D)$.*

Example 3.6. *Continuing with Example 3.1, we have $\mathsf{rajcode}(D) = (3, 3, 2, 1, 2)$ and $\mathsf{raj}(D) = 11$.*

Remark 3.7. *Recall that Pechenik, Speyer and Weigandt [PSW21] define the statistics $\mathsf{rajcode}(\cdot)$ and $\mathsf{raj}(\cdot)$ on permutations using increasing subsequences. We show that our $\mathsf{rajcode}$ and raj on Rothe diagrams agree with their definitions in Theorem 6.3. Therefore, our construction on Rothe diagrams is a diagrammatic way to compute the leading monomial and degree of $\widehat{\mathfrak{G}}_w$. In addition, we notice that positions of dark clouds in $\mathsf{snow}(RD(w))$ are connected to the Schensted insertion and Viennot's geometric construction. These connections are explored in §6.*

4. Snow diagrams of key diagrams and top Lascoux polynomials

In this section, we study top Lascoux polynomials via snow diagrams of key diagrams. With a slight abuse of notation, we define $\mathsf{rajcode}(\alpha) := \mathsf{rajcode}(D(\alpha))$ and $\mathsf{raj}(\alpha) := \mathsf{raj}(D(\alpha))$ for a weak composition α. We start by introducing some definitions.

Definition 4.1. *A weak composition α is called snowy if its positive entries are all distinct.*

Our main goal in this section is to establish Theorem 1.2:

Theorem 1.2. *Let α, γ be two weak compositions.*

(a) The polynomial $\widehat{\mathfrak{L}}_\alpha$ has leading monomial $x^{\mathsf{rajcode}(w)}$. If α is snowy, then $x^{\mathsf{rajcode}(w)}$ has coefficient 1 in $\widehat{\mathfrak{L}}_\alpha$.
(b) $\widehat{\mathfrak{L}}_\alpha$ is a scalar multiple of $\widehat{\mathfrak{L}}_\gamma$ if and only if $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$.
(c) Any $\alpha \in C_n$ has the same $\mathsf{rajcode}$ as exactly one $\alpha' \in C_n$ that is snowy.

This task is broken into four major lemmas established in the following four subsections. In Subsection 4.1, we use K-Kohnert diagram to establish the first major lemma:

Lemma 4.2. *The polynomial $\mathfrak{L}_\alpha$ has the term $x^{\mathsf{rajcode}(\alpha)}\beta^{\mathsf{raj}(\alpha)-|\alpha|}$.*

This is a good starting point. We know $\widehat{\mathfrak{L}}_\alpha$ has degree at least $\mathsf{raj}(\alpha)$. To show $\widehat{\mathfrak{L}}_\alpha$ indeed has degree $\mathsf{raj}(\alpha)$, we need the following equivalence relation on weak compositions.

Definition 4.3. *Let α and γ be two weak compositions. We say α is rajcode equivalent to γ, denoted as $\alpha \sim \gamma$, if $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$.*

Example 4.4. *Let $\alpha = (2, 0, 4, 3, 1)$ and $\gamma = (3, 1, 4, 3, 1)$. Then we have:*

$$D(\alpha) = \begin{array}{c|c|c|c|c|}
1 & \cdot & \cdot & \multicolumn{2}{c}{} \\
2 & \multicolumn{4}{c}{} \\
3 & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & \cdot & \cdot & \multicolumn{1}{c}{} \\
5 & \cdot & \multicolumn{3}{c}{}
\end{array}, \qquad \mathsf{snow}(D(\alpha)) = \begin{array}{c|c|c|c|c|}
1 & \cdot & \bullet & \circledast & \circledast \\
2 & \circledast & & \circledast & \circledast \\
3 & \cdot & \cdot & \cdot & \bullet \\
4 & \cdot & \cdot & \bullet & \multicolumn{1}{c}{} \\
5 & \bullet & \multicolumn{3}{c}{}
\end{array},$$

$$D(\gamma) = \begin{array}{c|c|c|c|c|}
1 & \cdot & \cdot & \cdot & \multicolumn{1}{c}{} \\
2 & \cdot & \multicolumn{3}{c}{} \\
3 & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & \cdot & \cdot & \multicolumn{1}{c}{} \\
5 & \cdot & \multicolumn{3}{c}{}
\end{array}, \qquad \mathsf{snow}(D(\gamma)) = \begin{array}{c|c|c|c|c|}
1 & \cdot & \bullet & \cdot & \circledast \\
2 & \cdot & & \circledast & \circledast \\
3 & \cdot & \cdot & \cdot & \bullet \\
4 & \cdot & \cdot & \bullet & \multicolumn{1}{c}{} \\
5 & \bullet & \multicolumn{3}{c}{}
\end{array}.$$

Be aware that the cell $(2, 2)$ is not in $\mathsf{snow}(D(\alpha))$ or $\mathsf{snow}(D(\gamma))$. Observe that $\mathsf{rajcode}(\alpha)$ and $\mathsf{rajcode}(\gamma)$ are both $(4, 3, 4, 3, 1)$, so $\alpha \sim \gamma$.

In Subsection 4.2, we study this equivalence relation. Then we show that snowy weak compositions form a complete set of representatives, which is the second major lemma.

Lemma 4.5. *For each equivalence class of $\sim$, there is a unique α such that α is snowy. Moreover, if $\gamma \sim \alpha$ and α is snowy, then $\gamma_r \geqslant \alpha_r$ for all r. In other words, a snowy weak composition is the unique entry-wise minimum in each equivalence class.*

In Subsection 4.3, we focus on $\widehat{\mathfrak{L}}_\alpha$ for snowy α and give a recursive description of $\widehat{\mathfrak{L}}_\alpha$, which leads to the third major lemma.

Lemma 4.6. *If α is snowy, then $x^{\mathsf{rajcode}(\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$ with coefficient 1.*

Finally, we devote the Subsection 4.4 to proving the last major lemma.

Lemma 4.7. *If $\alpha \sim \gamma$, then $\widehat{\mathfrak{L}}_\alpha = c\widehat{\mathfrak{L}}_\gamma$ for some $c \neq 0$.*

Once we have these four major lemmas, we can easily check Theorem 1.2.

Proof of Theorem 1.2. For statement (a) in Theorem 1.2, we can find $\gamma \sim \alpha$ such that γ is snowy by Lemma 4.5. Then the statement follows from Lemma 4.7 and Lemma 4.6.

For (b), the backward direction is just Lemma 4.7. For the forward direction, if $\widehat{\mathfrak{L}}_\alpha$ is a scalar multiple of $\widehat{\mathfrak{G}}_w$, then they have the same leading monomial. By the first statement, we have $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$.

Finally, (c) follows from Lemma 4.5.

□

4.1. **Existence of $x^{\mathsf{rajcode}(\alpha)}$ in $\mathfrak{L}_\alpha$.** In this subsection, we show the monomial $x^{\mathsf{rajcode}(\alpha)}\beta^{\mathsf{raj}(\alpha)-|\alpha|}$ exists in $\mathfrak{L}_\alpha$. We give an algorithm to construct a K-Kohnert diagram for α, which has the same underlying diagram as $\mathsf{snow}(D(\alpha))$. First, observe that $\mathsf{snow}(D(\alpha))$ contains no dark clouds if and only if α contains only zero entries. In this case, $\widehat{\mathfrak{L}}_\alpha = 1$ and $\mathsf{rajcode}(\alpha)$ only has zero entries. Our claim is immediate. In the rest of this subsection, we assume α is a weak composition with at least one positive entry, and thus $\mathsf{snow}(D(\alpha))$ has at least one dark cloud. To describe the algorithm, we introduce two useful moves on ghost diagrams.

Definition 4.8. *Let D be a ghost diagram. Let (r, c) be a non-ghost cell in D and let (r', c) be the highest empty space in column c. If $r' < r$, let $UP_{(r,c)}(D)$ be the diagram we get after moving (r, c) to (r', c). Let $UP^G_{(r,c)}(D)$ be the diagram we get after moving (r, c) to (r', c) and putting a ghost on (r, c) and all empty spaces between (r, c) and (r', c). If $r' > r$, then $UP^G_{(r,c)}(D) = UP_{(r,c)}(D) = D$.*

Remark 4.9. *Assume $UP_{(r,c)}$ or $UP^G_{(r,c)}$ moves a cell to (r', c). Then it can be achieved by a sequence of K-Kohnert moves if both the following conditions hold for each $r' < j \leqslant r$:*

- *If $(j, c) \notin D$, then D has no cell to the right of column c in row j.*
- *If $(j, c) \in D$, then it is not a ghost cell.*

Now we can describe the algorithm. Let $D^0 = D(\alpha)$. Recall by Remark 3.4, there is at most one dark cloud in each column of $\mathsf{snow}(D(\alpha))$. We can label all the dark clouds as $(r_1, c_1), \ldots, (r_m, c_m)$ where $c_1 < c_2 \cdots < c_m$. We iterate i from 1 to m for some $m \geqslant 1$. At iteration i, we start from D^{i-1}, and compute

$$D^i = UP^G_{(r_i,c_i)} \circ UP_{(r_i,c_i+1)} \cdots \circ UP_{(r_i,\alpha_{r_i})}(D^{i-1}).$$

Example 4.10. *Consider $\alpha = (1, 3, 4, 0, 4, 3)$, we compute its snow diagram and we have the dark clouds at $(2, 1), (3, 2), (6, 3), (5, 4)$. We compute D^4 according to the above algorithm.*

$$\mathsf{snow}(D(\alpha)) = \begin{array}{c|cccc} 1 & \cdot & \ast & \ast & \ast \\ 2 & \bullet & \cdot & \cdot & \ast \\ 3 & \cdot & \bullet & \cdot & \cdot \\ 4 & & & \ast & \ast \\ 5 & \cdot & \cdot & \cdot & \bullet \\ 6 & \cdot & \cdot & \bullet & \end{array} \quad D^0 = \begin{array}{c|cccc} 1 & \cdot & & & \\ 2 & \cdot & \cdot & \cdot & \\ 3 & \cdot & \cdot & \cdot & \cdot \\ 4 & & & & \\ 5 & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \end{array} \xrightarrow[(2,1)]{} \begin{array}{c|cccc} 1 & \cdot & \cdot & \cdot & \\ 2 & \cdot & & & \\ 3 & \cdot & \cdot & \cdot & \cdot \\ 4 & & & & \\ 5 & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \end{array}$$

$$\xrightarrow[(3,2)]{} \begin{array}{c|cccc} 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \\ 3 & \cdot & \mathsf{X} & & \\ 4 & & & & \\ 5 & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \end{array} \xrightarrow[(6,3)]{} \begin{array}{c|cccc} 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \\ 3 & \cdot & \mathsf{X} & \cdot & \\ 4 & & & \mathsf{X} & \\ 5 & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \mathsf{X} & \end{array} \xrightarrow[(5,4)]{} \begin{array}{c|cccc} 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot \\ 3 & \cdot & \mathsf{X} & \cdot & \mathsf{X} \\ 4 & & & \mathsf{X} & \mathsf{X} \\ 5 & \cdot & \cdot & \cdot & \mathsf{X} \\ 6 & \cdot & \cdot & \mathsf{X} & \end{array} = D^4.$$

We observe that in the previous example, D^4 has the same underlying diagram as $\mathsf{snow}(D(\alpha))$. This is true in general.

Lemma 4.11. *The D^m produced by the algorithm has the same underlying diagram as $\mathsf{snow}(D(\alpha))$.*

Proof. For a number c, we compare the column c of $\mathsf{snow}(D(\alpha))$ and D^m. If column c of $\mathsf{snow}(D(\alpha))$ has no dark cloud, then it is the same as the column c of $D(\alpha)$. In this case, the algorithm will not move any cells in column c. Thus, D^m and $D(\alpha)$ also agree in column c.

Now suppose $\mathsf{snow}(D(\alpha))$ has a dark cloud in column c, say at row r. In the underlying diagram of $\mathsf{snow}(D(\alpha))$, column c is obtained from the column c of $D(\alpha)$ by filling all spaces above row r. On the other hand, consider what the algorithm does on column c. It first might move cells above row r and then it fills all spaces weakly above row r. Thus, the column c in D^m is the same as the column c of $\mathsf{snow}(D(\alpha))$ if we ignore the labels. □

Next, we want to show D^m produced by the algorithm is in $\mathsf{KKD}(\alpha)$. We just need to check each $UP_{(r,j)}$ and $UP^G_{(r,c)}$ in each iteration is a sequence of K-Kohnert moves. To that end, we first make the following observation about the diagram D^i.

Lemma 4.12. *Let $c_0 = 0$. In the diagram D^i, if a cell is strictly to the right of column c_i, then there is a cell immediately on its left. In other words, the diagram D^i is left-justified if we ignore the first c_i columns.*

Proof. Prove by induction on i. The lemma holds for D^0, which is left-justified.

Assume D^{i-1} is left-justified if we ignore the first c_{i-1} columns, for some $i \geqslant 1$. Consider an arbitrary cell (r, c) in D^i with $c > c_i$. We show $(r, c-1)$ is in D^i by considering two possibilities.

- The cell (r, c) is not in D^{i-1}. Then during iteration i, a cell is moved to (r, c), which is the highest blank in column c of D^{i-1}. By our inductive hypothesis and $c-1 > c_{i-1}$, the highest blank in column $c-1$ of D^{i-1} is weakly lower than row r. Thus, $(r, c-1)$ is in D^i.
- Otherwise, (r, c) is in D^{i-1}. By our inductive hypothesis, $(r, c-1)$ is in D^{i-1}. If $r \neq r_i$, then we know any cells on row r are not moved during iteration i. Thus, $(r, c-1)$ is still in D^i. If $r = r_i$, then there are no empty spaces above (r, c) in D^{i-1}. By our inductive hypothesis, there is no empty spaces above $(r, c-1)$, so $(r, c-1)$ is still in D^i.

□

The above lemma shows that the diagram D^i is left-justified if we ignore the first c_i columns. We will use this property to show that D^m is in $\mathsf{KKD}(\alpha)$.

Proposition 4.13. *The above algorithm can be achieved by K-Kohnert moves, so $D^m \in \mathsf{KKD}(\alpha)$.*

Proof. We focus on iteration i of the algorithm. We check the operators on D^{i-1} can be achieved by K-Kohnert moves. We ignore all cells to the left of the column c_i in D^{i-1}. By the previous Lemma, this part of the diagram is left-justified. The highest empty space in column $c_i, \cdots, \alpha_{r_i}$ are going weakly up from left to right. Moreover, the condition in Remark 4.9 holds for all $(r_i, c_i), \cdots, (r_i, \alpha_{r_i})$.

Now $UP_{r_i, \alpha_{r_i}}$ can be achieved by K-Kohnert moves. After that, the condition in Remark 4.9 still holds for all $(r_i, c_i), \cdots, (r_i, \alpha_{r_i} - 1)$. Following this logic, this iteration can be achieved by K-Kohnert moves. □

We immediately have the desired result on the aforementioned monomial following Theorem 2.12.

Lemma 4.2. *The polynomial $\mathfrak{L}_\alpha$ has the term $x^{\mathsf{rajcode}(\alpha)}\beta^{\mathsf{raj}(\alpha)-|\alpha|}$.*

4.2. **Rajcode equivalent and snowy weak compositions.** In this subsection, we first investigate the rajcode equivalence by answering the following question: Are there any other equivalent descriptions of $\alpha \sim \gamma$? One answer involves $\mathsf{dark}(D(\alpha))$. With a slight abuse of notation, we define $\mathsf{dark}(\alpha)$ as $\mathsf{dark}(D(\alpha))$ for a weak composition α. Surprisingly, we can recover the underlying diagram of $\mathsf{snow}(D(\alpha))$ from $\mathsf{dark}(\alpha)$.

Lemma 4.14. *Let α be a weak composition. The underlying diagram of $\mathsf{snow}(D(\alpha))$ is:*

$$\bigcup_{(r,c) \in \mathsf{dark}(\alpha)} ([r] \times \{c\}) \cup (\{r\} \times [c]). \tag{4}$$

Proof. First, we show that the elements of the set (4) are cells in $\mathsf{snow}(D(\alpha))$. Take $(r, c) \in \mathsf{dark}(\alpha)$. We know $(r, c) \in D(\alpha)$. Since $D(\alpha)$ is left-justified, $\{r\} \times [c] \subseteq D(\alpha)$. Thus, these cells are in $\mathsf{snow}(D(\alpha))$. By the construction of $\mathsf{snow}(D(\alpha))$, the cells in $[r] \times \{c\}$ are also in $\mathsf{snow}(D(\alpha))$.

Now suppose there is a cell (r, c) in $\mathsf{snow}(D(\alpha))$ that is not in the set (4). Then there is no $r' > r$ with $(r', c) \in \mathsf{dark}(D)$, which implies (r, c) is not a snowflake in $\mathsf{snow}(D(\alpha))$. Thus, $(r, c) \in D(\alpha)$. Also, there is no $c' > c$ with $(r, c') \in \mathsf{dark}(D)$. By Remark 3.4, $(r, c) \in \mathsf{dark}(D)$. Thus, (r, c) is in the set (4), which is a contradiction. ☐

Even more surprisingly, we can recover $\mathsf{dark}(\alpha)$ from $\mathsf{rajcode}(\alpha)$.

Lemma 4.15. *Let α, γ be weak compositions. If $\mathsf{rajcode}(\alpha) = \mathsf{rajcode}(\gamma)$, then $\mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$.*

Proof. We prove the two diagrams $\mathsf{dark}(\alpha)$ and $\mathsf{dark}(\gamma)$ agree on each row r, by a reverse induction on r. The base case is immediate: For some r large enough, $\mathsf{dark}(\alpha)$ and $\mathsf{dark}(\gamma)$ agree on row r and underneath. Next we show that the value $\mathsf{rajcode}(\alpha)_r$ and cells in $\mathsf{dark}(\alpha)$ under row r determines whether $\mathsf{dark}(\alpha)$ has a cell on row r. Moreover, if such a cell exists, its column index is also determined.

Let $r \geqslant 1$. Define

$$C_r := \{c : \text{There are no dark clouds under } (r, c) \text{ in } \mathsf{snow}(D(\alpha)\}.$$

The complement of C_r is $\overline{C} := \mathbb{Z}_{>0} - C_r$. Clearly, $\overline{C_r} = \{c : (r', c) \in \mathsf{dark}(\alpha) \text{ for some } r' > r\}$. For $c \in \overline{C_r}$, (r, c) of $\mathsf{snow}(D(\alpha))$ is a snowflake or an unlabeled cell. If there is no dark cloud on row r of $\mathsf{snow}(D(\alpha)$, $\mathsf{rajcode}(\alpha)_r = |\overline{C_r}|$. Otherwise, we assume the dark cloud is at (r, c) for some $c \in C_r$. Then row r of $\mathsf{snow}(D(\alpha))$ has cells on (r, c') for $c' \in \overline{C_r}$ or $c' \leqslant c$. Suppose c is the i^{th} smallest number in C_r. We have $\mathsf{rajcode}(\alpha)_r = i + |\overline{C_r}|$.

Consequently, $\mathsf{rajcode}(\alpha)_r$ and $\mathsf{dark}(\alpha)$ under row r uniquely determines row r of $\mathsf{dark}(\alpha)$. If we assume $\mathsf{dark}(\alpha)$ and $\mathsf{dark}(\gamma)$ agree underneath row r as our inductive hypothesis, then they also agree on row r since $\mathsf{rajcode}(\alpha)_r = \mathsf{rajcode}(\gamma)_r$. The induction is finished. ☐

Now we have two equivalent ways of describing rajcode equivalence.

Proposition 4.16. *Let α and γ be two weak compositions. The following are equivalent:*

(1) $\alpha \sim \gamma$*;*
(2) $\mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$*.*
(3) The underlying diagrams of $\mathsf{snow}(D(\alpha))$ *and* $\mathsf{snow}(D(\gamma))$ *are the same;*

Proof. By Lemma 4.15, (1) implies (2). By Lemma 4.14, (2) implies (3). Clearly, (3) implies (1). ☐

Our next goal is to find representatives of rajcode equivalence. At the end of this subsection, we will see snowy weak compositions form a complete set of representatives. To understand snowy weak compositions, we start with the following observation.

Remark 4.17. *For a weak composition α, the following are equivalent:*

- *α is snowy.*
- *The rightmost cell in each row of $D(\alpha)$ are in different columns.*
- *The rightmost cell in each row of $D(\alpha)$ is a dark cloud in* $\mathsf{snow}(D(\alpha))$*.*

Before showing snowy weak compositions are representatives, we describe two of their upsides. First, snowy weak compositions are easy to work with since we can tell $\mathsf{dark}(\alpha)$, $\mathsf{rajcode}(\alpha)$ and $\mathsf{raj}(\alpha)$ in the following simple way.

Lemma 4.18. *Let α be a snowy weak composition. Then we have the following three claims*

(1) $\mathsf{dark}(\alpha) = \{(r, \alpha_r) : \alpha_r > 0\}$*,*
(2) $\mathsf{rajcode}(\alpha)_r = \alpha_r + |\{r' > r : \alpha_r < \alpha'_r\}|$*, and*

(3) $\mathsf{raj}(\alpha) = \sum_r (\alpha_r + |\{(r, r') : \alpha_r < \alpha'_r, r < r'\}|)$.

Proof. (1) follows from Remark 4.17. (2) follows from (1) and Lemma 4.14, and (3) immediately follows from (2). □

As a consequence, we have the following rule which tells us how $\mathsf{rajcode}(s_i\alpha)$ differs from $\mathsf{rajcode}(\alpha)$ when α is snowy.

Corollary 4.19. *Let α be a snowy weak composition and consider i with $\alpha_i > \alpha_{i+1}$. Then $\mathsf{rajcode}(s_i\alpha) = s_i\mathsf{rajcode}(\alpha) + e_i$, where e_i is the weak composition with 1 on its i^{th} entry and 0 elsewhere.*

The second upside of snowy weak compositions is that they are in bijection with Rook_+ defined in Subsection 2.5.

Lemma 4.20. *The following maps between $\{\alpha : \alpha \text{ is snowy}\}$ and Rook_+ are inverses of each other: For α snowy, we send it to $\mathsf{dark}(\alpha)$. For $R \in Rook_+$, we send it to α where*

$$\alpha_r = \begin{cases} 0 & \text{if row } r \text{ of } R \text{ is empty;} \\ c & \text{if } (r, c) \in R. \end{cases}$$

Proof. Follows from Remark 4.17. □

Now that we know these snowy weak compositions are very nice. We are ready to show that they are representatives of all equivalence classes.

Lemma 4.5. *For each equivalence class of $\sim$, there is a unique α such that α is snowy. Moreover, if $\gamma \sim \alpha$ and α is snowy, then $\gamma_r \geqslant \alpha_r$ for all r. In other words, a snowy weak composition is the unique entry-wise minimum in each equivalence class.*

Proof. Let γ be an arbitrary weak composition. First, we construct a snowy α such that $\alpha \sim \gamma$. We know $\mathsf{dark}(\gamma) \in \mathsf{Rook}_+$. We send it to a snowy α using the map in Lemma 4.20. Then $\mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$. By Proposition 4.16, $\alpha \sim \gamma$.

Next, take a number r. If $\alpha_r = 0$, then $\gamma_r \geqslant \alpha_r$ trivially. Otherwise, we know $(r, \alpha_r) \in \mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$. Thus, $\gamma_r \geqslant \alpha_r$.

Finally, we establish the uniqueness of this snowy α. Assume α' is a snowy weak composition such that $\alpha' \sim \gamma$. Then $\alpha'_r \geqslant \alpha_r$ and $\alpha_r \geqslant \alpha'_r$ for all r, so $\alpha = \alpha'$. □

Now we can explain why they got the name "snowy": a snowy weak composition has more snowflakes in its snow diagram than any others in its equivalence class. Say $\alpha \sim \gamma$ and α is snowy while γ is not. By Lemma 4.5, $|\alpha| < |\gamma|$. On the other hand, the number of snowflakes in $\mathsf{snow}(D(\alpha))$ (resp. $\mathsf{snow}(D(\gamma))$) is $\mathsf{raj}(\alpha) - |\alpha|$ (resp. $\mathsf{raj}(\gamma) - |\gamma|$). Since $\mathsf{raj}(\alpha) = \mathsf{raj}(\gamma)$, $\mathsf{snow}(D(\alpha))$ has more snowflakes than $\mathsf{snow}(D(\gamma))$.

4.3. **Top Lascoux polynomials of snowy weak compositions.** Now we have derived enough combinatorial insights on the equivalence relation $\sim$ and snowy weak compositions. Our next goal is to understand $\widehat{\mathfrak{L}}_\alpha$ when α is snowy. By Lemma 4.2, $\widehat{\mathfrak{L}}_\alpha$ has degree at least $\mathsf{raj}(\alpha)$. Next, we can show the degree of $\widehat{\mathfrak{L}}_\alpha$ equals to $\mathsf{raj}(\alpha)$ when α is snowy.

Lemma 4.21. *Let α be a snowy weak composition. The β-degree of $\mathfrak{L}_\alpha$ is $\mathsf{raj}(\alpha) - |\alpha|$, so the degree of $\widehat{\mathfrak{L}}_\alpha$ is $\mathsf{raj}(\alpha)$.*

Proof. We prove it by induction on α. For the base case, if α is weakly decreasing, then the β-degree of $\mathfrak{L}_\alpha$ is 0.

Now assume $\alpha_i < \alpha_{i+1}$ for some i. By Corollary 4.19, $\mathsf{raj}(s_i\alpha) = \mathsf{raj}(\alpha) - 1$. By our inductive hypothesis, assume the β-degree of $\mathfrak{L}_{s_i\alpha}$ is $\mathsf{raj}(s_i\alpha) - |\alpha| = \mathsf{raj}(\alpha) - 1 - |\alpha|$. By the recursive definition of Lascoux polynomials,

$$\mathfrak{L}_\alpha = \pi_i(\mathfrak{L}_{s_i\alpha}) + \beta\pi_i(x_{i+1}\mathfrak{L}_{s_i\alpha}).$$

The β degree in $\mathfrak{L}_\alpha$ is at most $\mathsf{raj}(\alpha) - |\alpha|$. Lemma 4.2 implies the β-degree of $\mathfrak{L}_\alpha$ is at least $\mathsf{raj}(\alpha) - |\alpha|$, so the inductive step is finished. □

Now we can describe $\widehat{\mathfrak{L}}_\alpha$ for snowy α recursively.

Lemma 4.22. *Let α be a snowy weak composition. Then*

$$\widehat{\mathfrak{L}}_\alpha = \begin{cases} x^\alpha & \text{if } \alpha_1 \geqslant \alpha_2 \geqslant \cdots \\ \pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases} \tag{5}$$

Proof. When α is weakly decreasing, our rule is immediate. Now assume $\alpha_i < \alpha_{i+1}$. By Corollary 4.19, $\mathsf{raj}(s_i\alpha) = \mathsf{raj}(\alpha) - 1$. We write $\mathfrak{L}_{s_i\alpha}$ as $g + \beta^{\mathsf{raj}(\alpha)-1-|\alpha|}\,\widehat{\mathfrak{L}}_{s_i\alpha}$ where the β-degree in g is less than $\mathsf{raj}(\alpha) - 1 - |\alpha|$. Now we write $\mathfrak{L}_\alpha$ as

$$\begin{aligned} \mathfrak{L}_\alpha &= \pi_i(\mathfrak{L}_{s_i\alpha}) + \beta\pi_i(x_{i+1}\mathfrak{L}_{s_i\alpha}) \\ &= \pi_i(\mathfrak{L}_{s_i\alpha}) + \beta\pi_i(x_{i+1}g) + \beta^{\mathsf{raj}(\alpha)-|\alpha|}\pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_i\alpha}) \end{aligned}$$

When we extract the coefficient of $\beta^{\mathsf{raj}(\alpha)-|\alpha|}$, the left-hand side is $\widehat{\mathfrak{L}}_\alpha$. On the right-hand side, the first two terms are ignored and we get $\pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_i\alpha})$. □

Combining Lemma 4.2 and Lemma 4.21, we know $x^{\mathsf{rajcode}(\alpha)}$ appears in $\widehat{\mathfrak{L}}_\alpha$ when α is snowy. Next, we show this monomial is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$. We start with the following observation about the operator $f \mapsto \pi_i(x_{i+1}f)$.

Remark 4.23. *Let γ be a monomial. We may describe the leading monomial of $\pi_i(x_{i+1}x^\gamma)$ and its coefficient as follows.*

- *If $\gamma_i > \gamma_{i+1}$, then $x_i x^{s_i\gamma}$ is the leading monomial with coefficient 1.*
- *If $\gamma_i = \gamma_{i+1}$, then $\pi_i(x_{i+1}x^\gamma) = 0$.*
- *If $\gamma_i < \gamma_{i+1}$, then $x_i x^\gamma$ is the leading monomial with coefficient -1.*

We can understand how the operator $f \mapsto \pi_i(x_{i+1}f)$ changes the leading monomial of polynomial f satisfying certain conditions.

Lemma 4.24. *Let f be a non-zero polynomial in the x-variables. Assume x^α is the leading monomial in f with coefficient $c \neq 0$. Pick an i such that $\alpha_i > \alpha_{i+1}$. Furthermore, assume for any monomial in f, its power of x_i is at most α_i. Then $x_i x^{s_i\alpha}$ is the leading monomial in $\pi_i(x_{i+1}f)$ with coefficient c.*

Proof. In this proof, we use "$\geqslant$" to denote the monomial order. Let Γ be the set of weak compositions γ such that x^γ appears in f. Let c_γ be the coefficient of x^γ in f. We may write f as $\sum_{\gamma\in\Gamma} c_\gamma x^\gamma$. Then $\pi_i(x_{i+1}f) = \sum_{\gamma\in\Gamma} c_\gamma\pi_i(x_{i+1}x^\gamma)$. By the remark above, $x_i x^{s_i\alpha}$ appears in $c_\alpha\pi_i(x_{i+1}x^\alpha)$ with coefficient $c_\alpha = c$. It is enough to show the following claim.

Claim: Take $\gamma \in \Gamma$ such that $\pi_i(x_{i+1}x^\gamma) \neq 0$ (i.e. $\gamma_i \neq \gamma_{i+1}$). Let $x^{\gamma'}$ be the leading monomial in $\pi_i(x_{i+1}x^\gamma)$. If $x^{\gamma'} \geqslant x_i x^{s_i\alpha}$, then $\gamma = \alpha$.

Proof: Assume $\alpha \neq \gamma$. Let k be the largest index such that the power of x_k differ in $x^{\gamma'}$ and $x_i x^{s_i\alpha}$. By $x^{\gamma'} \geqslant x_i x^{s_i\alpha}$, the power of x_k in $x^{\gamma'}$ is greater than the power of x_k in $x_i x^{s_i\alpha}$. We must have $k \leqslant i+1$. Otherwise, $x^\gamma > x^\alpha$, contradicting to x^α being the leading monomial in f.

Now we know γ', α and γ all agree after the $(i+1)^{th}$ entry. Then γ'_{i+1} is at least the power of x_{i+1} in $x_i x^{s_i\alpha}$, which is α_i. On the other hand, by $x^\gamma \leqslant x^\alpha$, $\gamma_{i+1} \leqslant \alpha_{i+1}$. Thus,

$$\gamma_{i+1} \leqslant \alpha_{i+1} < \alpha_i \leqslant \gamma'_{i+1}.$$

If $\gamma_i < \gamma_{i+1}$. Remark 4.23 implies $\gamma'_{i+1} = \gamma_{i+1}$, which is impossible. Thus, we must have $\gamma_i > \gamma_{i+1}$. By Remark 4.23 again, $\gamma'_{i+1} = \gamma_i$. By the assumptions in the statement of the lemma, $\gamma_i \leqslant \alpha_i$, so $\gamma'_{i+1} = \gamma_i = \alpha_i$.

Next, γ'_i is at least the power of x_i in $x_i x^{s_i\alpha}$, which is $\alpha_{i+1} + 1$. The remark above implies $\gamma'_i = \gamma_{i+1} + 1$. Thus, $\gamma_{i+1} \geqslant \alpha_{i+1}$. Recall that we also deduced $\gamma_{i+1} \leqslant \alpha_{i+1}$, so $\gamma_{i+1} = \alpha_{i+1}$.

Now we know $k < i$ and $\gamma_j = \alpha_j$ for $j = i$ or $i+1$. Thus, $\gamma_j = \alpha_j$ for all $j > k$, so $x^\gamma > x^\alpha$, which is a contradiction. □

Now we can establish our third major lemma.

Lemma 4.6. *If α is snowy, then $x^{\mathsf{rajcode}(\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$ with coefficient 1.*

Proof. We prove it by induction on α. If α is weakly decreasing, then $\mathfrak{L}_\alpha = x^\alpha = x^{\mathsf{rajcode}(\alpha)}$. Our claim is immediate.

Otherwise, assume $\alpha_r < \alpha_{r+1}$ for some i. Pick the largest such r. For our inductive hypothesis, assume $x^{\mathsf{rajcode}(s_r\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_{s_r\alpha}$ with coefficient 1.

By the maximality of r, $\alpha_{r+1} \geqslant \alpha_{r+2} \geqslant \alpha_{r+3} \geqslant \cdots$. Thus, in any K-Kohnert diagram of $s_r\alpha$, there cannot be more than α_{r+1} cells in row r. In other words, for any monomial of $\widehat{\mathfrak{L}}_{s_r\alpha}$, the power of x_r is at most α_{i+1}. The previous Lemma implies that $x_i x^{s_r \mathsf{rajcode}(s_r\alpha)}$ is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$ with coefficient 1. Finally, by Corollary 4.19, $x_i x^{s_r \mathsf{rajcode}(s_r\alpha)} = x^{\mathsf{rajcode}(\alpha)}$. □

4.4. **Proof of Lemma 4.7.** We first derive two consequences of $\alpha \sim \gamma$. We start with the following definition.

Definition 4.25. *Let D be a diagram. Then $\overline{D}$ is the diagram obtained by filling the spaces above each cell of D. In other words, $\overline{D} := \bigcup_{(r,c)\in D} [r] \times \{c\}$.*

Then $\overline{D(\alpha)}$ is completely determined by $\mathsf{dark}(\alpha)$.

Lemma 4.26. *Let α be a weak composition. Then $\overline{D(\alpha)} = \bigcup_{(r,c)\in \mathsf{dark}(\alpha)} [r] \times [c]$.*

Proof. We show each side is a subset of the other.

- Take $(r_1, c_1) \in D(\alpha)$. By Remark 3.4, there is $(r_2, c_2) \in \mathsf{dark}(\alpha)$ such that $r_2 \geqslant r_1$ and $c_2 \geqslant c_1$. Thus, $[r_1] \times \{c_1\} \subseteq [r_2] \times [c_2]$.
- Take $(r_1, c_1) \in \mathsf{dark}(\alpha)$. Thus, for any $c \in [c_1]$, $(r_1, c) \in D(\alpha)$. Then $[r_1] \times \{c\} \subseteq \overline{D(\alpha)}$, so $[r_1] \times [c_1] \subseteq \overline{D(\alpha)}$.

□

We have the following consequence of $\alpha \sim \gamma$.

Corollary 4.27. *If $\alpha \sim \gamma$, then $\overline{D(\alpha)} = \overline{D(\gamma)}$.*

Notice that the converse is not true. If $\alpha = (1,2)$ and $\gamma = (0,2)$, then $\overline{D(\alpha)} = [2] \times [2] = \overline{D(\gamma)}$. However, α and γ are not similar, since $\mathsf{dark}(\alpha) = \{(1,1),(2,2)\}$ and $\mathsf{dark}(\gamma) = \{(2,2)\}$.

Another nice consequence of $\alpha \sim \gamma$ one might expect is $s_r\alpha \sim s_r\gamma$. Unfortunately, this is not always true. It is easy to check $(0,1) \sim (1,1)$ but $s_1(0,1) = (1,0)$ and $s_1(1,1) = (1,1)$ are not similar. However, it is true when α and r satisfy the following condition.

Lemma 4.28. *Let α be a weak composition and $r \in \mathbb{N}$. Assume there exists c such that $(r,c) \notin \mathsf{snow}(D(\alpha))$ but $(r+1,c) \in \mathsf{snow}(D(\alpha))$. Then $\alpha_{r+1} > \alpha_r$. The diagram $\mathsf{dark}(s_r\alpha)$ is obtained from $\mathsf{dark}(\alpha)$ by switching row r and row $r+1$.*

Moreover, for any γ with $\gamma \sim \alpha$, we must have $\gamma_{r+1} > \gamma_r$ and $s_r\alpha \sim s_r\gamma$.

Proof. Since (r,c) is not in $\mathsf{snow}(D(\alpha))$, we can deduce two facts:

(1) There are no dark clouds under row r in column c, and

(2) $\alpha_r < c$.

By (1), the cell $(r+1, c)$ in $\mathsf{snow}(D(\alpha))$ is not a dark cloud or a snowflake. Thus, it is unlabeled and $(r+1, c) \in D(\alpha)$. By Remark 3.4, there must be a $c' > c$ such that $(r+1, c')$ is a dark cloud in $\mathsf{snow}(D(\alpha))$. This implies $\alpha_{r+1} > c$. By (2), we have $\alpha_{r+1} > \alpha_r$. Also by (2), the dark cloud in row r of $\mathsf{snow}(D(\alpha))$, if exists, is in the first $c-1$ columns. Thus, $\mathsf{dark}(s_r\alpha)$ is obtained from $\mathsf{dark}(\alpha)$ by switching row r and row $r+1$.

Now consider any $\gamma \sim \alpha$. By Proposition 4.16, $\mathsf{snow}(D(\gamma))$ and $\mathsf{snow}(D(\alpha))$ have the same underlying diagram. By the first part of this Lemma, $\mathsf{dark}(s_r\gamma)$ is obtained from $\mathsf{dark}(\gamma)$ by switching row r and row $r+1$. Since $\mathsf{dark}(\alpha) = \mathsf{dark}(\gamma)$, we have $\mathsf{dark}(s_r\alpha) = \mathsf{dark}(s_r\gamma)$, so $s_r\alpha \sim s_r\gamma$. □

These two consequences of $\alpha \sim \gamma$ allow us to prove the last main Lemma.

Lemma 4.7. *If $\alpha \sim \gamma$, then $\widehat{\mathfrak{L}}_\alpha = c\widehat{\mathfrak{L}}_\gamma$ for some $c \neq 0$.*

Proof. It is enough to assume γ is snowy and we proceed by induction on the number $\mathsf{raj}(\alpha)$. The base case is $\mathsf{raj}(\alpha) = 0$, which implies α only has 0s. Our claim is immediate.

Now assume $\mathsf{raj}(\alpha) > 0$. Consider the diagram $\overline{D(\alpha)}$. Clearly, the underlying diagram of any K-Kohnert diagram of α will be a subset of $\overline{D(\alpha)}$. In other words, any monomial in $\widehat{\mathfrak{L}}_\alpha$ must divide $x^{\mathsf{wt}(\overline{D(\alpha)})}$.

If the underlying diagram of $\mathsf{snow}(D(\alpha))$ is $\overline{D(\alpha)}$. Then $x^{\mathsf{wt}(\overline{D(\alpha)})}$ is the only monomial in $\widehat{\mathfrak{L}}_\alpha$. On the other hand, Corollary 4.27 gives $\overline{D(\alpha)} = \overline{D(\gamma)}$. By the same argument, $x^{\mathsf{wt}(D_{\max})}$ is the only monomial in $\widehat{\mathfrak{L}}_\gamma$. Our claim holds.

Otherwise, we can find $(r, c) \in \overline{D(\alpha)}$ but not in $\mathsf{snow}(D(\alpha))$. Find the (r, c) with the largest r. First, we know $(r, c) \notin D(\alpha)$, which implies $(r+1, c) \in \overline{D(\alpha)}$. By the maximality of r, $(r+1, c)$ is in $\mathsf{snow}(D(\alpha))$. We invoke Lemma 4.28 and conclude $\alpha_{r+1} > \alpha_r$, $\gamma_{r+1} > \gamma_r$ and $s_i\alpha \sim s_i\gamma$. Since γ is snowy, we know $\mathsf{raj}(s_r\gamma) = \mathsf{raj}(\gamma) - 1$, which implies $\mathsf{raj}(s_r\alpha) = \mathsf{raj}(\alpha) - 1$. By our inductive hypothesis, $\widehat{\mathfrak{L}}_{s_r\alpha} = c\widehat{\mathfrak{L}}_{s_r\gamma}$ for some $c \neq 0$.

We may write $\mathfrak{L}_{s_r\alpha}$ as $\beta^{\mathsf{raj}(s_r\alpha)-|\alpha|}\widehat{\mathfrak{L}}_{s_r\alpha} + g$, where g has β-degree less than $\mathsf{raj}(s_r\alpha) - |\alpha|$. Then

$$\begin{aligned}\mathfrak{L}_\alpha &= \pi_i(\mathfrak{L}_{s_r\alpha}) + \beta\pi_i(x_{i+1}\mathfrak{L}_{s_r\alpha})\\ &= \pi_i(\mathfrak{L}_{s_r\alpha}) + \beta\pi_i(x_{i+1}g) + \beta^{\mathsf{raj}(\alpha)-|\alpha|}\pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_r\alpha})\end{aligned}$$

The first two terms on the right-hand side have β degree less than $\mathsf{raj}(\alpha) - |\alpha|$. Thus, the β-degree in $\mathfrak{L}_\alpha$ is at most $\mathsf{raj}(\alpha) - |\alpha|$. By Lemma 4.2, the β-degree in $\mathfrak{L}_\alpha$ is $\mathsf{raj}(\alpha) - |\alpha|$. Extract the coefficient of $\beta^{\mathsf{raj}(\alpha)-|\alpha|}$ and get

$$\widehat{\mathfrak{L}}_\alpha = \pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_r\alpha}) = c\pi_i(x_{i+1}\widehat{\mathfrak{L}}_{s_r\gamma}) = c\widehat{\mathfrak{L}}_\gamma.$$

□

5. Vector space spanned by $\widehat{\mathfrak{G}}_w$

We now study the $\mathbb{Q}$-vector space spanned by $\widehat{\mathfrak{G}}_w$ for $w \in S_n$. In Subsection 5.1, we fix a positive n and study the graded vector space $\widehat{V_n} := \mathrm{span}\{\widehat{\mathfrak{G}}_w : w \in S_n\}$. We relate $\mathrm{Hilb}(\widehat{V_n}; q)$ to the q-analogue of Bell numbers introduced in Subsection 2.5. In Subsection 5.2, we move to study the vector space $\widehat{V} := \mathrm{span}\{\widehat{\mathfrak{G}}_w : w \in S_+\}$, which is the same as $\bigcup_{n \geqslant 1} \widehat{V_n}$. We will give an expression of $\mathrm{Hilb}(\widehat{V}; q)$.

5.1. The vector space $\widehat{V_n}$.

By work of Pechenik, Speyer and Weigandt [PSW21], the space $\widehat{V_n}$ has dimension B_n and basis $\{\widehat{\mathfrak{G}}_w : w \in S_n \text{ inverse fireworks}\}$. In this subsection, we will derive another basis of $\widehat{V_n}$ using $\widehat{\mathfrak{L}}_\alpha$.

Recall C_n is the set of weak compositions α such that α is at most $(n-1, \cdots, 2, 1)$ entry-wise. A monomial expansion of $\mathfrak{G}_w$ is given by Fomin and Kirillov [FK94]. It implies that if a monomial

$\beta^m x^\alpha$ appears in $\mathfrak{G}_w$ for some $w \in S_n$, then $\alpha \in C_n$. Using this fact, we obtain a refinement of Theorem 2.4.

Corollary 5.1. *For $w \in S_n$, $\mathfrak{G}_w$ expands positively into $\{\mathfrak{L}_\alpha : \alpha \in C_n\}$.*

Proof. By Theorem 2.4, we can expand $\mathfrak{G}_w$ into a sum of Lascoux polynomials. We just need to make sure for each $\mathfrak{L}_\alpha$ appearing in the expansion with a nonzero coefficient, we have $\alpha \in C_n$.

We know the monomial x^α is the leading monomial of κ_α, so x^α appears in $\mathfrak{L}_\alpha$. Since all coefficients in the sum are positive, we know $x^\alpha \beta^m$ appears in $\mathfrak{G}_w$ for some non-negative integer m. By the aforementioned fact, we have $\alpha \in C_n$. $\square$

By this corollary and Lemma 2.5, we have the following.

Corollary 5.2. *For $w \in S_n$, $\widehat{\mathfrak{G}}_w$ expands positively into $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n\}$.*

Now we are ready to give another basis of $\widehat{V_n}$:

Theorem 1.3. *The space $\widehat{V_n}$ is also the $\mathbb{Q}$-span of $\widehat{\mathfrak{L}}_\alpha$ with $\alpha \in C_n$. It has basis $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n$ is snowy$\}$*

Proof. By Corollary 5.2 and [PSW21, Theorem 1.4], $\widehat{V_n}$ is a subspace of $\mathsf{span}\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n\}$ with dimension B_n. It remains to check the number of snowy weak compositions in C_n is also B_n. In Lemma 4.20, we construct a bijection $\mathsf{dark}(\cdot)$ from snowy weak compositions in C_n to Rook_n, which has size B_n. $\square$

Now we have two different bases of $\widehat{V_n}$: One is (1) and the other is in Theorem 1.3. They give us two formulas of $\mathrm{Hilb}(\widehat{V_n}; q)$:

$$\mathrm{Hilb}(\widehat{V_n}; q) = \sum_{\substack{w \in S_n,\\ w \text{ is inverse}\\ \text{fireworks}}} q^{\mathsf{raj}(w)} = \sum_{\substack{\alpha \in C_n,\\ \alpha \text{ is snowy}}} q^{\mathsf{raj}(\alpha)}. \tag{6}$$

We are going to connect $\mathrm{Hilb}(\widehat{V_n}; q)$ with $B_n(q)$, a q-analogue of Bell numbers. To do this, we first transplant our notions on snowy weak compositions to non-attacking rook diagrams, which are easier to work with.

Definition 5.3. *Take $R \in \mathsf{Rook}_+$. Define the Northwest number of R, denoted as $\mathsf{NW}(R)$, to be $\mathsf{raj}(\alpha)$, where α is any weak composition with $\mathsf{dark}(\alpha) = R$.*

Note that $\mathsf{NW}(R)$ is independent from the choice of α. Equivalently, we may compute $\mathsf{NW}(R)$ as follows: For each $(r, c) \in R$, we mark all cells weakly above it and on its left. By Lemma 4.14, these marked cells agree with the underlying diagram of $\mathsf{snow}(D(\alpha))$ for any α with $\mathsf{dark}(\alpha) = R$. Then $\mathsf{NW}(R)$ is just the number of marked cells. This is why we call it the "Northwest number". Comparing this statistic with $\mathsf{GR}_n(\cdot)$ defined in Section 2.5, we have the following connection.

Remark 5.4. *Take $R \in \mathsf{Rook}_n$. Then $\mathsf{GR}_n(R) = |\mathsf{Stair}_n| - \mathsf{NW}(R) = \binom{n}{2} - \mathsf{NW}(R)$.*

Finally, we can derive an elegant expression for the degree generating function of $\widehat{V_n}$.

Theorem 1.4. *We have*

$$\mathrm{Hilb}(\widehat{V_n}; q) = q^{\binom{n}{2}} B_n(q^{-1}) = \mathsf{rev}(B_n(q)),$$

where $\mathsf{rev}(\cdot)$ is the operator that reverse the coefficients of a polynomial. In other words, it sends a polynomial $f(q)$ of degree d to $q^d f(q^{-1})$.

Proof. Apply the bijection $\mathsf{dark}(\cdot)$ to α on the right-hand side of Equation (6), together with (3) and the remark above, we obtain

$$\begin{aligned}\mathrm{Hilb}(\widehat{V_n};q) &= \sum_{\substack{\alpha \in C_n,\\ \alpha \text{ is snowy}}} q^{\mathsf{raj}(\alpha)} = \sum_{R\in\mathsf{Rook}_n} q^{\mathsf{NW}(R)} = \sum_{R\in\mathsf{Rook}_n} q^{\binom{n}{2}-\mathsf{GR}_n(R)} \\ &= q^{\binom{n}{2}} \sum_{R\in\mathsf{Rook}_n} q^{-\mathsf{GR}_n(R)} = q^{\binom{n}{2}} B_n(q^{-1}).\end{aligned}$$

Since $B_n(q)$ has degree $\binom{n}{2}$ by (3), we have $\mathrm{Hilb}(\widehat{V_n};q) = \mathsf{rev}(B_n(q))$. □

5.2. **The vector space $\widehat{V}$.** In this subsection, we study the vector space $\widehat{V} := \mathrm{span}\{\widehat{\mathfrak{G}}_w : w \in S_+\}$. It can also be viewed as $\bigcup_{n=1}^{\infty} \widehat{V_n}$. First, we show that top Lascoux also spans this space.

Theorem 5.5. *The span of $\widehat{\mathfrak{L}}_\alpha$ for $\alpha \in C_+$ is also $\widehat{V}$.*

Proof. By Corollary 2.6, $\widehat{V}$ is in the span of $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_+\}$. Now consider $\alpha \in C_+$. There exists n large enough such that $\alpha \in C_n$. Then $\widehat{\mathfrak{L}}_\alpha \in \widehat{V_n} \subset \widehat{V}$. □

Corollary 5.6. *There are two basis of $\widehat{V}$:*

$$\{\widehat{\mathfrak{G}}_w : w \in S_+ \text{ is inverse fireworks}\} \quad \text{and} \quad \{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_+ \text{ is snowy}\}$$

With these two bases, we have:

$$\mathrm{Hilb}(\widehat{V};q) = \sum_{\substack{w \in S_+,\\ w \text{ is inverse}\\ \text{fireworks}}} q^{\mathsf{raj}(w)} = \sum_{\substack{\alpha \in C_+,\\ \alpha \text{ is snowy}}} q^{\mathsf{raj}(\alpha)} = \sum_{R\in\mathsf{Rook}_+} q^{\mathsf{NW}(R)},$$

where the third expression is obtained by applying $\mathsf{dark}(\cdot)$ on α in the second expression. On the other hand, $\mathrm{Hilb}(\widehat{V};q)$ is the limit of $\mathrm{Hilb}(\widehat{V_n};q)$ as n goes to infinity. According to OEIS, coefficients in $B_n(q)$ are in A126347 and the coefficients of $\mathrm{Hilb}(\widehat{V};q)$ are in A126348. A formula for $\mathrm{Hilb}(\widehat{V};q)$ in OEIS is given by Jovovic: $\prod_{m>0}(1+\frac{q^m}{1-q})$. For completeness, we check this rule using our formula of $\mathrm{Hilb}(\widehat{V};q)$ involving snowy weak compositions.

Theorem 1.5. *We have*

$$\mathrm{Hilb}(\widehat{V};q) = \sum_{\alpha \text{ is snowy}} q^{\mathsf{raj}(\alpha)} = \prod_{m>0}\left(1+\frac{q^m}{1-q}\right)$$

Proof. Let $\mathsf{snowy}(M)$ be the set of all snowy weak compositions with the largest entry being at most M. It suffices to show

$$\sum_{\alpha\in\mathsf{snowy}(M)} q^{\mathsf{raj}(\alpha)} = \prod_{m>0}^{M}\left(1+\frac{q^m}{1-q}\right).$$

We prove it by induction on M. The claim is immediate when $M = 0$ as both sides are 1.

Now assume the claim above holds for some $M \geqslant 0$. Let $\mathsf{snowy}(M)_i$ be the set of all snowy weak compositions α such that its largest entry is $\alpha_i = M$. With this notation, we can express $\mathsf{snowy}(M)$ recursively:

$$\mathsf{snowy}(M) = \mathsf{snowy}(M-1) \bigsqcup \left(\bigsqcup_{i\geqslant 1} \mathsf{snowy}(M)_i\right).$$

Next, we define a map

$$\begin{aligned}\phi : \mathsf{snowy}(M-1) &\to \mathsf{snowy}(M)_1 \\ (\alpha_1,\alpha_2,\dots) &\mapsto (M,\alpha_1,\alpha_2,\dots)\end{aligned}$$

It is straightforward to see that ϕ is a bijection. Furthermore, we have $\mathsf{raj}(\phi(\alpha)) = \mathsf{raj}(\alpha) + M$. To get $\mathsf{snowy}(M)_i$ for $i > 1$, notice that the operator s_i on the set of weak compositions is a bijection between $\mathsf{snowy}(M)_i$ and $\mathsf{snowy}(M)_{i+1}$. For $\alpha \in \mathsf{snowy}(M)_i$, we have $\mathsf{raj}(s_i(\alpha)) = \mathsf{raj}(\alpha) + 1$ by Corollary 4.19. Inductively, we have

$$\sum_{\alpha \in \mathsf{snowy}(M)_i} q^{\mathsf{raj}(\alpha)} = q^{M+i-1} \sum_{\alpha \in \mathsf{snowy}(M-1)} q^{\mathsf{raj}(\alpha)}.$$

Finally,

$$\begin{aligned}\sum_{\alpha \in \mathsf{snowy}(M)} q^{\mathsf{raj}(\alpha)} &= \sum_{\alpha \in \mathsf{snowy}(M-1)} q^{\mathsf{raj}(\alpha)} + \left(\sum_{i \geqslant 1} q^{M+i-1}\right) \sum_{\alpha \in \mathsf{snowy}(M-1)} q^{\mathsf{raj}(\alpha)} \\ &= \left(1 + \sum_{i \geqslant 1} q^{M+i-1}\right) \sum_{\alpha \in \mathsf{snowy}(M-1)} q^{\mathsf{raj}(\alpha)} = \left(1 + \frac{q^M}{1-q}\right) \prod_{m>0}^{M-1} \left(1 + \frac{q^m}{1-q}\right).\end{aligned}$$

The inductive step is finished. ☐

6. Snow diagrams for Rothe diagrams

Now we move on to study the snow diagrams of Rothe diagrams. In Section 6.1, we prove that $\mathsf{rajcode}(\cdot)$ as defined in Definition 3.5 for Rothe diagrams is consistent to the one defined in [PSW21] for permutations. In Section 6.2, we show the dark clouds in the snow diagram of a Rothe diagram corresponds to the turning points in the Shadow Diagram for the same permutation. In Section 6.3, we study the snow diagrams for inverse fireworks permutations.

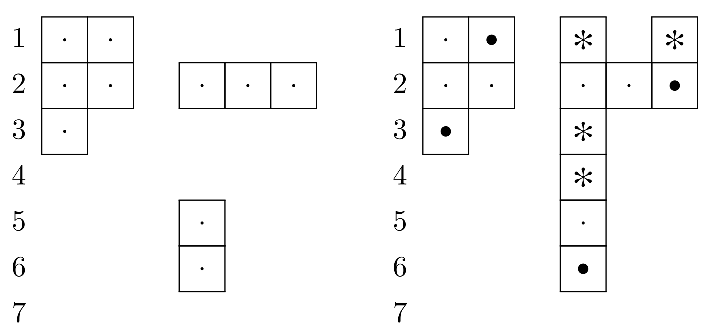
6.1. Rajcode of Rothe diagrams.

Pechenik, Speyer and Weigandt have defined $\mathsf{rajcode}$ on permutations (see Definition 2.1). In this subsection, we show that $\mathsf{rajcode}(w)$ in their definition agrees with the $\mathsf{rajcode}(RD(w))$ in our definition. To do so, we need a better understanding of $\mathsf{snow}(RD(w))$. We start by describing how the positions of dark clouds in $\mathsf{snow}(RD(w))$ are related to the Schensted insertion described in Subsection 2.4.

Proposition 6.1. *Take $w \in S_n$. Consider the Schensted insertion on w. The dark cloud in row r of $\mathsf{snow}(RD(w))$ can be described based on the insertion of $w(r)$.*

(1) If $w(r)$ is appended to the end of row one, then there is no dark cloud in row r of $\mathsf{snow}(RD(w))$;
(2) If $w(r)$ bumps c in row one, then (r, c) is a dark cloud in $\mathsf{snow}(RD(w))$.

Example 6.2. *Take $w \in S_7$ with one-line notation 3721564. Consider the Rothe diagram $RD(3721564)$ and its snow diagram:*



The Schensted insertion of w is presented in Example 2.14. We check Proposition 6.1 in the table below.

r	$w(r)$	*insertion of $w(r)$ in row one*	*position of $\bullet$ in row r of* $\mathsf{snow}(RD(w))$
7	4	*appended at the end of row one*	*row 7 has no $\bullet$*
6	6	*bumps 4 in row one*	*row 6 has $\bullet$ at $(6,4)$*
5	5	*appended at the end of row one*	*row 5 has no dark cloud*
4	1	*appended at the end of row one*	*row 4 has no dark cloud*
3	2	*bumps 1 in row one*	*row 3 has $\bullet$ at $(3,1)$*
2	7	*bumps 6 in row one*	*row 2 has $\bullet$ at $(2,6)$*
1	3	*bumps 2 in row one*	*row 1 has $\bullet$ at $(1,2)$*

Proof. We prove the statement by induction on r starting from $r = n$. The number $w(n)$ is inserted into the empty tableau. In this case, it is appended to the end of the first row. It is also clear that there can not be any dark cloud on row n of $\mathsf{snow}(RD(w))$.

Now suppose the statement holds for $r+1, r+2, \ldots, n$ for some $r \leqslant n-1$. Let P be the tableau right before the insertion of $w(r)$. By the inductive hypothesis, for each $r' > r$, $w(r')$ appears in row 1 of P if and only if there is no dark cloud in column $w(r')$ under row r of $\mathsf{snow}(RD(w))$. Now consider the insertion of $w(r)$.

(1) Case 1: $w(r)$ is appended to the end of row 1.
Assume toward contradiction that $(r, w(r'))$ is a dark cloud of $\mathsf{snow}(RD(w))$ for some $r' > r$. Then $w(r) > w(r')$. Moreover, there is no dark cloud in the column of $w(r')$ under row r, so $w(r')$ is in row 1 of P. Thus, $w(r)$ cannot be appended in row 1. Contradiction.

(2) Case 2: $w(r)$ bumps $w(r')$ in row 1 for some $r' > r$.
Then $w(r) > w(r')$. The cell $(r, w(r'))$ is in $RD(w)$. We need to show that it is a dark cloud in $\mathsf{snow}(RD(w))$. By Remark 3.4, we just need to make sure there is no dark cloud under it or on its right.
Suppose that there is a dark cloud in column $w(r')$ under row r. By the inductive hypothesis, $w(r')$ cannot appear in row 1 of P, which is a contradiction.
Finally, suppose there is a dark cloud on the right of $(r, w(r'))$. We may write this dark cloud as $(r, w(r''))$ with $w(r'') > w(r')$. Since it is a cell in $RD(w)$, we also have $r'' > r$ and $w(r) > w(r'')$. Since it is a dark cloud, there is no dark cloud under it. By the inductive hypothesis, $w(r'')$ is in row 1 of P. This is a contradiction: $w(r)$ should bump $w(r'')$ instead of $w(r')$ since $w(r) > w(r'') > w(r')$.

□

Now we can check $\mathsf{rajcode}(w)$ and $\mathsf{rajcode}(RD(w))$ agree.

Theorem 6.3. *Take $w \in S_n$. We have* $\mathsf{rajcode}(w) = \mathsf{rajcode}(RD(w))$.

Proof. Take $r \in [n]$. Consider row r of $\mathsf{snow}(RD(w))$. It contains $\mathsf{invcode}(w)_r$ cells that are not snowflakes. Let d_r be the number of dark clouds in $\mathsf{snow}(RD(w))$ that are southeastern of $(r, w(r))$. Clearly, d_r is also the number of snowflakes in row r of $\mathsf{snow}(RD(w))$. We have $\mathsf{rajcode}(w)_r = \mathsf{invcode}(w)_r + d_r$.

Consider the Schensted insertion of w. Let P be the tableau right before the insertion of $w(r)$. Define A as the number of elements in P that are larger than $w(r)$. We compute A in two ways.

- The tableau P consists of numbers $w(r+1), \ldots, w(n)$. There are $\mathsf{invcode}(w)_r$ of them less than $w(r)$, so $A = n - r - \mathsf{invcode}(w)_r$.
- Assume when inserting $w(r)$ to P, it goes to column c of row 1. Thus, $c-1$ is the number of entries in row 1 of P that are larger than $w(r)$. By Proposition 6.1, d_r is the number of entries under row 1 of P that are larger than $w(r)$. We have $A = c - 1 + d_r$. By Lemma 2.15, $c = \mathrm{LIS}^w(w(r))$, so $A = \mathrm{LIS}^w(w(r)) - 1 + d_r$.

Combining the two expressions of A yields

$$n - r - \mathsf{invcode}(w)_r = \mathrm{LIS}^w(w(r)) - 1 + d_r, \text{ so}$$

$$\mathsf{rajcode}(RD(w))_r = \mathsf{invcode}(w)_r + d_r = n - r + 1 - \mathrm{LIS}^w(w(r)) = \mathsf{rajcode}(w)_r.$$

□

6.2. **Dark Clouds of the Rothe Diagram via Viennot's geometric construction.** In 1977, Xavier Gérard Viennot gave a diagrammatic construction of the RSK correspondence in terms of shadow lines ([Vie77]). It is also known as the matrix-ball construction. We will show that the dark clouds in the snow diagram of a permutation can be obtained via Viennot's geometric construction. We denote $\mathsf{Row}_1(P(w))$ to be the first row of the tableau obtained by Schensted insertion on w.

For two cells $(i,j),(m,n) \in \mathbb{N} \times \mathbb{N}$, (m,n) lies in the shadow of (i,j) if and only if $m \leqslant i$ and $n \leqslant j$. This can be visualized by imagining shedding light from the Southeast. Notice that the usual convention imagines shedding light from the Northwest. We reverse this convention to make our results easier to state. To obtain the *shadow diagram* of $w \in S_n$, consider the points $(1, w(1)), \ldots, (n, w(n))$. Let $\left(i_1^{(1)}, w(i_1^{(1)})\right), \ldots, \left(i_{\ell_1}^{(1)}, w(i_{\ell_1}^{(1)})\right)$ be the points that are not in the shadow of any other point for some $\ell_1 \geqslant 1$ and $i_1^{(1)} > i_2^{(1)} > \cdots > i_{\ell_1}^{(1)}$. Then the first *shadow line* $L_1(w)$ is the boundary of the combined shadows of the points $\left(i_1^{(1)}, w(i_1^{(1)})\right), \ldots, \left(i_{\ell_1}^{(1)}, w(i_{\ell_1}^{(1)})\right)$. The rest of the $L_j(w)$ can be constructed recursively. Supposed $L_1, \ldots, L_{j-1}$ have been constructed, remove all points in the set $\{\left(i_k^{(p)}, w(i_k^{(p)})\right) : 1 \leqslant p \leqslant j-1, 1 \leqslant k \leqslant \ell_p\}$, then L_j is the boundary of the shadow of the remaining points of the points left, which we label as

$$\left(i_1^{(j)}, w(i_1^{(j)})\right), \ldots \left(i_{\ell_j}^{(j)}, w(i_{\ell_j}^{(j)})\right),$$

for some $\ell_j \geqslant 1$ and $i_1^{(j)} > i_2^{(j)} > \cdots > i_{\ell_j}^{(j)}$. Once there is no point left, the shadow lines we obtained form the shadow diagram for w.

Theorem 6.4 ([Vie77])**.** *Given $w \in S_n$ and suppose $L_1, \ldots, L_s$ are the shadow lines obtained from w until there is no point left. Then s equals the size of $\mathsf{Row}_1(P(w))$.*

For each L_j, it also consists $\ell_j - 1$ "turning points", which are points (x,y) of L_j such that $(x-1,y),(x,y-1) \notin L_j$, i.e.,

$$\left(i_2^{(j)}, w(i_1^{(j)})\right), \left(i_3^{(j)}, w(i_2^{(j)})\right), \ldots, \left(i_{\ell_j}^{(j)}, w(i_{\ell_j-1}^{(j)})\right).$$

In total, there are $n - |\mathsf{Row}_1(P(w))|$ turning points for each $w \in S_n$. There is a classical result connecting these turning points to the Schensted insertion.

Theorem 6.5 ([Vie77, Knu70])**.** *Let a shadow line L_j of a permutation w consists of points*

$$\left(i_1^{(j)}, w(i_1^{(j)})\right), \ldots \left(i_{\ell_j}^{(j)}, w(i_{\ell_j}^{(j)})\right)$$

for some $\ell_j \geqslant 1$ and $i_1^{(j)} > i_2^{(j)} > \cdots > i_{\ell_j}^{(j)}$. Then during Schensted insertion on w, when we insert $w(i_{k+1}^{(j)})$, it bumps $w(i_k^{(j)})$ from the first row.

Combining Proposition 6.1 and Theorem 6.5, we have the following.

Corollary 6.6. *Each of the turning points in the shadow diagram of w contains a dark cloud in $\mathsf{snow}(w)$. Any dark cloud in $\mathsf{snow}(w)$ is also a turning point in the shadow diagram of w.*

Example 6.7. *Consider $w = 3721564 \in S_7$. We present its Rothe diagram, its Shadow diagram, and the snow diagram of $RD(w)$. From Example 2.14, the Schensted insertion on w yields a tableau whose row 1 has three cells. Correspondingly, there are three shadow lines. The turning points of the shadow lines are $(3,1),(1,2),(6,4),(2,6)$, which are positions for dark clouds in $\mathsf{snow}(RD(w))$.*

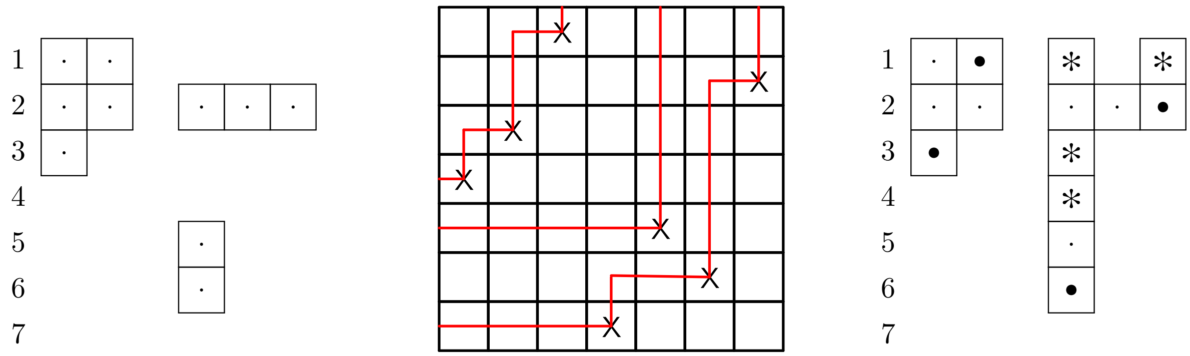


FIGURE 1. Left: RD(w); Middle: Shadow Diagram of w; Right: $\mathsf{snow}(RD(w))$

Remark 6.8. *A geometric interpretation for the rajcode is given in* [PSW21, Section 4] *in terms of the "blob diagrams." Specifically, the set of points in the same shadow line in the shadow line diagram is labeled as $B_n, B_{n-1}, \ldots$ from southeast to northwest. With the labeling on the blob diagrams, we can obtain the rajcode directly. That is, if $(i, w(i)) \in B_k$, then $\mathsf{rajcode}(w)_i = k - i$.*

6.3. **Inverse fireworks permutations.** Now we have seen that our snow diagrams are connected to the work of Pechenik, Speyer and Weigandt [PSW21]. Another interesting notion in their work is the inverse fireworks permutation. We see that they are also related to our snow diagrams.

Definition 6.9. [PSW21, Definition 3.5] *A permutation $w \in S_n$ is a fireworks permutation if its initial element in each decreasing run is increasing. A permutation $w \in S_n$ is an inverse fireworks permutation if w^{-1} is a fireworks permutation.*

Inverse fireworks permutations are crucial in [PSW21]. They are the representatives of equivalence classes, given by permutations with the same $\mathsf{rajcode}$. Our snowy weak compositions are analogs of these inverse fireworks permutations. Similar to snowy weak compositions, we can describe the condition of inverse fireworks permutations using the positions of dark clouds. First, we start with the following observation about the permutation diagram of an inverse fireworks permutation.

Lemma 6.10. *Let w be an inverse fireworks permutation. Consider each $r \in [n]$ such that row r of $RD(w)$ is not empty. The rightmost cell in row r of $RD(w)$ is $(r, w(r) - 1)$.*

Proof. Recall that $(r, w(r')) \in RD(w)$ if and only if $(r, r') \in \mathsf{Inv}(w)$ if and only if $(w(r'), w(r)) \in \mathsf{Inv}(w^{-1})$. Let $c = w(r)$. Clearly, cells in row r of $RD(w)$ are within the first $c - 1$ columns. It remains to check $(r, c-1) \in RD(w)$, which is equivalent to $(c - 1, c) \in \mathsf{Inv}(w^{-1})$.

Since row r of $RD(w)$ is nonempty, it must contain a cell (r, i) such that $(i, c) \in \mathsf{Inv}(w^{-1})$ for some $i \in [c-1]$. Since $w^{-1}(i) > w^{-1}(c)$ and w^{-1} is fireworks, $w^{-1}(c)$ can not be the initial element in its decreasing run. Therefore $w^{-1}(c-1) > w^{-1}(c)$ and we have $(c - 1, c) \in \mathsf{Inv}(w^{-1})$. □

We can characterize the inverse fireworks permutations using Rothe diagrams or the snow diagram of the permutation. This is similar to Remark 4.17, where we describe snowy weak compositions using dark clouds.

Proposition 6.11. *Take $w \in S_n$. The following are equivalent:*

(1) w is an inverse fireworks permutation.

(2) In $RD(w)$, the rightmost cells in each row are in different columns.
(3) In $\mathsf{snow}(RD(w))$, the rightmost cell in each row is a dark cloud.

Proof. The last two statements are clearly equivalent. Now we establish the equivalence of the first two statements.

Assume w is inverse fireworks. Take $r, r' \in [n]$ with $r \neq r'$ such that row r and row r' of $RD(w)$ are not empty. By Lemma 6.10, the rightmost cell in row r (resp. r') is at $(r, w(r) - 1)$ (resp. $(r', w(r') - 1)$). Clearly, $w(r) - 1 \neq w(r') - 1$, so we have our second statement.

Now we assume w is not inverse fireworks. We can find a number r in w^{-1} such that r is the initial element in its decreasing run, but r is less than r', the initial element of the previous decreasing run. Let $c' = w(r')$ and $c = w(r)$. Since $(c', c) \in \mathsf{Inv}(w^{-1})$, $(r, c') \in RD(w)$. Thus, row r of $RD(w)$ is not empty. Let (r, i) be the rightmost cell in row r. In other words, i is the largest such that $(i, c) \in \mathsf{Inv}(w^{-1})$. We have $c' \leqslant i < c - 1$. Consider the decreasing run before $w^{-1}(c)$: $w^{-1}(c') > w^{-1}(c' + 1) > \cdots > w^{-1}(c - 1)$. We see $(i, i + 1)$ is also in $\mathsf{Inv}(w^{-1})$. In row $w^{-1}(i + 1)$, the cell $(w^{-1}(i + 1), i)$ is the rightmost cell of its row. Thus, the second statement does not hold, and our proof is finished. ☐

With the above proposition, we can compute $\mathsf{rajcode}(w)$ easily if w is inverse fireworks. The following rule is similar to how we compute $\mathsf{rajcode}(\alpha)$ when α is snowy in Lemma 4.18.

Proposition 6.12. *Assume $w \in S_n$ is inverse fireworks. For each $r \in [n]$, $\mathsf{rajcode}(w)_r$ is the number of $r' > r$ such that $(r, r') \in \mathsf{Inv}(w)$ or $(r', r'') \in \mathsf{Inv}(w)$ for some r''.*

Proof. First, we know $\mathsf{rajcode}(w)_r = \mathsf{rajcode}(RD(w))_r$ is the number of cells in row r of $\mathsf{snow}(RD(w))$. The number of non-snowflake cells on this row is given by $|\{r' : (r, r') \in \mathsf{Inv}(w)\}|$.

Now we count the number of snowflakes in row r of $\mathsf{snow}(RD(w))$. It is the number of $r' > r$ such that row r' of $\mathsf{snow}(RD(w))$ has a dark cloud on the right of the column $w(r)$. By the previous Lemmas, row r' has a dark cloud at column $w(r') - 1$ if $RD(w)$ is nonempty in row r'. Thus, the number of snowflakes in row r of $\mathsf{snow}(RD(w))$ is the number of $r' > r$ such that $w(r') > w(r)$ and $(r', r'') \in \mathsf{Inv}(w)$ for some r''. ☐

7. Open Problems and Future Directions

We conclude with several open problems for future study. In Section 6.2, we present the connections between the following three constructions:

- Positions of dark clouds in $\mathsf{snow}(RD(w))$;
- First step of Viennot's geometric construction;
- Bumps in the first row during Schensted insertion.

Problem 7.1. *Find further connections between Viennot's geometric construction and* $\mathsf{snow}(RD(w))$.

Problem 7.2. *Find further connections between Schensted insertion and* $\mathsf{snow}(RD(w))$.

The Grothendieck to Lascoux expansion, proven in [SY21], involves finding certain tableaux and computing their right keys.

Problem 7.3. *Find a combinatorial formula for the expansion of Castelnuovo–Mumford polynomials into top Lascoux polynomials indexed by snowy weak compositions.*

Finding a combinatorial formula for the *structure constants* $c_{u,v}^w$ for Grothendieck polynomials, defined as

$$\mathfrak{G}_u \mathfrak{G}_v = \sum_w c_{u,v}^w \mathfrak{G}_w \,,$$

has been a long-standing open problem. These coefficients have a geometric interpretation: They are the intersection numbers for the Schubert classes in the connective K-theory. If we consider only the top-degree terms on both sides, we get the structure constants for Castelnuovo–Mumford polynomials, which we denote as $\widehat{c_{uv}^{w}}$, which are still non-negative integers. We can ask questions analogous to the structure constants problem.

Problem 7.4. *Find a combinatorial formula for* $\widehat{c_{uv}^{w}}$.

The Grothendieck polynomials $\mathfrak{G}_w(\boldsymbol{x})$ are a specialization of the double Grothendieck polynomials $\mathfrak{G}_w(\boldsymbol{x},\boldsymbol{y})$ by setting $y_1 = y_2 = \cdots = 0$. In [KM01], Knutson and Miller introduced pipe dream rules for both $\mathfrak{G}_w(\boldsymbol{x})$ and $\mathfrak{G}_w(\boldsymbol{x},\boldsymbol{y})$. For Castelnuovo–Mumford polynomials $\widehat{\mathfrak{G}_w}(\boldsymbol{x})$, we can think they correspond to a subset of pipe dreams for $\mathfrak{G}_w(\boldsymbol{x})$. In [PSW21], the authors proved a factorization of $\widehat{\mathfrak{G}_w}(\boldsymbol{x},\boldsymbol{y})$ into a $\boldsymbol{x}$-polynomial and a $\boldsymbol{y}$-polynomial, and they showed the the leading term is in fact,

$$\boldsymbol{x}^{\mathsf{rajcode}(w)}\boldsymbol{y}^{\mathsf{rajcode}(w^{-1})},$$

with coefficient 1 by constructing a pipe dream associated with it iteratively.

Problem 7.5. *Use the snow diagrams to give an explicit construction of pipe dreams for the leading term in* $\widehat{\mathfrak{G}_w}(\boldsymbol{x},\boldsymbol{y})$.

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