Séminaire Lotharingien de Combinatoire **XX** (2024)
Article #YY, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Analogues of two classical pipedream results on bumpless pipedreams

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Abstract. Schubert polynomials are distinguished representatives of Schubert cycles in the cohomology of the flag variety. Pipedreams (PD) and bumpless pipedreams (BPD) are two combinatorial models of Schubert polynomials. There are many classical results on PDs. For instance, Fomin and Stanley represented each PD as an element in the nil-Coexter algebra. Lenart and Sottile converted each PD into certain chains in the Bruhat order. This paper establishes the BPD analogues of both viewpoints. Our results lead to a bijection between PDs and BPDs via Lenart's growth diagram.

Keywords: Schubert polynomial, bumpless pipedreams, Fomin-Kirillov algebra, Bruhat orders

1 Introduction

Fix $n \in \mathbb{Z}_{\geqslant 0}$. For a permutation $w \in S_n$, Lascoux and Schützenberger [12] recursively define the *Schubert polynomial* $\mathfrak{S}_w$. The base case is $\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ where w_0 is the permutation with one-line notation $[n, n-1, \cdots, 1]$. To compute $\mathfrak{S}_w$ for other $w \in S_n$, we need the *divided difference operator* $\partial_i(f) := \frac{f - f(\cdots, x_{i+1}, x_i, \cdots)}{x_i - x_{i+1}}$. Let $s_i \in S_n$ denote the transposition that swaps i and $i+1$. Then for any $w \in S_n$ and $i \in [n-1]$:

$$\partial_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } w(i) > w(i+1), \\ 0 & \text{if } w(i) < w(i+1). \end{cases}$$

The Schubert polynomials represent Schubert cycles in flag varieties and have been extensively investigated. Schubert polynomials have two distinct combinatorial formulas involving "pipes": pipedreams (PD) [1, 3] and bumpless pipedreams (BPD) [11]. Both are fillings of grids with certain tiles. When we refer to cells of a grid, we use the matrix coordinates: row 1 is the topmost row and column 1 is the leftmost column. A *pipedream* is a filling of a staircase grid: The grid has a cell in row i column j for each $i + j \leqslant n + 1$. The rightmost cell in each row is . The rest of the cells can be (crossing) or (bump), but two pipes cannot cross more than once. A *bumpless pipedream (BPD)* is a

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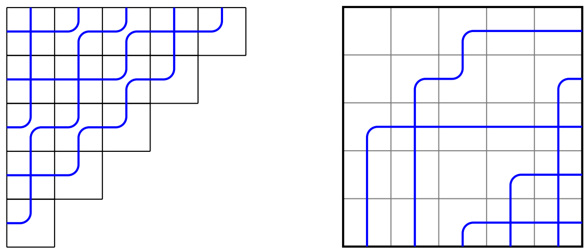
consistent filling of an $n \times n$ grid with six types of cells: , , , , and (blank). Pipes enter from each cell on the bottom and exit on the right edge. In addition, two pipes cannot cross more than once. The permutation associated to each PD (resp. BPD) can be read off as follows: Label the pipes $1, 2, \ldots, n$ along the top (resp. bottom) edge, follow the pipes, and read the labels from top to bottom on the left (resp. right) edge.

Example 1.1. When $n = 5$, we present a PD and a BPD associated with $[2, 5, 1, 4, 3]$:



Let $\mathsf{PD}(w)$ (resp. $\mathsf{BPD}(w)$) be the set of all PDs (resp. BPDs) associated with $w \in S_n$. For $P \in \mathsf{PD}(w)$ (resp. $P \in \mathsf{BPD}(w)$), the weight of P, denoted as $\mathsf{wt}(P)$, is a sequence of $n-1$ integers where the i^{th} entry is the number of (resp.) on row i. For instance, the PD and BPD in Example 1.1 both have weight $(2, 2, 0, 1)$. If $\alpha = (\alpha_1, \cdots, \alpha_{n-1})$ is a sequence of $n-1$ non-negative integers, we use x^α to denote the monomial $x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}$.

Theorem 1.2. *[1, 3, 11] For $w \in S_n$, $\mathfrak{S}_w = \sum_{P \in \mathsf{PD}(w)} x^{\mathsf{wt}(P)} = \sum_{D \in \mathsf{BPD}(w)} x^{\mathsf{wt}(D)}$.*

There is a recent surge of research connecting BPDs with PDs and finding BPD analogue of classical PD apparatus [7, 10, 8, 17]. This paper establishes the BPD analogue of two classical stories on PDs:

- The nil-Coexter algebra $\mathcal{N}_n$ is generated by $u_1, \cdots, u_{n-1}$. Fomin and Stanley [6] defined the following elements in $\mathbb{Q}[x_1, \cdots, x_{n-1}] \otimes \mathcal{N}_n$:

  $$A_i(x_i) := (1 + x_i u_{n-1})(1 + x_i u_{n-2}) \cdots (1 + x_i u_i) \text{ and } \mathfrak{S}^{\mathsf{PD}} := A_1(x_1) \cdots A_{n-1}(x_{n-1}).$$

  Combinatorially, after expanding $\mathfrak{S}^{\mathsf{PD}}$, each term $x^\alpha u_{i_1} \cdots u_{i_k}$ naturally corresponds to a $P \in \mathsf{PD}(w)$ with $\alpha = \mathsf{wt}(P)$ and $i_1 \cdots i_k$ is a reduced word of w. Algebraically, Fomin and Stanley proved $\mathfrak{S}^{\mathsf{PD}} = \sum_{w \in S_n} \mathfrak{S}_w u_{i_1} \cdots u_{i_l}$ where $i_1 \cdots i_l$ is any reduced word of w. Consequently, they obtain an operator theoretic proof of the PD fomula.

- The Bruhat order is a partial order on S_n. Lenart and Sottile [14] defined a bijection from $\mathsf{PD}(w)$ to chains $(w_1, w_2, \cdots, w_n)$ in the Bruhat order where $w_1 = w$, $w_n = w_0$ and there is an increasing i-chain from w_i to w_{i+1} for $i \in [n-1]$ (See Section 2.2).

Since the introduction of BPDs, finding a BPD analogue of the Fomin-Stanley construction has been an open problem. Instead of the nil-Coexter algebra, we consider the

Fomin-Kirillov algebra $\mathcal{E}_n$ [4]. It is generated by $d_{i,j}$ for $1 \leqslant i < j \leqslant n$ and has a right action on $\mathbb{Q}[S_n]$ denoted as $\odot$. Define the following elements in $\mathbb{Q}[x_1, \cdots, x_{n-1}] \otimes \mathcal{E}_n$:

$$R_i(x_i) := (x_i + d_{1,i+1} + \cdots + d_{i,i+1})(x_i + d_{1,i+2} + \cdots + d_{i,i+2}) \cdots (x_i + d_{1,n} + \cdots + d_{i,n}), \text{ and}$$

$$\mathfrak{S}^{\mathsf{BPD}} := w_0 \odot (R_1(x_1)R_2(x_2) \cdots R_{n-1}(x_{n-1})).$$

Combinatorially, after expanding $\mathfrak{S}^{\mathsf{BPD}}$, we show each term $x^\alpha w$ naturally corresponds to a $D \in \mathsf{BPD}(w)$ with $\alpha = \mathsf{wt}(D)$. Algebraically, we establish Theorem 4.3, obtaining an operator theoretic proof of the BPD formula.

Theorem 4.3. *We have* $\mathfrak{S}^{\mathsf{BPD}} = \sum_{w \in S_n} \mathfrak{S}_w w$.

A crucial tool to understand $\mathfrak{S}^{\mathsf{BPD}}$ is a novel encoding algorithm Φ that encodes each element of $\mathsf{BPD}(w)$ as partial fillings of a staircase grid which we call *flagged tableaux*. We denote the image of $\mathsf{BPD}(w)$ under Φ as $\mathsf{FT}(w)$. Each $T \in \mathsf{FT}(w)$ corresponds to a chain in the Bruhat order denoted as $\mathsf{chain}(T) = (w_n, \cdots, w_1)$. Then we establish Theorem 3.9, obtaining a BPD analogue of Lenart and Sottile's work.

Theorem 3.9. *The map* $\mathsf{chain}(\cdot)$ *is a bijection from* $\mathsf{FT}(w)$ *to chains* $(w_n, \cdots, w_1)$ *in the Bruhat order where* $w_n = w$, $w_1 = w_0$ *and there is an increasing i-chain from* w_{i+1} *to* w_i. *Consequently,* $\mathsf{chain} \circ \Phi$ *is a bijection from* $\mathsf{BPD}(w)$ *to such chains.*

In other words, PDs and BPDs can both be viewed as certain chains in the Bruhat order, exhibiting a duality. Finally, we use Lenart's growth diagram [13] to obtain a bijection between these chains, obtaining a bijection between $\mathsf{PD}(w)$ and $\mathsf{BPD}(w)$. We conjecture this bijection agrees with the existing bijection of Gao and Huang [7]. This conjecture has been verified on S_7.

Organization: In §2, we cover some necessary background. In §3, we define the encoding map $\Phi : \mathsf{BPD}(w) \to \mathsf{FT}(w)$ and establish Theorem 3.9. In §4, we construct our BPD analogue of the Fomin-Stanley construction. In §5, we use Lenart's growth diagram to build a bijection between $\mathsf{PD}(w)$ and $\mathsf{BPD}(w)$. In §6, we describe one conjecture that extends the chain formulas of $\mathfrak{S}_w$ to double Schubert polynomials.

2 Background

2.1 Fomin-Stanley construction

A *reduced word* of $w \in S_n$ is a word $i_1 i_2 \cdots i_l$ such that $w = s_{i_1} \cdots s_{i_l}$ and l is minimized. One can read off a reduced word of w from every $P \in \mathsf{PD}(w)$ as follows: Go through its crossings from top to bottom and right to left in each row. For a crossing in row r column c, read off $r + c - 1$. For instance, the PD in Example 1.1 gives 41324 which is a reduced word of $[2, 5, 1, 4, 3]$.

The *nil-Coexter algebra* $\mathcal{N}_n$ is generated by $u_1, \cdots, u_{n-1}$ satisfying:

$$\begin{cases} u_i^2 & = 0, \\ u_i u_j & = u_j u_i \text{ if } |i-j| \geqslant 2\ , \\ u_i u_{i+1} u_i & = u_{i+1} u_i u_{i+1} \text{ if } i \in [n-2]. \end{cases}$$

Consider $a = u_{i_1} \cdots u_{i_l} \in \mathcal{N}_n$, we have $a \neq 0$ if and only if $i_1 \cdots i_l$ is a reduced word of some $w \in S_n$. In this case, $a = u_{j_1} \cdots u_{j_{l'}}$ if and only if $j_1 \cdots j_{l'}$ is a reduced word for the same w. Fomin and Stanley [6] defined the following elements in $\mathbb{Q}[x_1, \cdots, x_{n-1}] \otimes \mathcal{N}_n$:

$$\begin{aligned} A_i(x_i) &:= (1 + x_i u_{n-1})(1 + x_i u_{n-2}) \cdots (1 + x_i u_i) \text{ for } i \in [n-1], \text{ and} \\ \mathfrak{S}^{\mathsf{PD}} &:= A_1(x_1) A_2(x_2) \cdots A_{n-1}(x_{n-1}). \end{aligned}$$

Combinatorially, $\mathfrak{S}^{\mathsf{PD}} = \sum_P x^{\mathsf{wt}(P)} u_{i_1} \cdots u_{i_l}$ where the sum runs over all PD and $i_1 \cdots i_l$ is the reduced word read off from the PD. Algebraically, Fomin and Stanley showed that

$$\mathfrak{S}^{\mathsf{PD}} = \sum_{w \in S_n} \mathfrak{S}_w u_{i_1} \cdots u_{i_l}, \tag{2.1}$$

where $i_1 \cdots i_l$ is an arbitrary reduced word of w. This formula would imply the PD formula in Theorem 1.2. Fomin and Stanley proved (2.1) by showing $\partial_i(\mathfrak{S}^{\mathsf{PD}}) = \mathfrak{S}^{\mathsf{PD}} u_i$ for any $i \in [n-1]$. This equation then reduces to $\partial_i(R_i(x_i) R_{i+1}(x_{i+1})) = R_i(x_i) R_{i+1}(x_{i+1}) u_i$. In §4, we present the BPD analogue of (2.1) and establish our equation in a similar way.

2.2 Bruhat order

For $1 \leqslant i < j \leqslant n$, we use $t_{i,j}$ to denote the permutation that swaps i and j. For $w \in S_n$, let $\ell(w) := |\{(i,j) : i < j, w(i) > w(j)|$. Let $\leqslant$ be the *Bruhat order* on S_n, where the cover relation is given by $u \lessdot w$ if $w = u t_{i,j}$ and $\ell(w) = \ell(u) + 1$. We say $C = (w_1, w_2, \cdots, w_d)$ is a *Bruhat chain* from w_1 to w_d if $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_d$. The length of C is $d-1$. The *weight* of C, denoted as $\mathsf{wt}(C)$, is a sequence of length $d-1$ where the i^{th} entry is $\ell(w_{i+1}) - \ell(w_i)$. The chain is *saturated* if $w_1 \lessdot w_2 \lessdot \cdots \lessdot w_d$. We may represent a saturated chain as

$$w_1 \xrightarrow{t_{a_1,b_1}} w_2 \xrightarrow{t_{a_2,b_2}} \cdots \xrightarrow{t_{a_{d-1},b_{d-1}}} w_d,$$

where $a_i < b_i$ and $w_{i+1} = w_i t_{a_i,b_i}$.

Take $k \in [n-1]$. We use $\leqslant_k$ to denote the *k-Bruhat order* on S_n. Its cover relation is given by $u \lessdot_k w$ if $u \lessdot w$ and $w = u t_{i,j}$ for some $i \leqslant k < j$. Similarly, we can define k-Bruhat chains and saturated k-Bruhat chains. For simplicity, we say "*k-chains*" in place of "k-Bruhat chains". The k-Bruhat order can be used to describe the Monk's rule [15]:

$\mathfrak{S}_w(x_1 + \cdots + x_k) = \sum_{w <_k u} \mathfrak{S}_u$ for any $w \in S_n$ and $k \in [n-1]$ such that $w(j) = n$ for some $j > k$. Sottile generalized the Monk's rule by considering multiplying $\mathfrak{S}_w$ with

$$h_d(x_1, \cdots, x_k) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant k} x_{i_1} \cdots x_{i_d},$$

where $k \in [n-1]$ and $d \in \mathbb{Z}_{>0}$. Say a saturated k-chain $w_1 \xrightarrow{t_{a_1,b_1}} w_2 \xrightarrow{t_{a_2,b_2}} \cdots \xrightarrow{t_{a_{d-1},b_{d-1}}} w_d$ is *increasing* if $w_1(a_1) < w_2(a_2) < \cdots < w_{d-1}(a_{d-1})$. In other words, the smaller number swapped is increasing. It is not hard to show for any $u, w \in S_n$ and $k \in [n-1]$, there is at most one increasing k-chain from u to w.

Theorem 2.1. *[16] Take $u \in S_n$ and $d \in \mathbb{Z}_{\geqslant 0}$. For any $k \in [n-1]$ such that $n, n-1, \cdots, n-d+1$ are among $w(k+1), \cdots, w(n)$, then*

$$\mathfrak{S}_u \times h_d(x_1, \cdots, x_k) = \sum_w \mathfrak{S}_w.$$

The sum is over all w such that there is an increasing k-chain from u to w with length d.

Lenart and Sottile [14] view PDs as certain Bruhat chains. We introduce the following definition to describe their chains in a more general way.

Definition 2.2. We say a Bruhat chain $C = (w_1, w_2, \cdots, w_l, w_{l+1})$ is *compatible* with a sequence $(k_1, \cdots, k_l)$ if there exists an increasing k_i-chain from w_i to w_{i+1} for each $i \in [l]$.

Lenart and Sottile [14] described a bijection from $\mathsf{PD}(w)$ to chains from w to w_0 compatible with $(1, 2, \cdots, n-1)$: Take $P \in \mathsf{PD}(w)$. For $i \in [n]$, let P_i be the pipedream obtained from P by changing all bumps above row i into crossings. Let w_i be the permutation associated with P_i. Then $(w_1, \cdots, w_n)$ is the resulting chain. In addition, if we change bumps in row i of P_i into crossings from left to right, permutations of the intermediate pipedreams will form the increasing i-chain from w_i to w_{i+1}.

Example 2.3. Let P be the pipedream in Example 1.1. Then its corresponding chain is $([2,5,1,4,3], [5,3,1,4,2], [5,4,1,3,2], [5,4,3,2,1], [5,4,3,2,1])$. The increasing 1-chain from $[2,5,1,4,3]$ to $[5,3,1,4,2]$ is given by: $[2,5,1,4,3] \xrightarrow{t_{1,5}} [3,5,1,4,2] \xrightarrow{t_{1,2}} [5,3,1,4,2]$.

If a pipedream P is sent to the chain C, then $\mathsf{wt}(C) = (n-1, \cdots, 1) - \mathsf{wt}(P)$ where the subtraction is entry-wise. Thus, this bijection recovers a result of Bergeron and Sottile:

Corollary 2.4. *[2] For $w \in S_n$, $\mathfrak{S}_w = \sum_C x^{(n-1,\cdots,1)-\mathsf{wt}(C)}$, where the sum is over all chains from w to w_0 compatible with $(1, 2, \cdots, n-1)$.*

We end this section by extending Corollary 2.4 using the following observation:

Proposition 2.5. *Pick $u, w \in S_n$, $k_1, k_2 \in [n-1]$ and $d_1, d_2 \in \mathbb{Z}_{\geqslant 0}$. The number of chains from u to w compatible with (k_1, k_2) and has weight (d_1, d_2) matches the number of chains from u to w compatible with (k_2, k_1) and has weight (d_2, d_1).*

Proof. By Theorem 2.1, the number of chains (u, v, w) compatible with (k_1, k_2) and has weight (d_1, d_2) is the coefficient of $\mathfrak{S}_w$ in $\mathfrak{S}_u \times h_{d_1}(x_1, \cdots, x_{k_1}) \times h_{d_2}(x_1, \cdots, x_{k_2})$. The proof is finished by the commutativity of polynomial multiplication. □

Since we have two sets with the same size, it would be natural to ask:

Problem 2.6. Find an explicit bijection between the two set of chains in Proposition 2.5.

In §5, we show Lenart's growth diagram [13] solves Problem 2.6 in a special case.

Combining Corollary 2.4 and Proposition 2.5, we deduce:

Corollary 2.7. *Take $w \in S_n$ and $\gamma \in S_{n-1}$. If $(d_1, \cdots, d_{n-1})$ is a sequence of numbers, let $\gamma^{-1}(d_1, \cdots, d_{n-1}) := (d_{\gamma^{-1}(1)}, \cdots, d_{\gamma^{-1}(n-1)})$. We also view γ as a sequence of numbers. Then $\mathfrak{S}_w = \sum_C x^{(n-1,\cdots,1)-\gamma^{-1}(\mathsf{wt}(C))}$, summing over all chains from w to w_0 compatible with γ.*

This corollary implies that we have a combinatorial formula of $\mathfrak{S}_w$ involving Bruhat chains for each choice of $\gamma \in S_{n-1}$. Under Lenart and Sottile's bijection, the PD formula is identified with the Bruhat chain formula when $\gamma = [1, 2, \cdots, n-1]$. In §3, we identify the BPD formula with the Bruhat chain formula when $\gamma = [n-1, n-2, \cdots, 1]$.

3 Encoding BPDs as flagged tableaux and chains

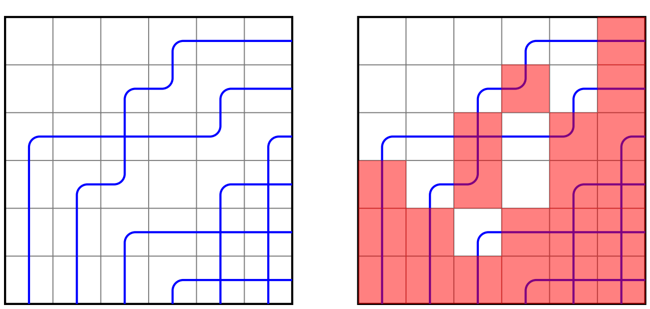
We first encode each BPD as the following combinatorial object.

Definition 3.1. A *flagged tableau* is a staircase grid with a cell in row i column j if $i + j \leqslant n$. Moreover, each cell in row i is empty or filled with a number in $[i]$.

We define an encoding map Φ from $\mathsf{BPD}(w)$ to the set of flagged tableaux.

Definition 3.2. Take $D \in \mathsf{BPD}(w)$ for some $w \in S_n$. For $i \in [n]$, there are $(i-1)$ pipes exiting from the top from row i of D, so there are $(i-1)$, and . We mark these cells, and then mark the rightmost unmarked cell in row i. There will be $n-i$ unmarked cells. To fill the cell in row i column j of $\Phi(D)$, we look at the j^{th} leftmost unmarked cell in row i of D. If it is a blank, we leave the cell in $\Phi(D)$ unfilled. Otherwise, it contains a pipe that ends in row p for some $p \leqslant i$. We fill the cell in $\Phi(D)$ by p.

Example 3.3. Assume $n = 6$. Take $D \in \mathsf{BPD}([2, 1, 6, 5, 3, 4])$ as depicted on the left. Then we perform the encoding algorithm and mark certain cells red. Finally, we obtain $\Phi(D)$.



$\Phi(D) =$

			1	1
		1	2	
2	2	2		
1				
5				

To precisely describe the image of $\mathsf{BPD}(w)$ under Φ, we need the following definition.

Definition 3.4. The *reading word* of a flagged tableau T, denoted as $\mathsf{word}(T)$, is a sequence of pairs obtained as follows. Go through entries of T from top to bottom, and right to left in each row. When we see the number i in column c, we write the pair $(i, n+1-c)$.

By the definition of flagged tableaux, for each pair in the reading word, the first entry is smaller than the second.

Example 3.5. In Example 3.3, $\mathsf{word}(\Phi(D)) = (1,2)(1,3)(2,3)(1,4)(2,4)(2,5)(2,6)(1,6)(5,6)$.

Let T be a flagged tableau with reading word $(a_1, b_1), \cdots, (a_d, b_d)$. For $i \in [d]$, we let $w_i = w_0 t_{a_1,b_1} \cdots t_{a_i,b_i}$. Then we say T is *associated* with the permutation w_d if

$$w_d \xrightarrow{t_{a_d,b_d}} w_{d-1} \xrightarrow{t_{a_{d-1},b_{d-1}}} \cdots \xrightarrow{t_{a_2,b_2}} w_1 \xrightarrow{t_{a_1,b_1}} w_0$$

is a saturated Bruhat chain. Let $\mathsf{FT}(w)$ consist of all flagged tableaux associated with w.

Example 3.6. In Example 3.3, $\Phi(D)$ is associated with $[2,1,6,5,3,4]$ because:

$$[2,1,6,5,3,4] \xrightarrow{t_{5,6}} [2,1,6,5,4,3] \xrightarrow{t_{1,6}} [3,1,6,5,4,2] \xrightarrow{t_{2,6}} [3,2,6,5,4,1] \xrightarrow{t_{2,5}} [3,4,6,5,2,1]$$
$$\xrightarrow{t_{2,4}} [3,5,6,4,2,1] \xrightarrow{t_{1,4}} [4,5,6,3,2,1] \xrightarrow{t_{2,3}} [4,6,5,3,2,1] \xrightarrow{t_{1,3}} [5,6,4,3,2,1] \xrightarrow{t_{1,2}} [6,5,4,3,2,1]$$

is a saturated Bruhat chain from $[2,1,6,5,3,4]$ to w_0. Notice that $D \in \mathsf{BPD}([2,1,6,5,3,4])$.

For a flagged tableau T, define the *weight* of T, denoted as $\mathsf{wt}(T)$, to be a sequence of $n-1$ numbers whose i^{th} entry is the number of blanks in row i. Then we have:

Proposition 3.7. *For $w \in S_n$, Φ is a weight-preserving bijection from $\mathsf{BPD}(w)$ to $\mathsf{FT}(w)$.*

We may turn T into a chain compatible with $(n-1, \cdots, 2, 1)$ as follows. Suppose T has reading word $(a_1, b_1), \cdots, (a_d, b_d)$ and set $w_i = w_0 t_{a_1,b_1} \cdots t_{a_i,b_i}$ for $i \in [d]$. Let m_i be the number of non-empty cells above row $i+1$ of T for $i = 0, 1. \cdots, n-1$. Clearly, $(w_{m_i}, w_{m_i - 1}, \cdots, w_{m_{i-1}})$ is an i-chain. Moreover, we can check it is an increasing i-chain. Then define $\mathsf{chain}(T) := (w_{m_{n-1}}, \cdots, w_{m_1}, w_0)$, which is compatible with $(n-1, \cdots, 2, 1)$.

Example 3.8. Let T be the $\Phi(D)$ in Example 3.3. Then $\mathsf{chain}(T)$ is

$$([2,1,6,5,3,4], [2,1,6,5,4,3], [3,1,6,5,4,2], [3,5,6,4,2,1], [4,6,5,3,2,1], [6,5,4,3,2,1]).$$

Theorem 3.9. *The map $\mathsf{chain}(\cdot)$ is a bijection from $\mathsf{FT}(w)$ to Bruhat chains from w to w_0 compatible with $(n-1, \cdots, 2, 1)$. Consequently, $\mathsf{chain} \circ \Phi$ is a bijection from $\mathsf{BPD}(w)$ to such chains.*

The bijection $\mathsf{chain} \circ \Phi$ is an analogue of Lenart and Sottile's bijection [14] on $\mathsf{PD}(w)$. Notice that for $D \in \mathsf{BPD}(w)$, if $\mathsf{wt}(D) = (\alpha_1, \cdots, \alpha_{n-1})$ then

$$\mathsf{wt}(\mathsf{chain}(\Phi(D))) = (1 - \alpha_{n-1}, \cdots, n-2-\alpha_2, n-1-\alpha_1).$$

Thus, we have identified the BPD formula of $\mathfrak{S}_w$ with the Bruhat chain formula in Corollary 2.7 with $\gamma = [n-1, \cdots, 2, 1]$.

4 Analogue of Fomin-Stanley construction on BPDs

We now construct $\mathfrak{S}^{\mathsf{BPD}}$, our analogue of $\mathfrak{S}^{\mathsf{PD}}$, as a generating function of the flagged tableaux, or equivalently BPDs. Instead of the nil-Coexter algebra $\mathcal{N}_n$, our construction uses the *Fomin-Kirillov algebra* [5] $\mathcal{E}_n$, generated by $\{d_{i,j} : 1 \leqslant i < j \leqslant n\}$ satisfying:

$$\begin{cases} d_{i,j}^2 & = 0 \text{ if } i < j \ , \\ d_{i,j}d_{j,k} & = d_{i,k}d_{i,j} + d_{j,k}d_{i,k} \text{ if } i < j < k \ , \\ d_{j,k}d_{i,j} & = d_{i,j}d_{i,k} + d_{i,k}d_{j,k} \text{ if } i < j < k \ , \\ d_{i,j}d_{k,l} & = d_{k,l}d_{i,j} \quad \text{if } i < j,\, k < l \text{ and } i, j, k, l \text{ distinct.} \end{cases}$$

Fomin and Kirillov described an action of $\mathcal{E}_n$ on $\mathbb{Q}[S_n]$. In this paper, we adopt a slightly different convention and consider a right action of $\mathcal{E}_n$ on $\mathbb{Q}[S_n]$. For $w \in S_n$,

$$w \odot d_{i,j} := \begin{cases} wt_{i,j} & \text{if } wt_{i,j} \lessdot w \\ 0 & \text{otherwise.} \end{cases}$$

Define $A := \mathbb{Q}[x_1, \cdots, x_{n-1}] \otimes \mathcal{E}_n$. It acts on $\mathbb{Q}[x_1, \cdots, x_{n-1}][S_n]$ from the right: $(fw) \odot (g \otimes e) = (fg)(w \odot e)$ for any $f, g \in \mathbb{Q}[x_1, \cdots, x_{n-1}]$, $w \in S_n$ and $e \in \mathcal{E}_n$. We may identify $\mathcal{E}_n$ and $\mathbb{Q}[x_1, \cdots, x_{n-1}]$ as subalgebras of A.

Definition 4.1. Take $i \in [n-1]$. For $i < j$, define $B_{i,j} \in \mathcal{E}_n$ as $B_{i,j} := d_{1,j} + \cdots + d_{i,j}$. Define $R_i(x_i) \in A$ as $R_i(x_i) := (x_i + B_{i,i+1})(x_i + B_{i,i+2}) \cdots (x_i + B_{i,n})$. Finally, define $\mathfrak{S}^{\mathsf{BPD}} \in \mathbb{Q}[x_1, \cdots x_{n-1}][S_n]$ as $\mathfrak{S}^{\mathsf{BPD}} := w_0 \odot (R_1(x_1)R_2(x_2) \cdots R_{n-1}(x_{n-1}))$.

We show $\mathfrak{S}^{\mathsf{BPD}}$ is a generating function of flagged tableaux, or equivalently all BPDs:

Proposition 4.2. *We have*

$$\mathfrak{S}^{\mathsf{BPD}} = \sum_{w \in S_n} \sum_{T \in \mathsf{FT}(w)} x^{\mathsf{wt}(T)} w = \sum_{w \in S_n} \sum_{D \in \mathsf{BPD}(w)} x^{\mathsf{wt}(D)} w.$$

Proof. If we expand $R_i(x_i)$, each term corresponds to one way of filling row i of a flagged tableau. The expression $(x_i + B_{i,j})$ in $R_i(x_i)$ corresponds to ways of filling the cell at row i and column $n+1-j$: x_i means to leave the box empty and $d_{p,j}$ means to fill it with p. If we expand $R_1(x_1) \cdots R_{n-1}(x_{n-1})$, for each term $x^\alpha d_{a_1,b_1} \cdots d_{a_k,b_k}$, there is a flagged tableau T with $\mathsf{wt}(T) = x^\alpha$ and $\mathsf{word}(T) = (a_1, b_1) \cdots (a_k, b_k)$. Let $w = w_0 \odot d_{a_1,b_1} \cdots d_{a_k,b_k}$. If $w = 0$, we know T is not associated with any permutation. Otherwise, $T \in \mathsf{FT}(w)$. Thus, we have the first equation. The second equation follows from Proposition 3.7. $\square$

Now we establish the BPD analogue of (2.1).

Theorem 4.3. *We have* $\mathfrak{S}^{\mathsf{BPD}} = \sum_{w \in S_n} \mathfrak{S}_w w$.

Our proof is similar to the arguments of Fomin and Stanley. Consider a right action of $\mathcal{N}_n$ on S_n with $w \odot u_i = wt_{i,i+1}$ if $w(i) < w(i+1)$ and $w \odot u_i = 0$ otherwise. We may extend this action to $\mathbb{Q}[x_1, \cdots, x_{n-1}][S_n]$ by setting $f \odot u_i = f$ for all $f \in \mathbb{Q}[x_1, \cdots, x_{n-1}]$. Similar to Fomin and Stanley's approach, Theorem 4.3 reduces to:

Proposition 4.4. *For each $i \in [n-1]$, $\partial_i(\mathfrak{S}) = \mathfrak{S} \odot u_i$.*

Proof Sketch. The left hand side is just $w_0 \odot R_1(x_1) \cdots \partial_i(R_i(x_i)R_{i+1}(x_{i+1})) \cdots R_{n-1}(x_{n-1})$. We turn the right hand side into $w_0 \odot R_1(x_1) \cdots R_i(x_i)\, u_{i,i+1}\, R_{i+1}(x_{i+1}) \cdots R_{n-1}(x_{n-1})$. Then we show $w_0 \odot R_1(x_1) \cdots R_{i-1}(x_{i-1})$ is in the span of terms $x^\alpha w$ where x^α is a monomial involving $x_1, \cdots, x_{i-1}$ and $w \in S_n$ satisfies $w(i+1) > \cdots > w(n)$. We just need

$$x^\alpha w \odot \partial_i((R_i(x_i)R_{i+1}(x_{i+1})) = x^\alpha w \odot R_i(x_i)\, u_{i,i+1}\, R_{i+1}(x_{i+1}) \text{ for such } x^\alpha w.$$

We then establish this equation via a complicated but routine computation. □

Fomin and Kirillov [4] defined the *Dunkl element* $\theta_i := -\sum_{j<i} d_{j,i} + \sum_{j>i} d_{i,j} \in \mathcal{E}_n$ for $i \in [n]$. They showed the Dunkl elements $\theta_1, \cdots, \theta_n$ commute with each other. We end this subsection by providing an alternative way to write $\mathfrak{S}^{\mathsf{BPD}}$ using Dunkl elements.

Proposition 4.5. *We have $\mathfrak{S}^{\mathsf{BPD}} = w_0 \odot \prod_{1 \leqslant i < j \leqslant n}(x_i - \theta_j)$. Notice that terms multiplied on the right hand side commute with each other, so the $\prod$ notation makes sense.*

Remark 4.6. Sergey Fomin kindly informed the author that $w_0 \odot \prod_{1 \leqslant i < j \leqslant n}(x_i - \theta_j)$ seems related to the following variation of Cauchy identity of Schubert polynomials:

$$\prod_{1 \leqslant i < j \leqslant n} (x_i - y_j) = \sum_{w \in S_n} \mathfrak{S}_w(x_1, \cdots, x_{n-1})\mathfrak{S}_{ww_0}(-y_n, \cdots, -y_2). \tag{4.1}$$

Indeed, by the Monk's rule, (4.1) is equivalent to $w_0 \odot \prod_{1 \leqslant i < j \leqslant n}(x_i - \theta_j) = \sum_{w \in S_n} \mathfrak{S}_w w$. In other words, Theorem 4.3 and Proposition 4.5 form an alternative proof of (4.1).

5 Bijection between pipedreams and bumpless pipedreams

In this section, we present a weight preserving bijection between $\mathsf{PD}(w)$ and $\mathsf{BPD}(w)$. By [14] and Theorem 3.9, we just need a weight reversing bijection between chains from w to w_0 compatible with $(1, \cdots, n-1)$ and those compatible with $(n-1, \cdots, 1)$.

This task can be done by Lenart's growth diagram [13], which can be viewed as the following algorithm. Given $k_1, k_2 \in [n-1]$ and chains C_1, C_2, where C_1 (resp. C_2) is a saturated k_1-chain from u to v (resp. k_2-chain from v to w), the algorithm outputs a saturated k_2-chain from u to v' and a saturated k_1-chain from v' to w. Moreover, the k_1-chain (resp. k_2-chain) in the output has the same length as C_1 (resp. C_2).

Assume $C_1 = (u_1, \cdots, u_{d_1})$ and $C_2 = (w_1, \cdots, w_{d_2})$ where $u_{d_1} = w_1$. We first draw:

$$u_1 \xrightarrow{k_1} u_2 \xrightarrow{k_1} \cdots \xrightarrow{k_1} u_{d_1-1} \xrightarrow{k_1} w_1 \xrightarrow{k_2} w_2 \xrightarrow{k_2} \cdots \xrightarrow{k_2} w_{d_2}.$$

We start from this labeled chain and apply a local move: Find a part of the chain that looks like $a \xrightarrow{k_1} b \xrightarrow{k_2} c$. We must have $a \lessdot_{k_1} b \lessdot_{k_2} c$. There exists a unique $b' \in S_n$ such that $b' \neq b$ and $a \lessdot b' \lessdot c$. If $a \lessdot_{k_2} b' \lessdot_{k_1} c$, we replace this part of the chain by $a \xrightarrow{k_2} b' \xrightarrow{k_1} c$. Otherwise, we must have $a \lessdot_{k_2} b' \lessdot_{k_1} c$ and we replace this part by $a \xrightarrow{k_2} b \xrightarrow{k_1} c$. We keep applying this local move until the labeled chain looks like:

$$u'_1 \xrightarrow{k_2} u'_2 \xrightarrow{k_2} \cdots \xrightarrow{k_2} u'_{d_2-1} \xrightarrow{k_2} w'_1 \xrightarrow{k_1} w'_2 \xrightarrow{k_1} \cdots \xrightarrow{k_1} w'_{d_1}.$$

Then we output the k_2-chain $(u'_1, \cdots, u'_{d_2-1}, w'_1)$ and the k_1-chain $(w'_1, \cdots, w'_{d_1})$.

Example 5.1. Say the inputs are: $k_1 = 2$, $k_2 = 3$, $C_1 = ([2,1,4,3],[2,4,1,3],[3,4,1,2])$, and $C_2 = ([3,4,1,2],[3,4,2,1])$. We start from the following labeled chain and apply local moves:

$$[2,1,4,3] \xrightarrow{2} [2,4,1,3] \xrightarrow{2} [3,4,1,2] \xrightarrow{3} [3,4,2,1].$$

$$[2,1,4,3] \xrightarrow{2} [2,4,1,3] \xrightarrow{3} [2,4,3,1] \xrightarrow{2} [3,4,2,1],$$

$$[2,1,4,3] \xrightarrow{3} [2,3,4,1] \xrightarrow{2} [2,4,3,1] \xrightarrow{2} [3,4,2,1].$$

Therefore, the outputs are $([2,1,4,3],[2,3,4,1])$ and $([2,3,4,1],[2,4,3,1],[3,4,2,1])$.

We may use Lenart's growth diagram to define a map $\mathsf{growth}_{k_1,k_2}$.

Definition 5.2. Take a chain (u, v, w) that is compatible with (k_1, k_2). Let C_1 (resp. C_2) be the increasing k_1-chain (resp. k_2-chain) from u to v (resp. v to w). Input C_1, C_2, k_1, k_2 to Lenart's growth diagram, obtaining a k_2-chain from u to v' and a k_1-chain from v' to w. Then define $\mathsf{growth}_{k_1,k_2}(u, v, w)$ as (u, v', w).

The map $\mathsf{growth}_{k_1,k_2}$ does not solve Problem 2.6. When (u, v, w) is compatible with (k_1, k_2), $\mathsf{growth}_{k_1,k_2}(u, v, w)$ might not be compatible with (k_2, k_1): By Example 5.1, we have

$$\mathsf{growth}_{2,3}([2,1,4,3],[3,4,1,2],[3,4,2,1]) = ([2,1,4,3],[2,3,4,1],[3,4,2,1]),$$

which is not compatible with $(3,2)$, but $([2,1,4,3],[3,4,1,2],[3,4,2,1])$ is compatible with $(2,3)$. Nevertheless, $\mathsf{growth}_{k_1,k_2}$ solves Problem 2.6 in the following special case.

Lemma 5.3. *Take* $1 \leqslant k_2 < k_1 \leqslant n-1$ *and* $u, w \in S_n$ *such that* $w(k_1+1) > w(k_1+2) > \cdots > w(n)$ *and* $w(j) = n+1-j$ *for each* $j \in [k_2]$. *Then* $\mathsf{growth}_{k_1,k_2}$ *is a weight reversing bijection from chains* (u, v, w) *compatible with* (k_1, k_2) *to chains* (u, v', w) *compatible with* (k_2, k_1).

Now we use $\mathsf{growth}_{k_1,k_2}$ to derive a map Growth. This map is defined on a chain $C = (w_n, \cdots, w_1)$ from w to w_0 compatible with $(n-1, \cdots, 2, 1)$. It first applies $\mathsf{growth}_{2,1}$, $\mathsf{growth}_{3,1}$, $\cdots$, $\mathsf{growth}_{n-1,1}$ to get a chain compatible with $(1, n-1, \cdots, 2)$. Then it applies $\mathsf{growth}_{3,2}, \cdots, \mathsf{growth}_{n-1,2}$ to get a chain compatible with $(1, 2, n-1, \cdots, 3)$. Eventually, it produces a chain compatible with $(1, 2, \cdots, n-1)$ defined as $\mathsf{Growth}(C)$. We can check when we apply each $\mathsf{growth}_{k_1,k_2}$, the condition in Lemma 5.3 is satisfied.

Proposition 5.4. *For $w \in S_n$, the map* Growth *is a weight-reversing bijection from* $\{$*chains from w to w_0 compatible with* $(n-1, \cdots, 1)\}$ *to* $\{$*chains from w to w_0 compatible with* $(1, \cdots, n-1)\}$.

By [14] and Theorem 3.9, Growth leads to a weight preserving bijection between $\mathsf{PD}(w)$ and $\mathsf{BPD}(w)$. We conjecture this map agrees with the existing bijection of Gao-Huang [7] and have checked our conjecture up to S_7.

Example 5.5. Consider the chain $([2,1,4,3],[2,3,4,1],[2,4,3,1],[4,3,2,1])$ which is compatible with $(3,2,1)$ and has weight $(1,1,2)$. We apply $\mathsf{growth}_{2,1}$ and then $\mathsf{growth}_{3,1}$ to get $([2,1,4,3],[4,1,3,2],[4,2,3,1],[4,3,2,1])$ which is compatible with $(1,3,2)$ and has weight $(2,1,1)$. Finally, use $\mathsf{growth}_{3,2}$ to get $([2,1,4,3],[4,1,3,2],[4,3,1,2],[4,3,2,1])$ which is compatible with $(1,2,3)$ and has weight $(2,1,1)$.

6 Extending Corollary 2.7 to double Schubert polynomials

The *double Schubert polynomial* $\mathfrak{S}_w(\mathbf{x},\mathbf{y})$ is in $x_1, \cdots, x_{n-1}$ and $y_1, \cdots, y_{n-1}$. It recovers $\mathfrak{S}_w$ after setting each y_i to 0 and can be computed using PDs and BPDs: For $P \in \mathsf{PD}(w)$ (resp. $\mathsf{BPD}(w)$), let $\mathsf{WT}(P)$ be the product over ⊞ (resp. □) in P, where the tile in row i column j gives $(x_i - y_j)$. By [9, 17], $\mathfrak{S}_w(\mathbf{x},\mathbf{y}) = \sum_{P \in \mathsf{PD}(w)} \mathsf{WT}(P) = \sum_{P \in \mathsf{BPD}(w)} \mathsf{WT}(P)$.

Take $\gamma \in S_{n-1}$ and let $C = (w_1, \cdots, w_n)$ be a chain compatible with γ. Define $\mathsf{WT}_\gamma(C)$ as $\prod_{i=1}^{n-1} \prod_t (x_{\gamma_i} - y_{w_i(t)})$, where t runs over all $t > \gamma_i$ such that $w_i(t) = w_{i+1}(t)$. After setting all y_i to 0, $\mathsf{WT}_\gamma(C)$ recovers $x^{(n-1,\cdots,1)-\gamma^{-1}(\mathsf{wt}(C))}$. The following conjecture extends Corollary 2.7 and has been checked for all $w \in S_n$ for $n \leqslant 8$ and all $\gamma \in S_{n-1}$:

Conjecture 6.1. *For $\gamma \in S_{n-1}$, we have* $\mathfrak{S}_w(\boldsymbol{x},\boldsymbol{y}) = \sum_{C:\text{chain from } w \text{ to } w_0 \text{ compatible with } \gamma} \mathsf{WT}_\gamma(C)$.

This conjecture agrees with the PD and BPD formula when $\gamma = [1, \cdots, n-1]$ and $\gamma = [n-1, \cdots, 1]$ respectively via the bijections in [14] and Theorem 3.9.

Acknowledgements

We thank Yibo Gao for suggesting this project. We thank Yibo Gao and Zachary Hamaker for many important suggestions, including using Lenart's Growth diagram to obtain a bijection between PDs and BPDs. We thank Yuhan Jiang and Brendon Rhoades for carefully reading an earlier version of this paper and providing valuable comments.

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