UNIVERSITY OF CALIFORNIA, SAN DIEGO

The Hecke algebra of type B at roots of unity, Markov traces and subfactors

A dissertation submitted in partial satisfaction of the
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in

Mathematics

by

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The dissertation of Rosa C. Orellana is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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To Pedri

Que Dios lo bendiga
hoy y siempre.

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ABSTRACT OF THE DISSERTATION

The Hecke algebra of type B at roots of unity, Markov traces and subfactors

by

Rosa C. Orellana
Doctor of Philosophy in Mathematics

University of California San Diego, 1999

Professor Hans Wenzl, Chair

The Hecke algebra of type B_n, $H_n(q, Q)$, is semisimple for generic values of q and Q. Its simple components are indexed by double partitions (λ, μ) of n.

We have constructed a nontrivial homomorphism from the specialized Hecke algebra of type B, $H_n(q, -q^k)$ onto a reduced Hecke algebra of type A for q not equal to 1. This homomorphism has proven to be a useful tool to reduce questions about the Hecke algebra of type B to the Hecke algebra of type A.

An immediate consequence of the existence of this homomorphism is that the Hecke algebra of type B appears as a commutant of the quantum group $U_q(\mathfrak{sl}(r))$. Using this homomorphism and the results from Wenzl [W1] on the Hecke algebra of type A, we have solved the following three problems:

A family of traces, depending on two parameters, has been defined by Geck and Lambropoulou [GL] motivated by their study of knots in a solid torus. We have computed the weights, i.e. values at minimal idempotents, of these traces. Wenzl [W1] obtained that the weights for the Hecke algebra of type A were specializations of Schur functions. Here we obtain a new class of functions labeled by pairs of Young diagrams. We give an expression of this function as products of Schur functions and a simple factor.

All simple modules of the Hecke algebra of type B at roots of unity have been constructed [DGM], however the dimensions are not known in general. Using this homomorphism we can explicitly describe many nontrivial simple modules. The dimensions of these modules can be computed using a generalization of the Littlewood-Richardson rule.

We have also been able to construct examples of subfactors from the inclusion of the Hecke algebra of type A into the Hecke algebra of type B.

Finally, using the results from the Hecke algebra of type B, we have been able to derive results for the Hecke algebra of type D.

Chapter 1

Introduction

Hecke algebras are of interest to a wide audience, since they arise naturally in the study of knots and links, quantum groups, and von Neumann algebras.

The Hecke algebra of type B is a two parameter deformation of the group algebra of the Weyl group of type B (also known as the hyperoctahedral group). We can think of the parameters as complex numbers. The representations of the Hecke algebra depend on rational functions in two parameters.

The Hecke algebra of type B_n, denoted by $H_n(q, Q)$, is semisimple whenever $Q \neq -q^k$, $k \in \{0, \pm 1, \ldots, \pm(n-1)\}$, and q is not a root of unity, see [DJ]. The simple components are indexed by ordered pairs of Young diagrams. These Hecke algebras can be defined as a finite dimensional quotient of the group algebra of the braid group of type B.

Motivated by their study of link invariants related to the braid group of type B, Geck and Lambropoulou [GL] have defined certain linear traces on the Hecke algebra of type B called Markov traces. Their definition is given inductively.

In this thesis we give an alternative way of computing this trace. Since the Hecke algebra of type B is semisimple, any linear trace can be written as a weighted linear combination of the irreducible characters (the usual trace). The coefficients in this linear expression are called weights. The weights are equal to the values of the trace at the minimal idempotents. Since the characters are known, it follows that the weights completely determine the trace. The weights are also indexed by ordered pairs of Young diagrams.

We have found the weight formula for the Markov trace defined by Geck and Lambropoulou [GL] for the Hecke algebra of type B. The weight formula can be written as a product of Schur functions and a simple factor. To prove this formula we construct a homomorphism from the specialization of the Hecke algebra of type B, $H_n(q, -q^{r_1+m})$, onto a reduced Hecke algebra of type A. Using this homomorphism we obtain that the Markov trace of the Hecke algebra of type B appears as a pullback of the Markov trace of the reduced Hecke algebra of type A.

A consequence of the above mentioned homomorphism is the existence of a duality between the quantum group $U_q(\mathfrak{sl}(r))$ and the specialized Hecke algebra $H_n(q, -q^{r_1+m})$.

In this thesis we also compute the weights for the Markov trace on the Hecke algebra of type D. We use the results of Hoefsmit [H] on the inclusion of the Hecke algebra of type D into the Hecke algebra of type B. We also use the results of Geck [G] on obtaining Markov traces of the Hecke algebra of type D from those of type B.

In [W1] Wenzl found examples of subfactors of the hyperfinite II_1 factor by studying the complex Hecke algebras of type A, denoted by $H_n(q)$. In this thesis we construct examples of subfactors obtained from the inclusion of the Hecke algebra of type A into the Hecke algebra of type B, $H_n(q, Q)$. To do this we must find the values of the parameters of the Hecke algebra of type B for which the inductive limit, i.e. $H_\infty(q, Q) = \bigcup_{n \geq 0} H_n(q, Q)$, has C^* representations. We show that there are C^* representations when $q = e^{2\pi i/l}$ and $Q = -q^k$ for some positive integers l and k.

We show that the surjective homomorphism described above is well-defined and onto when q is a root of unity. This implies that there exist quotients of the reduced Hecke algebra of type A which are isomorphic to quotients of $H_n(q, -q^{r_1+m})$ at roots of unity. These quotients are C^* algebras and we use them to construct the II_1 hyperfinite factor.

The Markov traces defined by Geck and Lambropoulou [GL] make the C^* algebras obtained from sequences of Hecke algebras of type B into a II_1 factor. Moreover, they also satisfy the commuting square property needed for the construction of subfactors which involves the conditional expectations of various subalgebras.

The subfactors obtained from the inclusion of the Hecke algebra of type A into the Hecke algebra of type B are equivalent to special cases of subfactors already obtained in [W1] for the Hecke algebras of type A. We compute the index and higher relative commutants for these subfactors. We found that the index is related to the Schur function of a rectangular diagram.

We also obtain intermediate subfactors of index two by studying the inclusion of the Hecke algebra of type D into the Hecke algebra of type B. We also study the inclusion of the Hecke algebra of type A into the Hecke algebra of type D. We compute the index for these subfactors.

This thesis is organized as follows. In Chapter 2 we give notation and general ideas used throughout the thesis. In Chapter 3, we give general information for the Hecke algebras and give the surjective homomorphism, we show that it is well-defined at roots of unity.

In Chapter 4, we give the weight function of the Markov trace defined by Geck and Lambropoulou [GL]. In Chapter 5, we show how to construct examples of subfactors and give the index and higher relative commutants. Finally, in Chapter 6, we show how the results obtained for the Hecke algebra of type B can be used to find similar results for the Hecke algebra of type D.

Chapter 2

Preliminaries

2.1 Partitions and Tableaux

A *partition* of n is a sequence of positive numbers $\alpha = [\alpha_1, \ldots, \alpha_k]$ such that

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq 0 \quad \text{and} \quad n = |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_k.$$

here $1 \leq k \leq n$. $|\alpha|$ is called the *weight* of α. The α_i's are called the *parts* of α, and the number of nonzero parts of α is called the *length* of α, denoted by $l(\alpha)$. If we say $l(\alpha) \leq r_1$, then we we will mean that there are $s \leq r_1$ nonzero parts and the remaining $r_1 - s$ are equal to zero. We use the notation $\alpha \vdash n$ to mean α is a partition of n.

A *Young diagram* is a pictorial representation of a partition α as an array of n boxes with α_1 boxes in the first row, α_2 boxes in the second row, and so on. We count the rows from top to bottom. We shall denote the Young diagram and the partitions by the same symbol α.

$$\alpha = [3, 2, 1] \longleftrightarrow$$

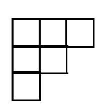

Figure 2.1: Example of a Young diagram.

We use the word *shape* interchangeably with the word partition. The set of all partitions of n is denoted by Λ_n. The square in the i-th row and j-th column of α is said to have *coordinates* (i, j).

If α and β are two partitions with $|\alpha| \leq |\beta|$, then we write $\alpha \subset \beta$ if $\alpha_i \leq \beta_i$ for all i. In this case we say that α is contained in β. If $\alpha \subset \beta$ then the set-theoretic difference $\beta - \alpha$ is called a *skew diagram*, β/α.

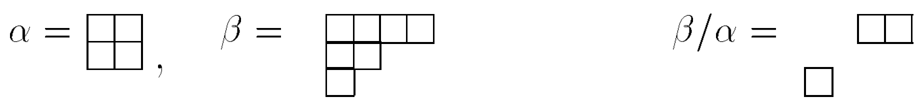

Figure 2.2: Example of a skew diagram.

Notice that in Figure 2.2, β/α can be interpreted as pair, $([2],[1])$. A *standard tableau* of shape α is a filling of the boxes with numbers $1, 2, \ldots, n$ such that the numbers in each row increase from left to right and in every column from top to bottom. The notation t^α will be used to denote a standard tableaux of shape α. We say t^α is contained in t^β, denoted $t^\alpha \subset t^\beta$, if t^α is obtained by removing appropriate boxes from t^β, i.e. the numbers $1, 2, \ldots, |\alpha|$ are in the same boxes in t^α and in t^β.

A *double partition* of size n, denoted by (α, β), is an ordered pair of partitions α and β such that $|\alpha| + |\beta| = n$. The length of a double partition is $l(\alpha, \beta) = l(\alpha) + l(\beta)$. A double partition can be associated with a pair of Young diagrams. The set of all pairs of Young diagrams with n boxes is denoted by Λ_n^D. We define as for single partitions $(\alpha, \beta) \subset (\delta, \gamma)$ if $\alpha_i \leq \delta_i$ and $\beta_j \leq \gamma_j$ for all i and j. That is, (α, β) is obtained from (δ, γ) by removing appropriate boxes of (δ, γ), and we say (α, β) is contained in (δ, γ).

$t^{(\alpha,\beta)} = (t^\alpha, t^\beta)$ is a pair of standard tableaux if the arrangement of the numbers $1, 2, \cdots, n$ is in increasing order in the rows and columns of both t^α and t^β. For any double partition (α, β), let $T_{(\alpha,\beta)}$ denote the set of standard tableaux of shape (α, β) and $\mathcal{T}_n$ denote the set of all pairs of tableaux with n boxes. For example, the diagram in Figure 2.3 is a pair of standard tableaux of shape $([2,1,1],[3,2])$

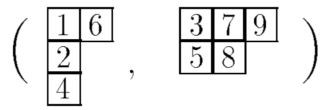

Figure 2.3: Example of a pair of tableaux

We say that a box in (α, β) has coordinates (i, j) if the box is in the i-th row and j-th column of either α or β. Two boxes in (α, β) can have the same coordinates if they occur in the same box in α as in β; for instance, the left-top-most box in α and the left-top-most box in β both have coordinates $(1, 1)$.

Definition 2.1.1. For any two boxes in a Young diagram or pair of Young diagrams with coordinates (i, j) and (s, t) respectively, define the axial distance, d from the box (i, j) to the box (s, t) as follows

$$d = (t - s) - (j - i). \tag{2.1}$$

The axial distance can be interpreted graphically as follows. If both boxes are in the same diagram. Start at the box (i, j) and proceed in a rectangular manner to the box (s, t), count +1 for each step to the right or up and -1 for every step to the left or down. The sum of these +1 and -1 give the axial distance. If the boxes are in different Young diagrams we superimpose the diagram β upon α and proceed as before.

Definition 2.1.2. The axial distance from the number m to the number l in the tableau $\tau^{(\alpha,\beta)}$ is the axial distance from the box containing m to the box containing l in (α, β).

The following observation will be used throughout the sequel. For $m, r_1, n \in \mathbb{N}$, let $m > n$ and $r_1 > n$. Consider a pair of Young diagrams (α, β) with n boxes. We construct from the pair (α, β) a Young diagram μ with $mr_1 + n$ boxes by adjoining a rectangular box with r_1 rows and m columns, as in Figure 2.4. The diagram described corresponds to the following partition:

$$\mu = [m + \alpha_1, m + \alpha_2, \ldots, m + \alpha_{r_1}, \beta_1, \beta_2, \ldots, \beta_{r_2}]$$

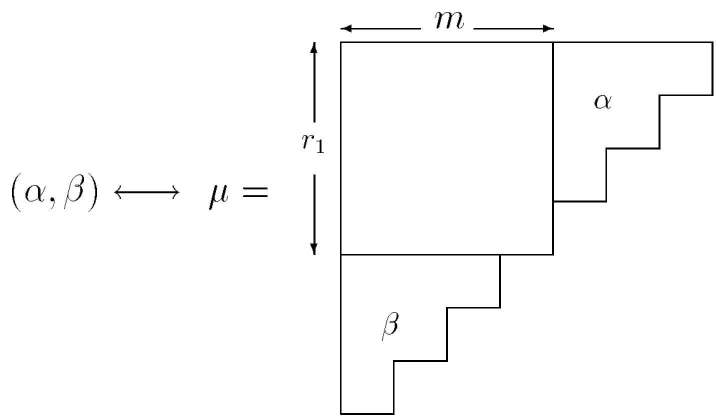

Figure 2.4: A correspondence between pairs of diagrams and one diagram

Observation: Let n, m, and r_1 be as above. Let $[m^{r_1}] \vdash f$. Then there is a 1-1 correspondence between pairs of Young diagrams with n boxes and Young diagrams containing $[m^{r_1}]$ with $n + f$ boxes.

2.2 Generalities

For convenience, by a *semisimple* algebra we mean a finite direct sum of full matrix rings. Let k be a field of characteristic 0 and let $k(x)$ denote the field of rational functions over k. The algebra of $n \times n$ matrices is denoted by $M_n(k)$ or just M_n. So if A and B are semisimple algebras, then we can write them as $A = \bigoplus A_i$ and $B = \bigoplus B_j$ with $A_i \cong M_{a_i}(k)$ and $B_j \cong M_{b_j}(k)$ for appropriate natural numbers a_i and b_j.

Let A be a subalgebra of B. Any simple B_j module is also an A module. Let g_{ij} be the number of simple A_i modules in its decomposition into simple A modules. The matrix $G = (g_{ij})$ is called the inclusion matrix for $A \subset B$.

The inclusion of A in B is conveniently described by a so-called *Bratteli diagram*. This is a graph with vertices arranged in 2 lines. In one line, the vertices are in 1-1 correspondence with the minimal direct summands A_i of A, in the other one with the summands B_j of B. Then a vertex corresponding to A_i is joined with a vertex corresponding to B_j by g_{ij} edges.

If A and B have the same identity, then the dimension b_j of the module B_j equals the sum of the dimensions of simple A_i-modules which are joined to B_j by edges (with multiplicities). In matrix notation if $\vec{b} = (b_j)$ and $\vec{a} = (a_i)$

$$\vec{b} = G\vec{a}$$

We can also interpret the numbers g_{ij} in the following way: let p_i be a minimal idempotent of A_i and let $p_i = \sum q_m$, where q_m's are mutually orthogonal minimal idempotents of B. This decomposition is not unique in general. But for any such decomposition there will be exactly g_{ij} idempotents in B_j. We obtain that

$$p_i B p_i \cong \bigoplus_j M_{g_{ij}}.$$

To illustrate the definitions above we consider the group algebras kS_{f-1} and kS_f of the symmetric group. It is well-known that the simple modules of kS_f are labeled by Young diagrams λ with f boxes. The *Young lattice* is the infinite graph whose vertices are labeled by all Young diagrams such that two vertices are connected by an edge if and only if the corresponding diagrams differ by exactly one box.

So the Bratteli diagram for $kS_{f-1} \subset kS_f$ is the subgraph of the Young lattice with the vertices labeled by Young diagrams with $f-1$ and f boxes. In Figure 2.5 we give an example of a Bratteli diagram for the inclusion of the complex group algebra of the symmetric group in three letters into the complex group algebra of the symmetric group in four letters, i.e., $\mathbb{C}S_3 \subset \mathbb{C}S_4$.

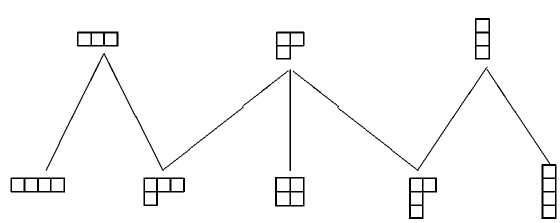

Figure 2.5: Bratteli diagram for $\mathbb{C}S_3 \subset \mathbb{C}S_4$

2.3 The Braid Groups

The braid group of type A, $\mathcal{B}_n(A)$, can be defined algebraically by generators $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_{n-1}$ and relations

$$\tilde{\sigma}_i\tilde{\sigma}_j = \tilde{\sigma}_j\tilde{\sigma}_i \quad \text{if } |i-j| > 1, \tag{2.2}$$

$$\tilde{\sigma}_i\tilde{\sigma}_{i+1}\tilde{\sigma}_i = \tilde{\sigma}_{i+1}\tilde{\sigma}_i\tilde{\sigma}_{i+1} \quad \text{if } 1 \leq i \leq n-2. \tag{2.3}$$

Similarly, we can define the braid group of type B, $\mathcal{B}_n(B)$, by generators $t, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and defining relations given by equations (2.2), (2.3),

$$\sigma_1 t \sigma_1 t = t \sigma_1 t \sigma_1, \tag{2.4}$$

and

$$t\sigma_i = \sigma_i t \quad \text{if } i > 1. \tag{2.5}$$

The elements in $\mathcal{B}_n(A)$ can be interpreted geometrically as braids in S^3. And elements in $\mathcal{B}_n(B)$ can be interpreted as braids in $\mathcal{B}_{n+1}(A)$ such that the first strand is fixed. We will usually boldface the fixed strand. Below we illustrate the generator $\tilde{\sigma}_i \in \mathcal{B}_n(A)$, the full-twist $\Delta_3^2 \in \mathcal{B}_3(A)$, and the generators $t, \sigma_i \in \mathcal{B}_n(B)$.

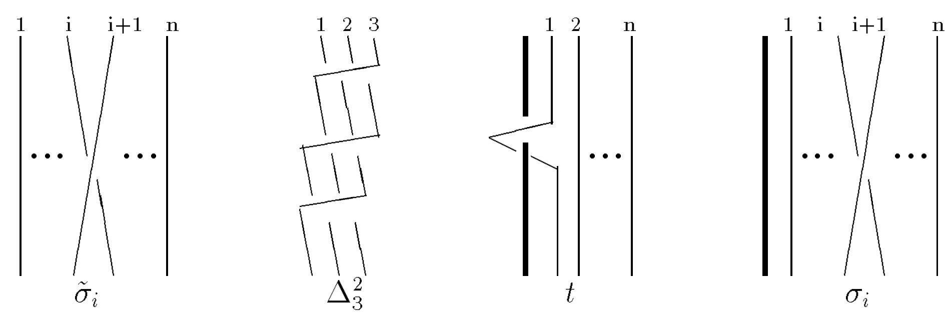

Figure 2.6: Examples of braids

In general, the full-twist Δ_f^2 in f strings, is a central element in $\mathcal{B}_f(A)$. Algebraically, $\Delta_f^2 = (\tilde{\sigma}_{f-1} \ldots \tilde{\sigma}_1)^f$. We now define a map from the generators of $\mathcal{B}_n(B)$

into $\mathcal{B}_{f+n}(A)$. We call the map $\tilde{\rho}_{f,n}$, and we define the image of the generators of $\mathcal{B}_n(B)$ as follows:

$$\tilde{\rho}_{f,n}(t) = \Delta_f^{-2}\Delta_{f+1}^2 \text{ and } \tilde{\rho}_{f,n}(\sigma_i) = \tilde{\sigma}_{f+i} \text{ for } i = 1, \ldots, n. \tag{2.6}$$

Pictorially, we have the following:

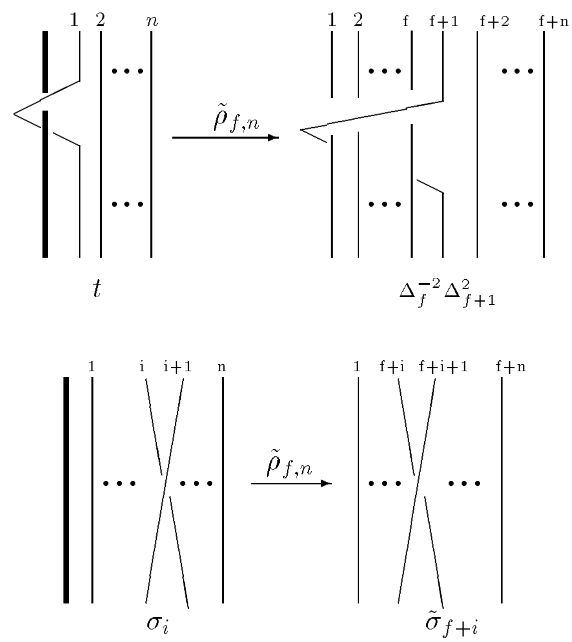

Figure 2.7: Pictorial definition of homomorphism, $\tilde{\rho}_{f,n}$

Proposition 2.3.1. *Let $n, f \in \mathbb{N}$ and $\tilde{\rho}_{f,n}$ be as defined in equation (2.6) for the generators of $\mathcal{B}_n(B)$. Then*

$$\tilde{\rho}_{f,n} : \mathcal{B}_n(B) \to \mathcal{B}_{f+n}(A)$$

extends to a well-defined group homomorphism.

Proof. It suffices to prove that $\tilde{\rho}_{f,n}$ preserves the relations of $\mathcal{B}_n(B)$. Notice that $\tilde{\rho}_{f,n}(t) = \tilde{\sigma}_f \ldots \tilde{\sigma}_2\tilde{\sigma}_1^2\tilde{\sigma}_2 \ldots \tilde{\sigma}_f$. Relations (2.2), (2.3) and (2.5) follow immediately from the definition of $\tilde{\rho}_{f,n}$ and the definition of $\mathcal{B}_n(A)$. To show that (2.4) holds we will use the commuting property of the full-twist.

$$\begin{aligned}\sigma_{f+1}(\Delta^2_{f+1}\Delta^{-2}_f)\sigma_{f+1}(\Delta^2_{f+1}\Delta^{-2}_f) &= \Delta^2_{f+2}\Delta^{-2}_{f+1}\Delta^2_{f+1}\Delta^{-2}_f \\ &= \Delta^2_{f+1}\Delta^{-2}_f\sigma_{f+1}\Delta^2_{f+1}\Delta^{-2}_f\sigma_{f+1}.\end{aligned}$$

It might be easier to see that this relation holds from the picture, see Figure 2.8. This completes the proof. □

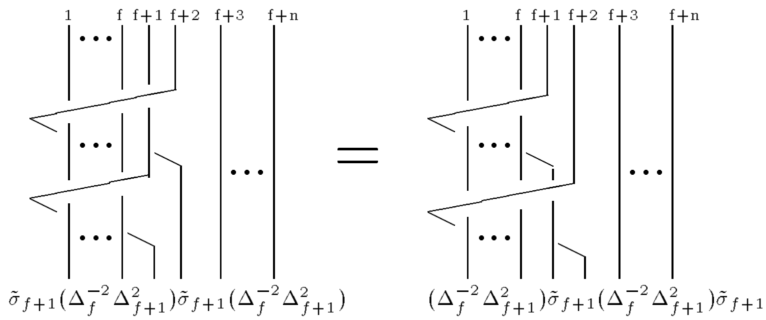

Figure 2.8: the fourth relation

Corollary 2.3.2. *The representations $\rho_{f,n}$ can be extended in a natural way to representations of the corresponding braid group algebras.*

Since the relations hold, one just extends by linearity.

2.4 Schur functions

For details and proofs on the results mentioned in this subsection we refer the reader to [M] Ch.I, sec. 3. Suppose $x_1, x_2, \dots x_r$ is a finite set of variables. For some partition α we set $x^\alpha = x_1^{\alpha_1}\dots x_r^{\alpha_r}$. Let $\delta = [r-1, r-2, \dots, 1, 0]$. We define the following determinant.

$$a_{\alpha+\delta} = \det(x_i^{\alpha_j+r-j})$$

This determinant is divisible in $\mathbb{Z}[x_1, \dots, x_r]$ by the Vandermonde determinant which is given by

$$a_\delta = \det(x_i^{r-j}).$$

Then the symmetric *Schur function* is defined as follows:

$$s_\alpha(x_1,\dots,x_r) = \frac{a_{\alpha+\delta}}{a_\delta}.$$

Take $x_i = q^{i-1}$ $(1 \le i \le r)$. If λ is a partition of length $\le r$, we have

$$s_\alpha(1,q,\dots,q^{r-1}) = q^{n(\alpha)} \prod_{1\le i<j\le r} \frac{1-q^{\alpha_i-\alpha_j+j-i}}{1-q^{j-i}} \tag{2.7}$$

where $n(\alpha) = \sum_{i=1}^{l(\alpha)} (i-1)\alpha_i$. We define the following Schur function as a normalization of equation (2.7):

$$s_{\alpha,r}(q) = \frac{s_\alpha(1,q,\dots,q^{r-1})}{s_{[1]}(1,q,\dots,q^{r-1})^{|\alpha|}}$$

Notice that $s_{\alpha,r}(q) = 0$ whenever $l(\alpha) > r$. Also if $\chi^\alpha(q)$ denotes the character of a matrix in $GL(r)$ with eigenvalues $q^{(-r+1)/2}$, $q^{(-r+3)/2}$, $\dots$, $q^{(r-1)/2}$ and if the number of boxes in α is n, $s_{\alpha,r}(q) = \chi^\alpha(q)/(\chi^{[1]}(q))^n$.

Let $r = r_1+r_2$; then the Schur function for $\mu = [m+\alpha_1,\dots,m+\alpha_{r_1},\beta_1,\dots,\beta_{r_2}]$ is given by

$$s_{\mu,r}(q) = q^{n(\mu)} \left(\frac{1-q}{1-q^r}\right)^{|\mu|} \prod_{1\le i<j\le r} \frac{(1-q^{\mu_i-\mu_j+j-i})}{(1-q^{j-i})}. \tag{2.8}$$

After some substitution and rearrangement of equation (2.8) we get the following equation for the Schur function of μ:

$$\begin{aligned} s_{\mu,r}(q) \quad &= \quad q^{mr_1(r_1-1)/2+r_1|\beta|} \left(\frac{1-q}{1-q^r}\right)^{|\mu|} s_\alpha(1,q,\dots,q^{r_1-1}) s_\beta(1,q,\dots,q^{r_2-1}) \\ & \prod_{i=1}^{r_1}\prod_{j=1}^{r_2} \frac{1-q^{m+r_1+\alpha_i-\beta_j+j-i}}{1-q^{r_1+j-i}}. \end{aligned} \tag{2.9}$$

The expression of the Schur function in (2.9) will be useful in the proof of the weight formula!

2.5 Traces

A *trace* is a functional $tr : B \to k$ such that $tr(ab) = tr(ba)$ for all $a, b \in B$. As there is up to scalar multiples only one trace on $M_n(k)$, any trace tr on $B = \bigoplus B_j$ is completely determined by a vector (t_j), where $t_j = tr(p_j)$ and p_j is a minimal idempotent of B_j.

The *annihilator ideal* J of tr is defined to be

$$J = (b \in B \,|\, tr(ab) = 0 \text{ for all } a \in B).$$

A trace tr in B is called *nondegenerate* if $J = 0$, or equivalently, if for any $b \in B$ there is a $b' \in B$ such that $tr(bb') \neq 0$. It can be shown that tr is nondegenerate if and only if $t_j \neq 0$ for each j.

The representation π_{tr} of B is defined on B/J by left multiplication. By the trace property it follows that

$$\pi_{tr}(B) \cong B/J.$$

Recall that if tr is a nondegenerate trace on B, the map $b \in B \to tr(b\cdot) \in B^*$ is an isomorphism between B and its dual B^*, where $tr(b\cdot)$ denotes the map $x \to tr(bx)$.

Let $A \subset B$ and tr be nondegenerate on both A and B. Using the isomorphism above for A and A^*, we obtain for every $b \in B$ a necessarily unique $\varepsilon_A(b) \in A$ such that $tr(b\cdot)|_A = tr(\varepsilon(b)\cdot)|_A$. The map $\varepsilon_A : B \to A$, $b \in B$ is called the trace preserving *conditional expectation* from B onto A, where the element $\varepsilon_A(b) \in A$ is uniquely determined by the equation

$$tr(\varepsilon_A(b)a) = tr(ba), \quad \text{for all } a \in A.$$

It follows from this equation and the faithfulness of tr that:

(1) $\varepsilon_A(a_1 b a_2) = a_1 \varepsilon_A(b) a_2$ for all $a_1, a_2 \in A$ and $b \in B$ and in particular $\varepsilon_A(a) = a$ for all $a \in A$.

(2) ε_A is nondegenerate, i.e. for all $0 \neq b \in B$ there is $b_1, b_2 \in B$ such that $\varepsilon_A(b_1 b) \neq 0$ and $\varepsilon_A(b b_2) \neq 0$.

Assume both A and B are C^* algebras, i.e. there exists a faithful representation of B on a Hilbert space such that both B and A are closed under the $*$ operation which assigns to a linear operator its adjoint.

A trace tr is *positive* if $tr(b^*b) \geq 0$ for all $b \in B$, in the finite dimensional case, tr is positive if an only if all components of the weight vector are nonnegative. If all components of the weight vector of the trace are positive, one has an inner product on B defined by

$$\langle a, b \rangle := tr(b^*a).$$

In this case, the conditional expectation ε_A can be interpreted as the orthogonal projection onto the subspace $A \subset B$. And it has the following additional properties:

$$\varepsilon_A(b^*) = \varepsilon_A(b)^*$$

$$\varepsilon_A(b^*b) \geq 0, \quad \text{for all } b \in B.$$

2.6 Subfactors

In this section we recall some definitions and basic results for constructing subfactors and for computing their invariants. For details and proofs of the following statements see [J2]. A *von Neumann algebra* A is a $*$-subalgebra of $B(H)$, bounded operators on the Hilbert space H, which contains 1 and is closed in the weak operator topology. A von Neumann algebra A whose center is trivial, i.e. $Z(A) = \mathbb{C} \cdot 1$ is called a *factor*. A II_1 *factor* is an infinite dimensional factor A which admits a normalized finite trace $tr : A \to F$ such that (i) $tr(1) = 1$; (ii) $tr(xy) = tr(yx)$, $x, y \in A$; and (iii) $tr(x^*x) \geq 0$, $x \in A$. This trace is unique. The *hyperfinite* II_1 factor is a separable II_1 factor which is approximately finite dimensional.

The trace induces a Hilbert norm on A, which is defined by $\|x\| = tr(x^*x)^{1/2}$, for all $x \in A$. Moreover, we can perform the GNS construction with respect to the trace and obtain a faithful representation of A on $L^2(A, tr)$, this Hilbert space is obtained as the closure of A in the norm induced by the trace. A acts by left multiplication operators on itself and the GNS representation is precisely this representation extended to $L^2(A, tr)$. Observe that the identity is the cyclic and separating vector. The representation of the II_1 factor A on $L^2(A, tr)$ is called the *standard form* of A.

From now on all factors and subfactors discussed will be II_1 factors. If A and B are a pair of factors, then A is a *subfactor* of B if A is a sub-von Neumann algebra of B, which is itself a factor and has the same identity as B, i.e. $1_A = 1_B$. The von Neumann algebra $A' \cap B$ is called the *relative commutant* of A in B.

If $A \subset B$ denotes the inclusion of II_1 factors with $1_A = 1_B$, we let tr be the unique normalized trace on B and observe that $tr|_A$ is the unique normalized trace on A by uniqueness of the trace. We define the orthogonal projection

$$e_A : L^2(B, tr) \to L^2(A, tr|_A)$$

by

$$e_A(xv) = \varepsilon_A(x)v, \quad v \in L^2(B, tr) \text{ and } \ x \in B$$

where ε_A is the trace preserving conditional expectation. We denote by $\langle B, e_A \rangle$ the von Neumann algebra generated by B and e_A on $L^2(B, tr)$. This is called the *basic construction.* In particular, if A is a factor, then so is $\langle B, e_A \rangle$. If it is a finite factor, we define the index $[B : A]$ of A in B to be the number $1/tr(e_A)$, where tr denotes the unique normalized trace on $\langle B, e_A \rangle$. If $\langle B, e_A \rangle$ is not finite, the index is defined to be infinite.

In this thesis we will study examples of subfactors constructed using the following set-up.

(i) Let (B_n) be an ascending sequence of C^* algebras with B_n a proper subalgebra of B_{n+1} for all $n \in \mathbb{N}$. Let furthermore tr be a positive finite extremal trace on its inductive limit B_∞ and let π_{tr} be the GNS construction with respect to tr. Then it is well-known that the weak closure B of $\pi_{tr}(B_\infty)$ is isomorphic to R, the hyperfinite II_1 factor.

(ii) Let (A_n) be an ascending sequence of C^*-subalgebras such that $A_n \subset B_n$ and the weak closure A of $\pi_{tr}(A_\infty)$ is a subfactor.

(iii) Consider the following square:

$$\begin{array}{ccc} B_n & \overset{\varepsilon_{B_n}}{\longleftarrow}\!\!\subset & B_{n+1} \\ \varepsilon_{A_n}\downarrow & & \downarrow\varepsilon_{A_{n+1}} \\ A_n & \subset & A_{n+1} \end{array}$$

where $\varepsilon_{A_{n+1}}$, ε_{A_n} and ε_{B_n} are the trace preserving conditional expectations onto A_{n+1}, A_n and B_n respectively. We require that this diagram commutes, i.e.,

$$\varepsilon_{A_{n+1}}\varepsilon_{B_n} = \varepsilon_{A_n} \quad \text{for all } n \in \mathbb{N}$$

This condition is called the *commuting square property.*

The interesting case for us is when the sequences $(A_n) \subset (B_n)$ have periodic Bratteli diagrams. The sequence (A_n) is *periodic* with period k if there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ the inclusion matrix for $A_{n+k} \subset A_{n+k+1}$ is the same (after relabeling of the central projections) as the one for $A_n \subset A_{n+1}$.

We say that $(A_n) \subset (B_n)$ is periodic if both (A_n) and (B_n) are periodic with same period k and if also the inclusion matrices for $A_{n+k} \subset B_{n+k}$ and $A_n \subset B_n$ are the same. If the inclusion matrices for $A_n \subset B_n$, $A_n \subset A_{n+1}$ and $B_n \subset B_{n+1}$ become periodic for all $n \geq n_0$ for some n_0, then the index $[B : A]$ of the subfactor A is the square of the norm of the inclusion matrix for $A_n \subset B_n$ for all $n \geq n_0$.

There are finer invariants for the subfactor $A \subset B$ than the index. Let $B^{(1)} = \langle B, e_A \rangle$ be obtained by the basic construction, then it is known by [J2] that

$$[B : A] = [B^{(1)} : B].$$

We also have that the inclusion matrix for $B \subset B^{(1)}$ is given by the transpose of the inclusion matrix, G, of $A \subset B$. Now iterate the basic construction to obtain a tower $A \subset B \subset B^{(1)} \subset B^{(2)} \subset \cdots$ of II_1 factors. Let

$$C_i = A' \cap B^{(i)}$$

be the relative commutant of A in $B^{(i)}$. Then the structure of the algebras $C_1, C_2, \cdots$ is an invariant of subfactors of B. The C_i's are called *higher relative commutants* of $A \subset B$.

2.7 Quantum groups

We now give some definitions about quantum groups, see [Ji1, Ji2, D]. A quantum group is a q deformation of the universal enveloping algebra of a finite or affine Lie algebra.

Let Φ be the root system corresponding to a semisimple Lie algebra $\mathfrak{g}$, with Cartan matrix $A = (a_{ij})_{1 \le i,j \le m}$. Let $\cdot$ be the invariant form on the root lattice and normalize so that $\alpha_i \cdot \alpha_i = 2$ for all short roots α_i. Let X be the weight lattice of $\mathfrak{g}$ which we assume to be embedded in the $\mathbb{Q}$-span of the root lattice, and extend the dot product to X by linearity. Let α_i, $i = 1, \cdots, m$ be the simple roots of Φ. Then, for $q \in \mathbb{C}$, one defines the quantum group $U_q(\mathfrak{g})$ by generators X_i^+, X_i^-, k_i and k_i^{-1}, $1 \le i \le m$ and relations

$$k_i k_j = k_j k_i, \qquad\qquad k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_i X_j^{\pm} k_i^{-1} = q^{\pm d_i a_{ij}/2} X_j^{\pm},$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q^{d_i} - q^{-d_i}},$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^{\nu} \frac{[1-a_{ij}]_{q^{d_i}}!}{[\nu]_{q^{d_i}}! [1-a_{ij}-\nu]_{q^{d_i}}!} (X_i^{\pm})^{1-a_{ij}-\nu} (X_j^{\pm}) (X_i^{\pm})^{\nu} = 0 \quad i \neq j,$$

where

$$[m]_t! = \prod_{j=1}^{m} \frac{t^j - t^{-j}}{t - t^{-1}} \quad \text{for any } t \in \mathbb{C}$$

$U_q(\mathfrak{g})$ also has a comultiplication Δ defined by

$$\Delta(X_i^{\pm}) = k_i \otimes X_i^{\pm} + X_i^{\pm} \otimes k_i^{-1}, \quad \text{and} \quad \Delta(k_i) = k_i \otimes k_i$$

The general representation theory of a quantum group is similar to the one of the corresponding classical algebra if q is not a root of unity.

For each finite dimensional vector space V one obtains a solution $R(x) \in \mathrm{End}(V \otimes V)$ of the quantum Yang-Baxter equation (QYBE) depending on a parameter x, see [D]. This R-matrix can be understood to be a deformation of the flip P, i.e. $P(v \otimes w) = w \otimes v$, which lies in the commutant of the second tensor power of the given representation of the quantum group.

Chapter 3

Hecke Algebras

3.1 Hecke algebras of type A

In this section we summarize results by Wenzl [W1] about the representation theory of the Hecke algebra of type A.

The Hecke algebra of type A_{n-1}, denoted by $H_n(q)$, is the free complex algebra with 1 and generators $g_1, g_2, \ldots, g_{n-1}$ and parameter $q \in \mathbb{C}$ with defining relations

(H1) $\quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad$ for $i = 1, 2, \ldots, n-2$;

(H2) $\quad g_i g_j = g_j g_i, \quad$ whenever $\mid i - j \mid \geq 2$;

(H3) $\quad g_i^2 = (q-1) g_i + q \quad$ for $i = 1, 2, \ldots, n-1$.

It is well-known that $H_n(q) \cong \mathbb{C}S_n$ if q is not a root of unity, where $\mathbb{C}S_n$ is the group algebra of the symmetric group, S_n, (see [Bou], p. 54-56). It follows from this that $H_n(q)$ has dimension $n!$. Similarly as for the symmetric group, we can label the irreducible representations of $H_n(q)$ by Young diagrams. If $\mu \vdash n$, then (π_μ, V_μ) denotes the irreducible representation of $H_n(q)$ indexed by μ. Here V_μ is the vector space with orthonormal basis given by $\{v_{t^\mu}\}$ where t^μ is a standard tableau of shape μ. These representations can be considered as q-analogs of Young's orthogonal representations of the Symmetric group, [H, W1]. We give a brief description of these representations.

Let $d(t^\mu, i) = c(i+1)-c(i)-r(i+1)+r(i)$ where $c(i)$ and $r(i)$ denote respectively, the column and row of the box containing the number i, we will refer to $d(t^\mu, i)$ as the *axial distance* from i to $i+1$. For $d \in \mathbb{Z} \setminus \{0\}$, let

$$a_d(q) = \frac{q^d(1-q)}{1-q^d} \quad \text{and} \quad c_d(q) = \frac{\sqrt{(1-q^{d+1})(1-q^{d-1})}}{1-q^d}. \tag{3.1}$$

Then one defines π_μ on the vector space V_μ by

$$\pi_\mu(g_i)v_{t^\mu} = a_d(q)v_{t^\mu} + c_d(q)v_{g_i(t^\mu)} \tag{3.2}$$

where $d = d(t^\mu, i)$ and $g_i(t^\mu)$ is the tableau obtained by interchanging i and $i+1$ in t^μ. This representation follows from the ones defined in [W1] by setting $\pi_\mu(g_i) = (q+1)\pi_\mu(e_i) - Id_{V_\mu}$, where e_i is the eigenprojection corresponding to the characteristic value q of g_i. Moreover $\pi_n = \bigoplus_{\mu \vdash n} \pi_\mu$ is a faithful representation of $H_n(q)$. Thus we have that

$$H_n(q) \cong \bigoplus_{\mu \vdash n} \pi_\mu(H_n(q)),$$

where $\pi_\mu(H_n(q)) \cong M_{f^\mu}$ and M_{f^μ} is the ring of complex $f^\mu \times f^\mu$ matrices, and f^μ is the number of standard tableaux of shape μ.

The Hecke algebras of type A satisfy the following embedding of algebras

$$H_1(q) \subset \cdots \subset H_n(q) \subset H_{n+1}(q) \subset \cdots .$$

Thus, we define the inductive limit of the Hecke algebra, $H_n(q)$, by

$$H_\infty(q) = \bigcup_{n \geq 0} H_n(q).$$

The full-twist, Δ_f^2, is a central element in $H_n(q)$ and it is defined algebraically by $\Delta_f^2 := (g_{f-1} \cdots g_1)^f$. The following lemma describes the action of the full-twist on the Hecke algebra of type A_{n-1}.

Lemma 3.1.1. *Let α_λ be the scalar by which the full-twist acts in the irreducible Hecke algebra representation labeled by λ. Then*

$$\alpha_\lambda = q^{n(n-1)-\sum_{i<j}(\lambda_i+1)\lambda_j} \tag{3.3}$$

For the proof of this lemma see [W2] pg.261.

Let t^μ be a Young tableau with n boxes and $(t^\mu)'$ be the Young tableau obtained from t^μ by removing the box containing n. The map $t^\mu \to (t^\mu)'$ defines a bijection between T_μ and $\bigcup_{\mu'\subset\mu} T_{\mu'}$, where T_μ denotes the set of all standard tableaux of shape μ. Therefore, we have the following decomposition of modules

$$V_\mu\Big|_{H_{n-1}(q)} = \bigoplus_{\mu'\subset\mu} V_{\mu'} \tag{3.4}$$

The interesting case for defining subfactors is when the parameter q is a root of unity, $q \neq 1$. In what follows we will describe the semisimple quotients of $H_n(q)$ which are associated with $\mathfrak{sl}(r)$ for $1 < r < l$.

Let $r, l \in \mathbb{N}$ and $l > r$, then an (r,l)-*diagram* is a Young diagram μ with r rows such that $\mu_1 - \mu_r \leq l - r$, we denote the set of all (r,l) diagrams of size n by $\Lambda_n^{(r,l)}$. An (r,l) *tableau* of shape $\mu \in \Lambda_n^{(r,l)}$, t^μ, is a standard tableau such that if we remove the box containing n the Young subdiagram, μ' with $n-1$ boxes is in $\Lambda_{n-1}^{(r,l)}$ and $t^{\mu'}$ is an (r,l) tableau. The set of (r,l) tableaux of shape μ is denoted by $T_\mu^{(r,l)}$.

For each $\mu \in \Lambda_n^{(r,l)}$ let $V_\mu^{(r,l)}$ be the vector space with basis $\{v_\tau\}$ indexed by elements of $T_\mu^{(r,l)}$. If q is a primitive l-th root of unity, Wenzl [W1] defined the linear map $\pi_\mu(g_i)$ as in equation (3.2) for all vectors v_τ for which d is not divisible by l. The restriction of this map to $V_\mu^{(r,l)}$ will be denoted by $\pi_\mu^{(r,l)}$, i.e., for $\tau \in T_\mu^{(r,l)}$

$$\pi_\mu^{(r,l)}(g_i)v_\tau = b_d(q)v_\tau + c_d v_{s_i\tau}.$$

Theorem 3.1.2 (Wenzl [W1], Corollary 2.5). *Let q be a primitive l-th root of unity with $l \geq 4$. Then there exists for every $\mu \in \Lambda_n^{(r,l)}$ a semisimple irreducible* *representation* $\pi_\mu^{(r,l)}$ *of* $H_n(q)$. *Then*

$$\pi_n^{(r,l)} : x \in H_n(q) \to \bigoplus_{\mu \in \Lambda_n^{(r,l)}} \pi_\mu^{(r,l)}(x)$$

is semisimple but generally not a faithful representation. Also representations corresponding to different (r,l) *diagrams are nonequivalent.*

We have well-defined representations $\pi^{(r,l)}$ of $H_\infty(q)$ given by

$$\pi^{(r,l)}(x) = \pi_n^{(r,l)}(x), \quad \text{if } x \in H_n(q).$$

The representation $\pi^{(r,l)}$ is an approximately finite representation, i.e.,

$$A_n := \pi^{(r,l)}(H_n(q))$$

are finite dimensional C^*-algebras. Furthermore, the representation $\pi^{(r,l)}$ is a unitary representation.

Wenzl showed that the ascending sequence of finite dimensional C^*-algebras (A_n) is periodic with period r.

When $q = e^{2\pi i/l}$, Wenzl obtains from the Hecke algebras, $H_n(q)$, an AF algebra with periodic Bratteli diagram for the sequence

$$\pi^{(r,l)}(H_1(q)) \subset \cdots \subset \pi^{(r,l)}(H_n(q)) \subset \pi^{(r,l)}(H_{n+1}(q)) \subset \cdots .$$

3.2 Hecke algebra of type B

The Hecke algebra $H_n(q,Q)$ of type B_n is the free complex algebra with generators $t, \tilde{g}_1, \ldots, \tilde{g}_{n-1}$ and parameters $q, Q \in \mathbb{C}$ the generators $\tilde{g}_i$'s satisfy (H1)-(H3) as in the definition of the Hecke algebra of type A and the following relations:

(B1) $t^2 = (Q-1)t + Q$;

(B2) $t\tilde{g}_1 t\tilde{g}_1 = \tilde{g}_1 t\tilde{g}_1 t$;

(B3) $t\tilde{g}_i = \tilde{g}_i t, \qquad \text{for } i \geq 2.$

Clearly, the Hecke algebra of type A is a subalgebra of $H_n(q,Q)$. It is known that $H_n(q,Q) \cong \mathbb{C}\mathcal{H}_n$ (the complex group algebra of the hyperoctahedral group). Hoefsmit [H] has written down explicit irreducible representations of $H_n(q,Q)$ indexed by ordered pairs of Young diagrams.

The Hecke algebras of type B satisfy the following embedding of algebras

$$H_0(q,Q) \subset H_1(q,Q) \subset \cdots \subset H_n(q,Q) \subset \cdots .$$

The inductive limit of the Hecke algebra of type B is defined by

$$H_\infty(q,Q) := \bigcup_{n\geq 0} H_n(q,Q).$$

Right and Double Coset Representatives

The fact that q is invertible in $\mathbb{C}(q,Q)$ implies that the generators $\tilde{g}_i$ are also invertible in $H_n(q,Q)$. In fact, the inverse of the generators is given by

$$\tilde{g}_i^{-1} = q^{-1}\tilde{g}_i + (q^{-1}-1)1 \in H_n(q,Q)$$

This implies that the following element is well-defined in $H_n(q,Q)$:

$$t_i^{'} = \tilde{g}_i \cdots \tilde{g}_1 t \tilde{g}_1^{-1} \cdots \tilde{g}_i^{-1}.$$

We use these elements to define the set $\mathcal{D}_n$ as a subset of $H_n(q,Q)$. If $n=1$, we let $\mathcal{D}_1 := \{1,t\}$. For $n \geq 2$ we have

$$\mathcal{D}_n := \{1, \tilde{g}_{n-1}, t_{n-1}^{'}\}.$$

These are known as the *distinguished double coset representatives* of $H_{n-1}(q,Q)$ in $H_n(q,Q)$. Also the set of *right coset representatives* of $H_{n-1}(q,Q)$ in $H_n(q,Q)$ is defined as follows:

$$\mathcal{R}_n := \{1,\ t'_{n-1},\ \tilde{g}_{n-1}\tilde{g}_{n-2}\ldots\tilde{g}_{n-k},\ \tilde{g}_{n-1}\tilde{g}_{n-2}\ldots\tilde{g}_{n-k}t'_{n-k-1}\ (1\leq k\leq n-1)\}.$$

Note that $\mathcal{R}_1 = \{1,t\}$. The elements $\{r_1r_2\ldots r_n | r_i \in \mathcal{R}_i\}$ form a $\mathbb{C}(q,Q)$-basis of $H_n(q,Q)$, see [GL].

Observe that (H3) and (B1) imply that t and $\tilde{g}_i$ have at most 2 eigenvalues each, hence also at most 2 projections corresponding to these eigenvalues. The presentation of $H_n(q,Q)$ with these projections as generators is given below.

So let for $q \neq -1$ and $Q \neq -1$

$$e_t = \frac{(Q-t)}{(Q+1)} \quad \text{and} \quad e_i = \frac{(q-g_i)}{(q+1)} \quad \text{for } i = 1, \cdots, n-1$$

be the projections corresponding to the eigenvalue -1. Then $g_i = q(1-e_i) - e_i = q-(q+1)e_i$. So $\langle 1, t, g_1, g_2, \ldots, g_{n-1}\rangle = \langle 1, e_t, e_1, e_2, \ldots, e_{n-1}\rangle$ and the defining relations (H1)-(H3)and (B1)-(B3) of $H_n(q,Q)$ translate to

(PH1) $e_ie_{i+1}e_i - q/(q+1)^2e_i = e_{i+1}e_ie_{i+1} - q/(q+1)^2e_{i+1}$ for $i = 1, 2, \ldots, n-2$;

(PH2) $e_ie_j = e_je_i$, whenever $\mid i-j \mid \geq 2$;

(PH3) $e_i^2 = e_i$ for $i = 1, 2, \ldots, n-1$;

(PH4) $e_t^2 = e_t$;

(PH5) $e_te_1e_te_1 - (Q+q)/(q+1)(Q+1)e_te_1 = e_1e_te_1e_t - (Q+q)/(q+1)(Q+1)e_1e_t$;

(PH6) $e_te_i = e_ie_t$ for $i \geq 2$.

3.3 Representations of the Hecke algebra of type B

In this section we give a short summary of the representation theory of $H_n(q,Q)$. We briefly describe the semi-orthogonal representations of $H_n(q,Q)$ defined by Hoefsmit [H]. Hoefsmit constructed for each double partition (α, β) of n an irreducible representation $(\pi_{(\alpha,\beta)}, V_{(\alpha,\beta)})$ of $H_n(q,Q)$ of degree $\binom{n}{|\alpha|}f^\alpha f^\beta$ where f^α is the number of standard tableaux of shape α.

Let $T_{(\alpha,\beta)}$ denote the set of pairs of standard tableaux of shape (α,β). We define the complex vector space $V_{(\alpha,\beta)}$ with orthonormal basis given by $\{v_\tau \mid \tau \in$

$T_{(\alpha,\beta)}\}$. The following notation and definitions are needed to define the action of the generators of $H_n(q,Q)$ on $V_{(\alpha,\beta)}$.

Let (α,β) be a pair of Young diagrams and $\tau = (t^\alpha, t^\beta)$ be a pair of standard tableaux of shape (α,β). Define the *content* of a box b as follows:

$$\mathrm{ct}(b) = \begin{cases} Qq^{j-i} & \text{if } b \text{ is in position } (i,j) \text{ in } t^\alpha \\ -q^{j-i} & \text{if } b \text{ is in position } (i,j) \text{ in } t^\beta. \end{cases}$$

Now define for each $1 \le i \le n-1$

$$(g_i)_\tau = \frac{q-1}{1 - \frac{ct(\tau(i))}{ct(\tau(i+1))}}$$

where $\tau(i)$ denotes the coordinates of the box containing the number i. Notice that $(g_i)_\tau$ depends only on the position of i and $i+1$. We are now ready to define the action of the generators on $V_{(\alpha,\beta)}$.

$$\begin{aligned} tv_\tau &= ct(\tau(1))v_\tau \\ g_i v_\tau &= (g_i)_\tau v_\tau + (q - (g_i)_\tau)v_{s_i(\tau)} \quad \text{for } i = 1, \cdots, n-1 \end{aligned} \tag{3.5}$$

where $s_i(\tau)$ is the standard tableau obtained from τ by switching i and $i+1$ in τ. If i and $i+1$ do not occur in the same row or column of t^α or t^β, then $s_i(t^\alpha, t^\beta)$ is again a pair of standard tableaux. Let V be the span of $\{v_\tau, v_{s_i(\tau)}\}$. Obviously, V is g_i-invariant. The action of $g_i\big|_V$ is given by the following 2×2 matrix

$$\begin{pmatrix} (g_i)_\tau & (q - (g_i)_\tau) \\ (q - (g_i)_{s_i(\tau)}) & (g_i)_{s_i(\tau)} \end{pmatrix}. \tag{3.6}$$

Finally, we have that if i and $i+1$ occur in the same row then $g_i v_\tau = qv_\tau$; and if i and $i+1$ occur in the same column then $g_i v_\tau = -v_\tau$.

Theorem 3.3.1 (Hoefsmit [H], Thm.2.2.7). *The modules $V_{(\alpha,\beta)}$, where (α,β) runs over all double partitions of n, form a complete set of nonisomorphic irreducible modules for $H_n(q,Q)$.*

Remark: Let $q \neq -1$ and $Q \neq -1$. One can easily obtain the representations for the spectral projections defined in Section 3.2. Recall the equations $e_t = \frac{Q-t}{Q+1}$ and $e_i = \frac{q-g_i}{q+1}$ for $i = 1, \cdots n-1$. The matrix representations of these projections are obtained by via the substititution $(g_i)_\tau = q - (q+1)(e_i)_\tau$.

Recall that we have the same definition of axial distance for pairs of partitions.

$$d = d(\tau^{(\alpha,\beta)}, i) = c(i+1) - c(i) + r(i) - r(i+1).$$

We have two possibilities for the denominators of $(e_i)_\tau$.

$$(e_i)_\tau = \begin{cases} (1+q)(1-q^d) & \text{if } i \text{ and } i+1 \text{ are both in } t^\alpha \text{ or } t^\beta \\ (1+q)(1+Qq^d) & \text{otherwise.} \end{cases} \tag{3.7}$$

Thus, the representations are undefined if and only if $(e_i)_\tau$ is undefined. This implies that if $Q \neq -q^k$ for $k \in \mathbb{Z}$ and if q is not an l-th root of unity for $1 \leq l \leq n-1$ then the representations are well-defined in $V_{(\alpha,\beta)}$ for all pairs (α, β) and $i = 1, 2, \ldots, n-1$. Notice that $(e_i)_\tau$ is also undefined when $d = 0$ and both i and $i+1$ are in t^α or t^β, but this never happens if τ is a pair of standard tableaux, see [W1] Lemma 2.11.

Observe that the map $\tau \to \tau'$ (where τ' is obtained from τ by removing the box containing n) defines a bijection between $\mathcal{T}_{(\alpha,\beta)}$ and $\bigcup_{(\alpha,\beta)' \subset (\alpha,\beta)} \mathcal{T}_{(\alpha,\beta)'}$. So in particular we have

$$V_{(\alpha,\beta)}\Big|_{H_{n-1}(q,Q)} \cong \bigoplus_{(\alpha,\beta)' \subset (\alpha,\beta)} V_{(\alpha,\beta)'}$$

where $(\alpha, \beta)'$ is a pair of Young tableaux obtained by removing one box from either α or β. From the definition of $\pi_{(\alpha,\beta)}$ and $\pi_{(\alpha,\beta)'}$ we see that this equation yields the decomposition of $V_{(\alpha,\beta)}$ as an $H_{n-1}(q, Q)$-module

$$\pi_{(\alpha,\beta)}\Big|_{H_{n-1}(q,Q)} \cong \bigoplus_{(\alpha,\beta)' \subset (\alpha,\beta)} \pi_{(\alpha,\beta)'}. \tag{3.8}$$

3.4 Representations of the Hecke algebra of type B onto a reduced Hecke algebra of type A

In this section we will show that for specializations of the Hecke algebra of type B there is a homomorphism onto a reduced Hecke algebra of type A. Before proving this assertion we introduce some necessary background.

Fix an positive integer n and let $m, r_1 \in \mathbb{N}$ be such that $m > n$ and $r_1 > n$. For these integers, set $\lambda = [m^{r_1}]$. Then as we observed in Chapter 2, Section 2.1, there is a one-to-one correspondence between double partitions of n and partitions of $n + mr_1$ containing λ. We fix a standard tableau t^λ. We also assume throughout that q is not a root of unity.

Recall that the representations of $H_n(q, Q)$ depend on rational functions with denominators $(Qq^d + 1)$ where $d \in \{0, \pm 1, \ldots, \pm(n-1)\}$. If $Q = -q^{r_1+m}$ then $1 - q^{r_1+m+d} \neq 0$ whenever $d \neq -(r_1 + m)$. It follows that all representations of $H_n(q, -q^{r_1+m})$ are well-defined whenever $r_1 + m > n$. Thus the specialized algebra $H_n(q, -q^{r_1+m})$ is well-defined and semisimple.

3.4.1 Homomorphism of $H_n(q, -q^{r_1+m})$ onto a reduced algebra of $H_{n+f}(q)$

If $p \in H_f(q)$ is an idempotent then the *reduced algebra* of $H_f(q)$ with respect to this idempotent is

$$pH_f(q)p := \{pap \mid a \in H_f(q)\}.$$

In ([W1], Cor. 2.3) Wenzl defined a special set of minimal idempotents of $H_f(q)$ indexed by the standard tableaux. The sum of these idempotents is 1. These minimal idempotents are well-defined whenever $H_f(q)$ is semisimple.

Let $\lambda \vdash f$ and t^λ be a standard tableau of shape λ. Then p_{t^λ} denotes the minimal idempotent indexed by t^λ. Thus, the reduced algebra with respect to p_{t^λ}

decomposes as follows:

$$p_{t^\lambda} H_{n+f}(q) p_{t^\lambda} \cong \bigoplus_{\substack{\lambda\subset\mu \\ \mu\vdash n+f}} \pi_\mu(p_{t^\lambda})\pi_\mu(H_{n+f}(q))\pi_\mu(p_{t^\lambda}).$$

Notice that if μ is a partition of $n+f$ and μ does not contain λ then $\pi_\mu(p_{t^\lambda})$ is the zero matrix. In particular, if we choose λ to be rectangular, we have that the reduced algebra $p_{t^\lambda} H_{f+1}(q) p_{t^\lambda}$ has only two nonzero irreducible modules indexed by partitions $[m+1, m^{r_1-1}]$ and $[m^{r_1}, 1]$, since these are the only partitions of $f+1$ which contain λ.

Recall that in Proposition (2.3.1) we showed that there is a homomorphism $\tilde{\rho}_{f,n}$ from the braid group $\mathcal{B}_n(B)$ into the braid group $\mathcal{B}_{f+n}(A)$. In what follows we will show that it is possible to extend $\tilde{\rho}_{f,n}$ to a homomorphism of the associated Hecke algebras.

Lemma 3.4.1. *Fix a positive integer n and choose $m, r_1 \in \mathbb{N}$ such that $m > n$ and $r_1 > n$. Set $f = mr_1$ and assume $\lambda = [m^{r_1}]$ and $\gamma = [m^{r_1}, 1]$. Let α_λ and α_γ be as in Lemma 3.1.1. Choose a minimal idempotent p_{t^λ} in $H_f(q)$. We define a map $\rho_{f,n}$ for the generators of $H_n(q, -q^{r_1+m})$ as follows:*

$$\rho_{f,n}(1) = p_{t^\lambda}, \qquad \rho_{f,n}(t) = -\frac{\alpha_\lambda}{\alpha_\gamma} p_{t^\lambda} \Delta_f^{-2} \Delta_{f+1}^2 \qquad \text{and} \qquad \rho_{f,n}(\tilde{g}_i) = p_{t^\lambda} g_{i+f}$$

for $i = 1, \ldots, n-1$, (see Figure 3.1 for a pictorial definition). Then $\rho_{f,n}$ extends to a well-defined homomorphism of algebras, $\rho_{f,n} : H_n(q, -q^{r_1+m}) \to p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$.

Proof. It is enough to check that $\rho_{f,n}$ preserves the relations of the Hecke algebra of type B. First notice that since $p_{t^\lambda} \in H_f(q)$, it commutes with g_j for all $j > f$. From this observation and the defining relations of $H_{n+f}(q)$ it follows that we only need to check the relations involving $\rho_{f,n}(t)$. For relations (B2) and (B3) (see Section 3.2), notice that p_{t^λ} commutes with the full-twist Δ_{f+1}^2 and Δ_f^{-2} and with g_{f+1}. Thus these two relations follow from Proposition (2.3.1).

It remains to be shown that $\rho_{f,n}(t)$ has two eigenvalues: -1 and $Q = -q^{r_1+m}$. In particular, we want to show that $\rho_{f,n}(t)$ acts by a scalar on two irreducible modules of $H_{f+1}(q)$ and by 0 on all others. Since λ is rectangular, this means that there are only two partitions of $f+1$ containing it.

By Lemma 3.1.1, we have that the full-twist Δ^2_{f+1} acts by the scalar α_β (resp. α_γ) on the irreducible module indexed by β (resp. γ). Also Δ_f^{-2} acts by α_λ^{-1} on the irreducible modules of $H_{f+1}(q)$ which are indexed by Young diagrams which contain λ. Therefore, we have that $\Delta_f^{-2}\Delta^2_{f+1}$ acts by $\alpha_\beta\alpha_\lambda^{-1}$ on the module V_β and by $\alpha_\gamma\alpha_\lambda^{-1}$ on the module V_γ. Therefore, we have that $-\frac{\alpha_\lambda}{\alpha_\gamma}p_{t^\lambda}\Delta_f^{-2}\Delta^2_{f+1}$ acts by -1 on V_γ and by $-\frac{\alpha_\beta}{\alpha_\gamma}$ on V_β, and by zero on all other modules, since p_{t^λ} kills all modules which do not contain λ.

In order to determine the constant $-\frac{\alpha_\beta}{\alpha_\gamma}$ we only need to substitute the partitions $\beta=[m+1,m^{r_1-1}]$ and $\gamma=[m^{r_1},1]$ in the formula given in Lemma 3.1.1. Thus we obtain

$$-\frac{\alpha_\beta}{\alpha_\gamma} = -q^{-\sum_{i<j}(\beta_i+1)\beta_j+\sum_{i<j}(\gamma_i+1)\gamma_j} = -q^{r_1+m}.$$

This concludes this proof. $\square$

The following figure gives a pictorial definition of the homomorphism $\rho_{f,n}$.

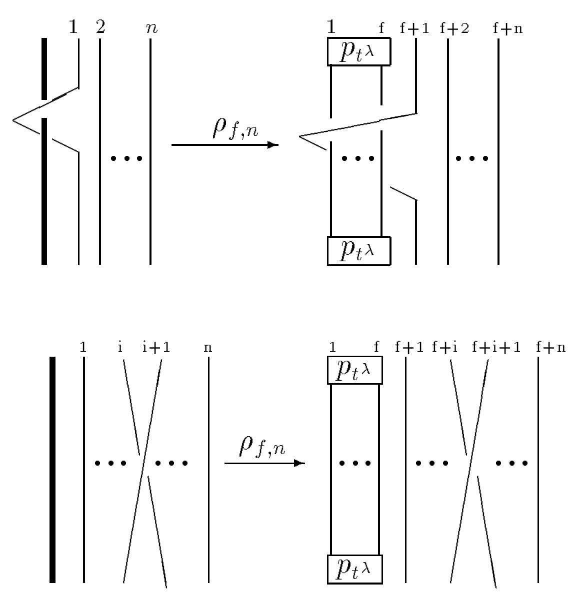

Figure 3.1: homomorphism for Hecke algebras

We have shown that $\rho_{f,n}$ is a homomorphism. We will also show that it is onto.

In particular, we will show that the irreducible representations of the reduced algebra are also irreducible representations of $H_n(q, -q^{r_1+m})$.

Notice that for $\mu \vdash n+f$ there is a 1-1 correspondence between standard skew tableaux of shape μ/λ and tableaux t^μ which contain t^λ. For this reason we denote by $T_{\mu/\lambda}$ the set of standard tableaux which contain t^λ. Notice that the order of $T_{\mu/\lambda}$ is equal to the number of standard skew tableaux of shape μ/λ. For our choice of m, i.e. $m > n$, we have that μ/λ will consist of two parts which can be interpreted as a double partition, say (δ, γ). In this case, the order of $T_{\mu/\lambda}$ is $\binom{n}{|\delta|} f^\delta f^\gamma$, where f^γ is the number of standard tableaux of shape γ; this formula is given by Hoefsmit [H].

Define $V_{\mu/\lambda}$ as the complex vector space with orthonormal basis $\{v_{t^\mu} \mid t^\mu \in T_{\mu/\lambda}\}$. Notice that $V_{\mu/\lambda}$ is a subspace of V_μ and $p_{t^\lambda} V_\mu = V_{\mu/\lambda}$, where V_μ has basis indexed by standard tableaux of shape μ.

Observation: Let $\mu \vdash n+f$ and V_μ be an irreducible $H_{n+f}(q)$ module. Then $p_{t^\lambda} V_\mu$ is an irreducible $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ module.

This observation implies that there is a set of irreducible representations of $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ indexed by Young diagrams with $n+f$ boxes, which contain λ. It follows from equation (3.4) that

$$p_{t^\lambda} V_\mu \Big|_{p_{t^\lambda} H_{n+f-1}(q) p_{t^\lambda}} \cong \bigoplus_{\substack{\lambda \subset \mu' \\ \mu' \subset \mu}} V_{\mu'/\lambda} \tag{3.9}$$

where μ' has $n+f-1$ boxes and $p_{t^\lambda} V_{\mu'} = V_{\mu'/\lambda}$.

We let z_μ denote the minimal central idempotents of $H_{n+f}(q)$, and $z_{\mu/\lambda}$ denote the minimal central idempotents of $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$. Notice that $z_{\mu/\lambda}$ has rank equal to the number of skew standard tableaux of shape μ/λ.

Theorem 3.4.2. *Let f, n be as in Lemma 3.4.1 and assume that q is not a root of unity. Then $\rho_{f,n}$ as defined in Lemma 3.4.1 is an onto homomorphism.*

Proof. In Lemma 3.4.1 we showed that $\rho_{f,n}$ is a homomorphism. Thus it only remains to show that it is onto. The proof is by induction on n. For $n = 1$, we have

$$\rho_{f,1} : H_1(q, -q^{r_1+m}) \to p_{t^\lambda} H_{f+1}(q) p_{t^\lambda}.$$

Since $\lambda \vdash f$ is a rectangular diagram there are only two Young diagrams with $f+1$ boxes which contain λ, i.e. $[m+1, m^{r_1-1}]$ and $[m^{r_1}, 1]$. As we showed in Lemma 3.4.1 the action of $\rho_{f,1}(t)$ on the representation indexed by $[m+1, m^{r_1-1}]$ (resp. $[m^{r_1}, 1]$) is $-q^{r_1+m}$ (resp. -1). And both representations are 1 dimensional.

For $n = 1$, we have that the algebra $H_1(q, -q^{r_1+m})$ has two irreducible representations indexed by $([1], \emptyset)$ and $(\emptyset, [1])$. Both of these representations are 1 dimensional and $t \in H_1(q, -q^{r_1+m})$ acts by a scalar on these representations. The action of t on $V_{([1],\emptyset)}$ (resp. $V_{(\emptyset,[1])}$) is $-q^{r_1+m}$ (resp. -1). Since q is not a root of unity, these representations are irreducible and nonequivalent. This shows that $\pi_{([1],\emptyset)} \cong \pi_{[m+1,m^{r_1-1}]}$ and $\pi_{(\emptyset,[1])} \cong \pi_{[m^{r_1},1]}$.

Assume that for $n > 1$ we have $\rho_{f,n}$ is onto. Then for all $\nu \vdash n+f$ containing λ, $V_{\nu/\lambda}$ is an irreducible $H_n(q, -q^{r_1+m})$, and if $\mu \vdash n+f+1$ is such that $\lambda \subset \mu$, we have

$$V_{\mu/\lambda}\Big|_{H_n(q,-q^{r_1+m})} \cong \bigoplus_{\substack{\lambda \subset \mu' \\ \mu' \subset \mu}} V_{\mu'/\lambda},$$

as in equation (3.9). This implies $V_{\mu/\lambda}$ is an $H_{n+1}(q, -q^{r_1+m})$-module.

The remainder of this proof is similar to the proof of irreducibility of modules of the Hecke algebra of type A in [W1], Theorem 2.2.

Let $\mu \vdash n+f+1$ be as described above. By the induction assumption, $H_n(q, -q^{r_1+m})$ is a semisimple algebra with minimal central idempotents $z_{\mu'/\lambda}$. Let $0 \neq W \subset V_{\mu/\lambda}$ be an $H_{n+1}(q, -q^{r_1+m})$ module. But $V_{\mu/\lambda}$ decomposes as an $H_n(q, -q^{r_1+m})$ module into the direct sum of irreducible modules $V_{\mu'/\lambda}$ since $\rho_{f,n}$ is onto $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$. Thus, there exists a $\mu' \vdash n+f$ such that $V_{\mu'/\lambda} \subset W$. Let $\tilde{\mu}' \neq \mu'$ be another Young diagram with $n+f$ boxes such that $\lambda \subset \tilde{\mu}' \subset \mu$. There is exactly one $\mu'' \vdash n+f-1$ contained in both μ' and $\tilde{\mu}'$ such that μ'' contains λ. Let $t^\mu \in T_{\mu/\lambda}$ be such that $(t^\mu)' \in T_{\mu'/\lambda}$ and $(t^\mu)'' \in T_{\mu''/\lambda}$. Then $(g_{n+f}(t^\mu))' \in T_{\tilde{\mu}'/\lambda}$

and therefore

$$\pi_\mu(z_{\tilde{\mu}'/\lambda})\pi_\mu(g_{n+f})v_{t^\mu} = c_d v_{g_{n+f}(t^\mu)} \in V_{\tilde{\mu}'/\lambda}$$

where d is the axial distance in t^μ between $n+f$ and $n+f+1$. Since q is not a root of unity then c_d is well-defined and nonzero, see equation (3.1) for the definition of c_d. Hence the irreducible $H_n(q, -q^{r_1+m})$-module, $V_{\tilde{\mu}'/\lambda}$, is contained in W. But $\tilde{\mu}'$ was arbitrary, therefore $W \supset \bigoplus_{\mu' \subset \mu} V_{\mu'}$.

Next we show that the $V_{\mu/\lambda}$ are mutually nonisomorphic $H_{n+1}(q, -q^{r_1+m})$-modules. As we observed above this is true for $n = 1$. For $n = 2$ there are five irreducible modules; 4 are one dimensional and 1 is two dimensional. We must check that the one dimensional modules are nonequivalent. By the definition of the action of t and g_{f+1} we have that t acts by $-q^{r_1+m}$ on $V_{[m+2,m^{r_1-1}]}$ and $V_{[m+1,m+1,m^{r_1-2}]}$. But g_{f+1} acts by q on $V_{[m+2,m^{r_1-1}]}$ and by -1 on $V_{[m+1,m+1,m^{r_1-2}]}$. In a similar way we can show that $V_{[m^{r_1},2]}$ and $V_{[m^{r_1},1^2]}$ are nonequivalent. And since t acts by -1 on these last two modules, we have that they are nonequivalent to the former two. If $n > 2$ and μ and $\tilde{\mu}$ are two distinct partitions of $n+f+1$ which contain λ, then there exists a $\mu' \supset \lambda$ such that $\mu' \subset \mu$ but $\mu' \not\subset \tilde{\mu}$. The proof of this fact is found in [W1] Lemma 2.11. We would also like to remark that restricting to Young diagrams containing λ does not affect the result. Hence, $V_{\mu/\lambda}$ and $V_{\tilde{\mu}/\lambda}$ differ already as $H_n(q, -q^{r_1+m})$ modules. □

Remark: Notice that this theorem formalizes the idea we indicated in Section 2.1 about associating pairs of Young diagrams (α, β) with one Young diagram where we adjoin the $m \times r_1$ rectangle, see Figure 2.4.

It is well-known that there exits a duality between the quantum group $U_q(\mathfrak{sl}(r))$ and the Hecke algebra of type A. This duality is the quantum analogue of the Schur-Weyl duality between the general linear group, $GL(n)$, and the symmetric group, S_n, (see [D], [Ji1], and [Ji2]).

The following is an easy Corollary of Theorem 3.4.2.

Corollary 3.4.3. *The diagonal action of $U_q(\mathfrak{sl}(r))$ and the action of the specialized Hecke algebra of type B, $H_n(q, -q^{r_1+m})$ on $V_\lambda \otimes V^{\otimes n}$ have the double centralizing* *property in* $End(V_\lambda \otimes V^{\otimes n})$.

The proof of this corollary follows immediately from the duality between the Hecke algebra of type A and $U_q(\mathfrak{sl}(r))$.

Remark: To relate the above to the literature we make the following remark. However, this remark will not be used in the sequel. For definitions we refer the reader to [Ji1].

The results of this section imply that there is an R-matrix representation of $H_n(q, -q^{r_1+m})$ on $V_\lambda \otimes V^{\otimes n}$, where V is the fundamental module of $U_q(\mathfrak{sl}(r))$ and V_λ is the irreducible module corresponding to λ. In this R-matrix representation t acts on $V_\lambda \otimes V = V_\beta \oplus V_\gamma$ (γ and β as defined in the proof of Lemma 3.4.1) as a scalar. And $\tilde{g}_i \to R_i$ is given by the same R-matrix as for the Hecke algebra of type A where R_i acts on the i-th and $(i+1)$-th copies of V.

3.5 The Hecke algebra of type B at roots of unity

In the previous section we observed that the irreducible representations of $H_n(q, Q)$ depend on rational functions with denominator $(Qq^d + 1)$ or $(1 - q^d)$ where $d \in \mathbb{Z}$. Thus some of the representations will be undefined when $Q = -q^k$ for some $k \in \mathbb{Z}$ or when q is a root of unity. It is the objective of this section to describe the simple decomposition of a quotient of the Hecke Algebra of type B when $Q = -q^k$ and q is an l-th root of unity.

In Section 3.4.1 we defined for $r_1,\ m \in \mathbb{N}$ such that $r_1 > n$ and $m > n$ an onto homomorphism from the specialized Hecke algebra of type B, $H_n(q, -q^{r_1+m})$, onto a reduced Hecke algebra of type A, $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$, where p_{t^λ} is an idempotent indexed by t^λ, a standard tableau corresponding to $\lambda = [m^{r_1}]$, i.e.,

$$\rho_{f,n} : H_n(q, -q^{r_1+m}) \longrightarrow p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}.$$

In what follows we will show that there is a well-defined surjective homomorphism when q is a root of unity and $Q = -q^{m+r_1}$ if we map into a well-defined quotient of $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$.

By Theorem 3.1.2 when q is an l-th root of unity then $\pi_n^{(r,l)}(H_n(q))$ is a well-defined quotient of the Hecke algebra of type A which is semisimple. The simple components are indexed by (r,l)-diagrams. We will denote this quotient by $H_n^{(r,l)}(q)$.

Recall that for an (r,l) diagram the set $T_\lambda^{(l)} \subset T_\lambda$ consists of tableaux $t^\lambda \in T_\lambda$ such that $(t^\lambda)' \in T_{\lambda'}^{(r,l)}$ for an (r,l)-diagram $\lambda' \in \Lambda_{n-1}^{(r,l)}$. In [W1] Wenzl showed that there exist well-defined minimal idempotents of $H_n^{(r,l)}(q)$ for every element in $T_\lambda^{(r,l)}$. We denote these idempotents by $p_{t^\lambda}^{(r,l)}$. In particular, we have the following reduced algebra $p_{t^\lambda}^{(r,l)} H_n^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$. Throughout the sequel we will only be interested in the case when $\lambda = [m^{r_1}]$. Notice that λ is (r,l)-diagram if $m \leq l-r$. Now we choose a Young tableaux $t^\lambda \in T_\lambda^{(r,l)}$ such that $p_{t^\lambda}^{(r,l)}$ is well-defined. Define a map from the generators of $H_n(q, -q^{r_1+m})$ into the reduced algebra $p_{t^\lambda}^{(r,l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$ as follows:

$$\begin{aligned} \hat{\rho}_{f,n}(1) &= p_{t^\lambda}^{(r,l)} \\ \hat{\rho}_{f,n}(t) &= -\frac{\alpha_\lambda}{\alpha_\gamma} p_{t^\lambda}^{(r,l)} \Delta_f^{-2} \Delta_{f+1}^2 \\ \hat{\rho}_{f,n}(g_i) &= p_{t^\lambda}^{(r,l)} g_{f+i}, \quad \text{for } i = 1, \dots n-1. \end{aligned}$$

Theorem 3.5.1. *Let m, r_1, r_2, $l \in \mathbb{N}$, $l \geq 4$ and $r = r_1 + r_2 < l$. Assume q is a primitive l-th root of unity and $Q = -q^{m+r_1}$ with $r_1 < m + r_1 \leq l - r_2$. Then $\hat{\rho}_{f,n}$ as defined above is a nontrivial onto homomorphism.*

Proof. For the proof that $\hat{\rho}_{f,n}$ is a homomorphism see Lemma (3.4.1). Since $\hat{\rho}_{f,n}$ is well-defined at roots of unity.

We now show that $\hat{\rho}_{f,n}$ is onto. The proof is by induction on n. For $n = 1$, we have $\hat{\rho}_{f,1} : H_1(q, -q^{r_1+m}) \rightarrow p_{t^\lambda}^{(r,l)} H_{f+1}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$. Since $\lambda \vdash f$ is a rectangular diagram there are only two Young diagrams with $f+1$ boxes which contain λ, i.e., $[m+1, m^{r_1-1}]$ and $[m^{r_1}, 1]$. Note that $V_{[m+1,m^{r_1}-1]}^{(r,l)}$ is well-defined as long as $m+1 \leq l-r$ and $V_{[m^{r_1},1]}^{(r,l)}$ is well-defined as long as $r_2 > 0$. The action of $\hat{\rho}_{f,1}(t)$ on the representation indexed by $[m+1, m^{r_1-1}]$ (resp. $[m^{r_1}, 1]$) is $-q^{r_1+m}$ (resp. -1). And both representations are 1 dimensional.

The algebra $H_1(q, -q^{r_1+m})$ has two irreducible representations indexed by the diagrams $([1], \emptyset)$ and $(\emptyset, [1])$. Both of these representations are 1 dimensional and $t \in H_1(q, -q^{r_1+m})$ acts by a scalar on these representations. The action of t on $V_{([1],\emptyset)}$ (resp. $V_{(\emptyset,[1])}$) is $-q^{r_1+m}$ (resp. -1). Since q is not a (r_1+m)-th root of unity, these representations are irreducible and nonequivalent. This shows that $\pi_{([1],\emptyset)} \cong \pi^{(r,l)}_{[m+1,m^{r_1-1}]}$ and $\pi_{(\emptyset,[1])} \cong \pi^{(r,l)}_{[m^{r_1},1]}$ whenever the representations are well-defined.

Assume that for $n > 1$ we have $\hat{\rho}_{f,n}$ is onto. If $\nu \vdash n+f$ is an (r,l)-diagram such that ν contains λ, then $V^{(r,l)}_{\nu/\lambda}$ is an irreducible $H_n(q, -q^{r_1+m})$-module. Now let $\mu \vdash n+f+1$ be an (r,l)-diagram which contains λ, then

$$V^{(r,l)}_{\mu/\lambda}\Big|_{H_n(q,-q^{r_1+m})} \cong \bigoplus_{\lambda\subset\mu'\subset\mu} V^{(r,l)}_{\mu'/\lambda},$$

as in equation (3.9). Clearly $V^{(r,l)}_{\mu/\lambda}$ is a representation of $H_{n+1}(q, -q^{r_1+m})$.

The irreducibility can be shown exactly as in [[W1], Theorem 2.2 and Corollary 2.5]. The fact that representations belonging to different Young diagrams are inequivalent is also shown as in [[W1], Theorem 2.2 and Lemma 2.11]. □

This theorem constructs a semisimple quotient of $H_n(q, -q^{r_1+m})$ which we denote by $H^{(r,l)}_n(q, -q^{r_1+m})$.

Observation: There is a 1-1 correspondence between pairs of Young diagrams (α, β) satisfying the condition $\alpha_{r_1} - \beta_1 \geq -m$ with $l(\alpha) \leq r_1$ and Young diagrams containing a rectangular diagram $[m^{r_1}]$, see Figure 3.2

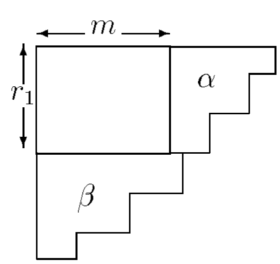

Figure 3.2: Correspondence between pairs of Young diagrams and a Young diagram

Now we define a subset , $_n(l, m, r)$ of the set of double partitions. We will show that the quotient $H_n^{(r,l)}(q,-q^{r_1+m})$ which is isomorphic to the image of $\hat{\rho}_{f,n}$ is indexed by the ordered pairs of Young diagrams which we now define.

Definition 3.5.2. Let $m,l,r \in \mathbb{N}$ with $r \le l-1$. A pair of Young diagrams (α,β) such that $l(\alpha) \le r_1$ and $l(\beta) \le r_2$ is called a *(m,l,r)-diagram* if

(1) $\alpha_1 - \beta_{r_2} \le l - r - m$ and

(2) $\alpha_{r_1} - \beta_1 \ge -m$.

Let $\Gamma_n(l,m,r)$ denote the set of all (m,l,r)-diagrams with n boxes.

We have the following corollary of Theorem 3.5.1.

Corollary 3.5.3. *(i) Let* $\mu = [m+\alpha_1, \cdots, m+\alpha_{r_1}, \beta_1, \cdots \beta_{r_2}]$. *There exists a 1-1 correspondence between* $\mu \in \Lambda_{n+f}^{(r,l)}$ *and* $(\alpha,\beta) \in \Gamma(l,m,r)$.
(ii) If the representation indexed by (a,β) *is well-defined, then the bijection in (i) is compatible with the homomorphism of Theorem 3.5.1.*

Proof. (i) Recall that $\mu \in \Lambda_{n+f}^{(r,l)}$ implies that $\mu_1 - \mu_r \le l - r$, where $l(\mu) \le r = r_1 + r_2$. By substituting $\mu_1 = \alpha_1 + m$ and $\mu_r = \beta_{r_2}$ one gets $\alpha_1 - \beta_{r_2} + r_2 \le l - r_1 + m$ which is condition (1) in the definition of the elements in $\Gamma(l,m,r)$. The other condition is easily seen by the definition of a Young diagram. $\mu_{r_1} \le \mu_{r_1+1}$ implies condition (2) $\alpha_{r_1} - \beta_1 > -m$. Clearly, having $(a,\beta) \in \Gamma(l,m,r)$ one can construct μ by adjoining the box $[m_1^r]$.

(ii) By (i) we have two indexing sets for the irreducible representations of $H_n^{(r,l)}(q,-q^{r_1+m})$. If $(\pi_\mu^{(r,l)}, V_\mu^{(r,l)})$ is a well-defined irreducible representation then we can also index it with a pair $(\alpha,\beta) \in \Gamma(l,m,r)$. Furthermore, if we restric $V_\mu^{(r,l)}$ to $H_{n-1}^{(r,l)}(q,-q^{r_1+m})$ we obtain the decomposition

$$V_\mu^{(r,l)}\Big|_{H_{n-1}^{(r,l)}(q,-q^{r_1+m})} = \bigoplus_{\mu' \leftrightarrow \mu} V_{\mu'}^{(r,l)}$$

where $\mu' \in \Lambda_{n-1}^{(r,l)}$ and $\mu' \supset \lambda$ by Theorem 3.5.1. Note that μ' can be associated with a pair $(a,\beta)' \in \Gamma_{n-1}(l,m,r)$ and $V_{(\alpha,\beta)}^{(r,l)}$ can be associated with $V_{mu'}^{(r,l)}$ whenever the representations are well-defined. Therefore, the bijection in (i) is compatible with the homomorphism $\hat{\rho}_{f,n}$. $\square$

In Figure 3.3 we illustrate the statements of Corollary 3.5.3 using the Bratteli diagrams for the example, $l=5$, $m=2$, $r_1=1$ and $r_2=2$. In this case $\lambda=[2]$.

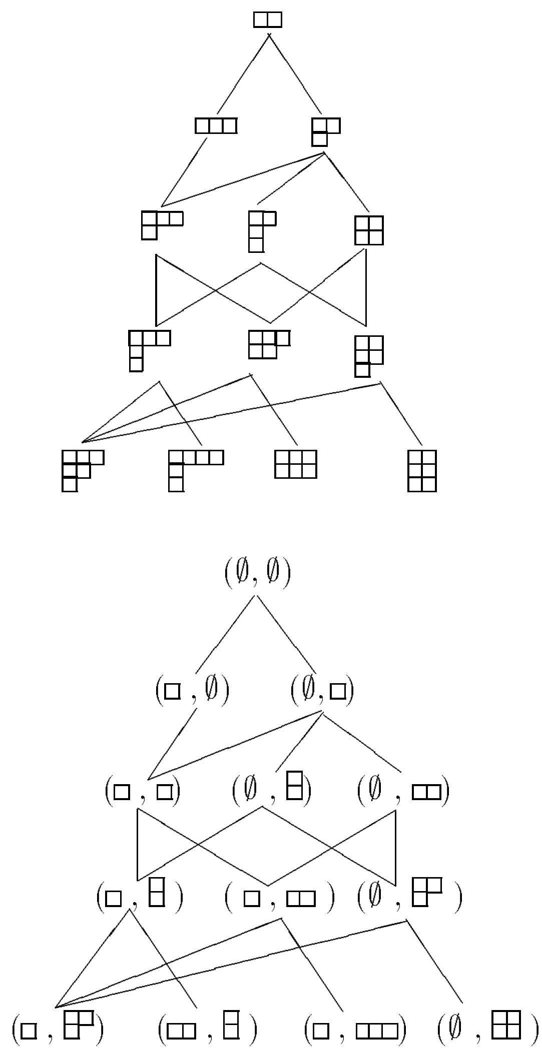

Figure 3.3: Bratteli Diagrams for $p^{(5)}_{t[2]}H^{(3,5)}_n(q)p^{(5)}_{t[2]}$ and $H^{(5)}_n(q,-q^3)$

Fix $m,l,r\in\mathbb{N}$ with $l\geq 4$ and let $q=e^{2\pi i/l}$. Set

$$B_n := H^{(r,l)}_n(q,-q^{r_1+m}) = \bigoplus_{(\alpha,\beta)\in\Gamma_n(l,m,r)} \pi^{(r,l)}_{(\alpha,\beta)}(H_n(q,-q^{r_1+m})). \tag{3.10}$$

Here we used the identification in Corollary 3.5.3. With this identifications we can define the representation

$$\pi^{(r,l)} : H_\infty^{(r,l)}(q, -q^{r_1+m}) \longrightarrow B_\infty$$

of the corresponding inductive limits by

$$\pi^{(r,l)}(x) = \bigoplus_{\lambda \in \Gamma_n(m,l,r)} \pi_{(\alpha,\beta)}^{(r,l)}(x) \tag{3.11}$$

for all $x \in H_n^{r,l}(q, -q^{r_1+m})$.

In [W1] Wenzl showed that if q is an l-th root of unity we have that the inclusion diagrams for the Hecke algebras of type A at roots of unity eventually become periodic with period r (the maximum number of rows allowed).

Lemma 3.5.4. *If the inclusion diagrams for* $\cdots \subset H_{n-1}^{(r,l)}(q) \subset H_n^{(r,l)}(q) \subset \cdots$ *has period* r*, then the inclusion diagram for*

$$\cdots \subset p_{t^\lambda}^{(r,l)} H_{n-1}^{(r,l)}(q) p_{t^\lambda}^{(r,l)} \subset p_{t^\lambda}^{(r,l)} H_n^{(r,l)}(q) p_{t^\lambda}^{(r,l)} \subset \cdots$$

has period r.

The proof of this Lemma follows immediately from the definition of reduced algebra.

Corollary 3.5.5. *The inclusion diagram*

$$\cdots \subset H_{n-1}^{(r,l)}(q, -q^{r_1+m}) \subset H_n^{(r,l)}(q, -q^{r_1+m}) \subset \cdots$$

has period r *whenever* $\cdots \subset p_{t^\lambda}^{(r,l)} H_{n-1}^{(r,l)}(q) p_{t^\lambda}^{(r,l)} \subset p_{t^\lambda}^{(r,l)} H_n^{(r,l)}(q) p_{t^\lambda}^{(r,l)} \subset \cdots$ *has period* r.

Proof. We have shown above that the quotient $H_n^{(r,l)}(q, -q^{r_1+m})$ of the Hecke algebra of type B is isomorphic to the reduced algebra $p_{t^\lambda}^{(r,l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$. Thus periodicity follows from this isomorphism.□

Results in this chapter have been adapted from the paper Weights of Markov traces on Hecke algebras R. C. Orellana, 1999. The dissertation author was the primary investigator and single author of these papers.

Chapter 4

Markov traces and the weight formula

4.1 Markov traces

In this section we give the necessary background on Markov traces. We refer the reader to [GL] or [G] for details.

A *trace* function on $H_\infty(q,Q)$ is a $\mathbb{C}(q,Q)$-linear map $\phi : H_\infty(q,Q) \longrightarrow \mathbb{C}(q,Q)$ such that $\phi(hh') = \phi(h'h)$ for all $h, h' \in H_\infty(q,Q)$. This definition is in fact valid for any associative algebra over a commutative ground ring. In the case of the group algebra it is clear that every trace function is constant on the conjugacy classes of the underlying group. Notice that $\phi(hh') - \phi(h'h) = 0$; this means that $\phi(hh' - h'h) = \phi([h,h']) = 0$, which implies that the commutators are in the kernel of a trace function in an algebra.

The weights we are going to give in this paper correspond to a trace that satisfies the following definition.

Definition 4.1.1. Let $z \in \mathbb{C}(q,Q)$ and $\mathrm{tr} : H_\infty(q,Q) \longrightarrow \mathbb{C}(q,Q)$ be an $\mathbb{C}(q,Q)$-linear map. Then tr is called a *Markov trace* (with parameter z) if the following conditions are satisfied:

(1) tr is a trace function on $H_\infty(q,Q)$;

(2) $\mathrm{tr}(1)=1$ (normalization);

(3) $\mathrm{tr}(hg_n)=z\,\mathrm{tr}(h)$ for all $n\geq 1$ and $h\in H_n(q,Q)$.

The name of these traces comes from their invariance under the Markov moves for closed braids. Remember that the Hecke algebra is a quotient of the braid group algebra. We note that all generators g_i (for $i=1,2,\cdots$) are conjugate in $H_\infty(q,Q)$. In particular, any trace function on $H_\infty(q,Q)$ must have the same value on these elements. This explains why the parameter z is independent of n in rule (3) of this definition.

Geck and Pfeiffer [GP] showed that a trace function on the Hecke algebra is uniquely determined by its value on basis elements corresponding to representatives of minimal length in the various conjugacy classes of the underlying Coxeter group. The representatives of minimal length are of the form $d_1\cdots d_n$, where d_i is a distinguished double coset representative of $H_i(q,Q)$ with respect to $H_{i-1}(q,Q)$.

Let tr be a Markov trace with parameter z, and let $d_i\in\mathcal{D}_i$ (set of double coset representatives) for $i=1,\cdots,n$. Then

$$\mathrm{tr}(d_1\cdots d_n)=z^a\,\mathrm{tr}(t'_0t'_1\cdots t'_{b-1})$$

where a is the number of factors d_i which are equal to g_{i-1} and b is the number of factors which are equal to t'_{i-1}. Thus, tr is uniquely determined by its parameter z and the values on the elements in the set $\{t'_0t'_1\cdots t'_{i-1}\,|\,i=1,2,\cdots\}$.

Conversely, given z, y_1, $y_2,\cdots\in\mathbb{C}(q,Q)$ there exist a unique Markov trace on $H_\infty(q,Q)$ such that $\mathrm{tr}(t'_0t'_1\cdots t'_{k-1})=y_k$ for all $k\geq 1$. For a proof of these results see [GL], Theorem 4.3.

We are particularly interested in the special case when $y_i=y^i$ for all $i\in\mathbb{N}$, in this case there are only two parameters. We have that if d_i is a distinguished double coset representative then $\mathrm{tr}(d_ix)=\xi\,\mathrm{tr}(x)$ where $\xi=y$ or z. The proof of the following proposition is found in [GL].

Proposition 4.1.2. *Let $z, y \in \mathbb{C}(q,Q)$ and $\mathrm{tr} : H_\infty(q,Q) \longrightarrow \mathbb{C}(q,Q)$ be a Markov trace with parameter z such that $\mathrm{tr}(t_0' t_1' \cdots t_{k-1}') = y^k$ for all $k \geq 1$ then*

$$\mathrm{tr}(ht_{n,0}') = y\,\mathrm{tr}(h) \textit{ for all } n \geq 0 \textit{ and } h \in H_n(q,Q)$$

where $t_{n,0}' = g_n \cdots g_1 t g_1^{-1} \cdots g_n^{-1}$ or $g_n^{-1} \cdots g_1^{-1} t g_1 \cdots g_n$.

Notice that the converse is trivially true. We will compute the weight vectors for this Markov trace on $H_n(q,Q)$.

4.2 The weight formula

In this section we define for every pair of partitions, (α, β), a rational function in q and Q, $W_{(\alpha,\beta)}(q,Q)$. We will show that this function gives the weights for the Markov trace defined by Geck and Lambropoulou [GL] for the Hecke algebra of type B. If we denote the weights by $\omega_{(\alpha,\beta)}$ then the Markov trace, tr, can be written as follows:

$$\mathrm{tr}(x) = \sum_{(\alpha,\beta)\vdash n} \omega_{(\alpha,\beta)} \chi^{(\alpha,\beta)}(x), \tag{4.1}$$

where $x \in H_n(q,Q)$ and $\chi^{(\alpha,\beta)}$ is the character (the usual trace) of the irreducible representation of $H_n(q,Q)$ indexed by (α,β).

Let r_1, $r_2 \in \mathbb{N}$. First we define a rational function in q and Q for any arbitrary double partition (α,β) such that $l(\alpha) \leq r_1$ and $l(\beta) \leq r_2$. If $l(\alpha) = s < r_1$ then $\alpha_i = 0$ for $i = s+1, \cdots, r_1$, similarly for β. Let $r = r_1 + r_2$.

$$\begin{aligned} W_{(\alpha,\beta)}(q,Q) \;:=\; & \frac{q^{n(\alpha)+n(\beta)}}{(1+q+\cdots+q^{r-1})^{|\alpha|+|\beta|}} \prod_{1\leq i<j\leq r_1} \frac{1-q^{\alpha_i-\alpha_j+j-i}}{1-q^{j-i}} \\ & \times \prod_{1\leq i<j\leq r_2} \frac{1-q^{\beta_i-\beta_j+j-i}}{1-q^{j-i}} \prod_{i=1}^{r_1}\prod_{j=1}^{r_2} \frac{Qq^{\alpha_i-i}+q^{\beta_j-j}}{Qq^{-i}+q^{-j}} \end{aligned} \tag{4.2}$$

Notice that this function can be expressed as a product of Schur functions and an additional simple factor

$$W_{(\alpha,\beta)}(q,Q)=q^{r_1|\beta|}\frac{s_\alpha(1,q,\cdots,q^{r_1-1})s_\beta(1,q,\cdots,q^{r_2-1})}{s_{[1]}(1,q,\cdots,q^{r_1+r_2-1})^{|\alpha|+|\beta|}}\prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\frac{(1+Qq^{\alpha_i-\beta_j+j-i})}{(1+Qq^{j-i})}. \tag{4.3}$$

Recall that $s_\alpha(1,q,\cdots,q^{r_1-1})=q^{n(\alpha)}\prod_{1\le i<j\le r_1}\frac{1-q^{\alpha_i-\alpha_j+j-i}}{1-q^{j-i}}$ is the symmetric Schur function defined in Section 2.4, equation (2.7). From (4.3) we can see that $W_{(\alpha,\beta)}(q,Q)=0$ if $l(\alpha)>r_1$ or $l(\beta)>r_2$.

Observation: Let $1-r_1\le s\le r_2-1$. Assume that q is not a root of unity and $Q\neq -q^{-s}$. Then $W_{(\alpha,\beta)}(q,Q)$ is an analytic rational function.

The rectangular Young diagram with r rows which has m boxes in the first r_1 rows and 0 boxes in the remaining rows, i.e. $[m^{r_1}]$, has Schur function given by

$$s_{[m^{r_1}],r}(q)=\frac{q^{mr_1(r_1-1)/2}\prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\frac{1-q^{m+r_1+j-i}}{1-q^{r_1+j-i}}}{s_{[1]}(1,q,\cdots,q^{r-1})^{mr_1}}.$$

We are going to assume that for a fixed positive integer n and $m,r_1\in\mathbb{N}$, we have $m>n$ and $r_1>n$. Then for any double partition of n, (α,β), we have $\mu=[m+\alpha_1,\cdots,m+\alpha_{r_1},\beta_1,\cdots,\beta_{r_2}]$ is a partition of $n+mr_1$. Then by equation (2.9) from Section 2.4 we have the following equation

$$\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}=q^{r_1|\beta|}\frac{s_\alpha(1,q,\cdots,q^{r_1-1})s_\beta(1,q,\cdots,q^{r_2-1})}{s_{[1]}(1,q,\cdots,q^{r})^{|\alpha|+|\beta|}}\prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\frac{1-q^{m+r_1+\alpha_i-\beta_j+j-i}}{1-q^{m+r_1+j-i}}.$$

Observe that we have the following equality:

$$\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}=W_{(\alpha,\beta)}(q,-q^{r_1+m}). \tag{4.4}$$

Notice that $W_{(\alpha,\beta)}(q,-q^{r_1+m})$ is well-defined since $r_1+m>r_1$ for $m\in\mathbb{N}$, and as observed before $W_{(\alpha,\beta)}(q,Q)$ is undefined for $Q=-q^{-s}$ where $1-r_1<s<r_2-1$. So $W_{(\alpha,\beta)}(q,-q^{r_1+m})$ is an analytic rational function.

Lemma 4.2.1.
$$W_{(\alpha,\beta)}(q,Q) = \sum_{(\alpha,\beta)\leftrightarrow(\gamma,\eta)} W_{(\gamma,\eta)}(q,Q)$$
where $(\alpha,\beta) \leftrightarrow (\gamma,\eta)$ *means that* (γ,η) *is obtained by adding one box to* (α,β)*.*

Proof. Assume $\mu = [m+\alpha_1, \cdots, m+\alpha_{r_1}, \beta_1, \cdots, \beta_{r_2}]$. By Littlewood-Richardson rule for Schur functions (see [M]) we have the following:
$$s_{[1],r}s_{\mu,r} = \sum_{\substack{\mu\subset\nu \\ |\nu|=|\mu|+1}} s_{\nu,r}$$
where $s_{[1],r}(q) = 1$. Now divide both sides of this equation by $s_{[m^{r_1}],r}(q)$ and we get by equation (4.4).
$$W_{(\alpha,\beta)}(q,-q^{r_1+m}) = \sum_{(\alpha,\beta)\subset(\gamma,\eta)} W_{(\gamma,\eta)}(q,-q^{r_1+m}). \tag{4.5}$$
Since $W_{(\alpha,\beta)}(q,Q)$ is an analytic rational function and the above equation holds for all $Q = -q^{r_1+m}$, thus we have that
$$W_{(\alpha,\beta)}(q,Q) = \sum_{(\alpha,\beta)\leftrightarrow(\gamma,\eta)} W_{(\gamma,\eta)}(q,Q). \tag{4.6}$$
holds for all values of Q. □

In [W1] Wenzl showed that the weights of the Markov trace (with parameter $z = q^r\frac{(1-q)}{(1-q^r)}$) on the Hecke algebra of type A are given by the symmetric Schur function $s_{\mu,r}(q)$ defined in Section 2.4.

Let $\lambda = [m^{r_1}] \vdash f$. Since we assumed that $m > n$ and $r_1 > n$, then we have that for all $\mu \vdash n+f$ we have that μ/λ can be interpreted as a double partition (α,β) of n. Now we fix t^λ a standard tableau of shape λ. Then the reduced algebra $p_{t^\lambda}H_{n+f}(q)p_{t^\lambda}$ has p_{t^λ} as the identity. The Markov trace for the reduced algebra is given by the renormalized Markov trace of $H_{n+f}(q)$. By renormalization we mean that we must divide the trace of $H_{n+f}(q)$ by the trace of the identity, i.e. $\mathrm{tr}(p_{t^\lambda}) = s_{\lambda,r}(q)$, of the reduced algebra. Therefore, we have that $\frac{s_{\mu,r}(q)}{s_{\lambda,r}(q)}$ are the weights of $p_{t^\lambda}H_{n+f}(q)p_{t^\lambda}$. Notice that this implies that $W_{(\alpha,\beta)}(q,Q)$ specializes to the weights for the reduced algebra when $Q = -q^{r_1+m}$.

Lemma 4.2.2. *Let $g_{n-1} \in H_n(q,Q)$, $z = \frac{q^r(1-q)}{(1-q^r)}$ and $W_{(\alpha,\beta)}(q,Q)$ as defined in equation (4.2). Then for any $x \in H_n(q,Q)$*

$$\mathrm{tr}(x) = \sum_{(\alpha,\beta)\vdash n} W_{(\alpha,\beta)}(q,Q)\chi^{(\alpha,\beta)}(x) \tag{4.7}$$

defines a well-defined trace which satisfies the Markov property, i.e. $\mathrm{tr}(hg_{n-1}) = z\,\mathrm{tr}(h)$, *where* $h \in H_{n-1}(q,Q)$.

Proof. It is clear that tr is indeed a trace. We must show that it satisfies the Markov property. We have assumed that $\mu/\lambda = (\alpha,\beta)$, then $W_{(\alpha,\beta)}(q,-q^{r_1+m}) = \frac{s_{\mu,r}(q)}{s_{\lambda,r}(q)}$. We also have that $\chi^{\mu/\lambda} = \chi^{(\alpha,\beta)}\Big|_{Q=-q^{r_1+m}}$ since in the proof of Theorem 3.4.2 we showed that $V_{\mu/\lambda}$ is an irreducible module of $H_n(q,-q^{r_1+m})$. Thus we have that

$$\sum_{\mu\vdash n+f} \frac{s_{\mu,r}(q)}{s_{\lambda,r}(q)}\chi^{\mu/\lambda}(x) = \sum_{(\alpha,\beta)\vdash n} W_{(\alpha,\beta)}(q,-q^{r_1+m})\chi^{(\alpha,\beta)}(x)$$

defines a Markov trace for the reduced algebra $p_{t^\lambda}H_{n+f}(q)p_{t^\lambda}$ with parameter $z = \frac{q^r(1-q)}{(1-q^r)}$. By Theorem 3.4.2, $\rho_{f,n} : H_n(q,-q^{r_1+m}) \to p_{t^\lambda}H_{n+f}(q)p_{t^\lambda}$ is an onto homomorphism. In particular, the irreducible modules of $p_{t^\lambda}H_{n+f}(q)p_{t^\lambda}$ are also irreducible modules of $H_n(q,-q^{r_1+m})$. Therefore, we have that the above equation also defines a trace which satisfies the Markov property for $H_n(q,-q^{r_1+m})$. We know that $W_{(\alpha,\beta)}(q,Q)$ and $\chi^{(\alpha,\beta)}$ are analytic functions; thus by the identity theorem in complex analysis, since the weights work for all $Q = -q^{r_1+m}$, they must work for all Q. □

Lemma 4.2.1 and Lemma 4.2.2 imply that the function $W_{(\alpha,\beta)}(q,Q)$ defined in equation (4.2) is a weight function for a Markov trace with parameter $z = q^r(1-q)/(1-q^r)$. In Section 4.1 we noted that a Markov trace on the Hecke algebra of type B is uniquely determined by a parameter z and by the values on the set $\{t_0't_1'\cdots,t_{k-1}' \,|\, k \geq 1\}$. Therefore, we still need to compute the values of tr on this set in order to completely determine the trace defined by the weights. To compute these values, we need the following definitions. Most of these definitions were given in Sections 2.5 and 2.6, but we repeat some of them for the reader's convenience.

For any pair $A \subset B$ of semisimple finite algebras and a trace, tr, nondegenerate on both A and B, recall that one can define the conditional expectation $\varepsilon_A : B \to A$ is defined by $\mathrm{tr}(\varepsilon_A(b)a) = \mathrm{tr}(ba)$ for all $a \in A$ and $b \in B$. ε_A is well-defined and unique.

Let B be represented via left multiplication on itself, where we write $L^2(B, \mathrm{tr})$ to denote the representation space B to distinguish it from the algebra B. We use b_ξ to denote the elements in $L^2(B, \mathrm{tr})$.

One obtains from ε_A an idempotent $e_A : L^2(B, \mathrm{tr}) \to L^2(B, \mathrm{tr})$ defined by $e_A b_\xi = \varepsilon_A(b)_\xi$. The idempotent e_A can be thought of as an orthogonal projection onto A with respect to the bilinear form $(b_\xi, c_\xi) \to \mathrm{tr}(bc)$. Recall that the algebra $\langle B, e_A \rangle$ generated by B and e_A is Jones basic construction for $A \subset B$. For the proof of the following theorem see [J1].

Theorem 4.2.3. *Let A, B, ε_A, e_A, and* tr *be as defined above. Then*

(a) The algebra $\langle B, e_A \rangle$ is isomorphic to the centralizer $End_A B$ of A acting by left multiplication on B. In particular, it is semisimple.

(b) There is a 1-1 correspondence between the simple components of A and $End_A B$ such that if $p \in A_i$ is a minimal idempotent, pe_A is a minimal idempotent of $\langle B, e_A \rangle_i$.

(c) $e_A b e_A = \varepsilon_A(b) e_A$ for all $b \in B$.

In our case we have the pair of semisimple algebras $H_{n+f-1}(q) \subset H_{n+f}(q)$. The corresponding orthogonal projection is $p_{[1^r]}$, where r is the maximum length of the partitions indexing the irreducible representations. It might also be helpful to keep Figure 4.1 in mind, since it clearly shows the veracity of the next lemma.

Before stating the lemma, we would like to define a tensor product of Hecke algebras given in [GW] by Goodman and Wenzl. We have $H_n(q) \otimes H_m(q) \subset H_{n+m}(q)$ defined by $a \otimes b = a(\mathrm{shift}_n(b))$ for $a \in H_n(q)$ and $b \in H_m(q)$, where shift_n is the

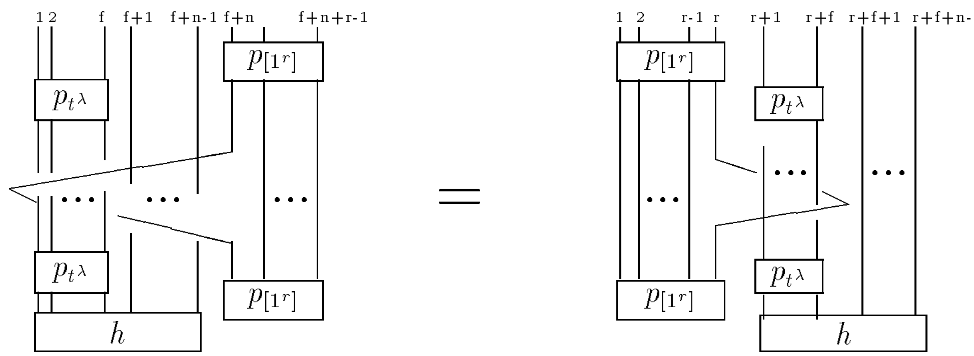

Figure 4.1: Lemma 4.5

operator which sends g_i to g_{i+n} for all i. In particular, this tensor product allows us to multiply minimal idempotents using the generalized Littlewood-Richardson rule in [GW]. Denote by 1_n the identity in $H_n(q)$.

Lemma 4.2.4. *A Markov trace on the Hecke Algebra of type A induces a Markov trace on the Hecke algebra of type B, which satisfies the condition that* $\mathrm{tr}(t'_n x) = y\,\mathrm{tr}(x)$*, where* $x \in H_n(q, Q)$*. In particular, this implies that* $y_k = y^k$ *for all* $k \in \mathbb{N}$*.*

Proof. Let tr be a Markov trace for $H_{n+f}(q)$ and $\mathrm{tr}_{p_{t^\lambda}}$ be the Markov trace corresponding to the reduced algebra. We have shown that the weights for the reduced algebra define a Markov trace for $H_n(q, -q^{r_1+m})$.

Assume that $\varepsilon_{H_{n+f-1}(q)} : H_{n+f}(q) \to H_{n+f-1}(q)$ is the unique conditional expectation with respect to the Markov trace defined by the weight function. Thus, $\varepsilon_{H_{n+f-1}(q)}(g_{n+f}h) = zh$ for all $h \in H_{n+f-1}(q)$ and $z = q^r \frac{(1-q)}{(1-q^r)}$.

We denote the image of t'_n under the homomorphism ρ_{n+f} by $p_{t^\lambda}\tau_{n+f}$. Consider the element $p_{t^\lambda}\tau_{n+f}h$ where $h \in H_{n+f-1}(q)$. We would like to compute the conditional expectation of $p_{t^\lambda}\tau_{n+f}h$. Observe that

$$\varepsilon_{H_{n+f-1}(q)}(p_{t^\lambda}\tau_{n+f}h) = \varepsilon_{H_{n+f-1}(q)}(p_{t^\lambda}\tau_{n+f})h.$$

Therefore it suffices to compute the conditional expectation for $p_{t^\lambda}\tau_{n+f}$. Consider the following

$$(p_{t^\lambda} \otimes 1_{n-1} \otimes p_{[1^r]})(\tau_{n+f} \otimes 1_{r-1})(p_{t^\lambda} \otimes 1_{n-1} \otimes p_{[1^r]})(h \otimes 1_r)$$

The above expression is equal to the left hand side of the following equation.

$$[((p_{[1^r]} \otimes p_{t^\lambda})(1_{r-1} \otimes \tau_{n+f})(p_{[1^r]} \otimes p_{t^\lambda})) \otimes 1_{n-1}](1_r \otimes h)$$
$$= (const)(p_{[1^r]} \otimes p_{t^\lambda} \otimes 1_{n-1})(1_r \otimes h)$$

Since we are restricted to r rows we have by the generalization of the Littlewood-Richardson rule, see [GW], that $p_{[1^r]} \otimes p_{t^\lambda}$ is a minimal idempotent. But Theorem 4.2.3 (c) implies that $(const)p_{t^\lambda} = \varepsilon_{H_{n+f-1}}(q)(\tau_{n+f}p_{t^\lambda})$. Thus,

$$\mathrm{tr}_{p_{t^\lambda}}(\tau_{n+f}h) = (const)\,\mathrm{tr}_{p_{t^\lambda}}(h).$$

This implies that we have a Markov trace with the property $\mathrm{tr}(t'_n h) = (const)\,\mathrm{tr}(h)$ for all $h \in H_{n+f-1}(q)$. But this trace is also a trace for $H_n(q, -q^{r_1+m})$. Since the above property holds for all $Q = -q^{r_1+m}$, then it must hold for all Q. Thus our assertion is proved. □

Theorem 4.2.5. *Let $r_1, r_2 \in \mathbb{N}$ and set $r = r_1 + r_2$. If* tr *is a Markov trace on the Hecke algebra of type B, with parameter $z = q^r \frac{(1-q)}{(1-q^r)}$, such that $\mathrm{tr}(t'_0 t'_1 \cdots, t'_{k-1}) = y^k$ for $k \geq 1$. Then the weights are given by $W_{(\alpha,\beta)}(q,Q)$ as defined in equation (4.2) with $y = \frac{(q^{r_2}Q+1)(1-q^{r_1})}{(1-q^r)} - 1$.*

Proof. The fact that the weights define a Markov trace follows from Lemmas 4.2.1, 4.2.2 and 4.2.4. It remains to show that the weight formula is indeed given by this values of $y = \frac{(Qq^{r_2}+1)(1-q^{r_1})}{(1-q^{r_1+r_2})} - 1$. By Lemma 4.2.1 it suffices to compute $\mathrm{tr}(t)$ in $H_1(q)$. This is a straight forward computation, we have

$$W_{([1],\emptyset)}(q,Q) = \frac{(1-q^{r_1})(1+Qq^{r_2})}{(1-q^r)(1+Q)} \quad \text{and} W_{(\emptyset,[1])}(q,Q) = \frac{q^{r_1}(1-q^{r_1})(1+Qq^{-r_1})}{(1-q^r)(1+Q)}.$$

Also $\chi^{([1],\emptyset)}(t) = Q$ and $\chi^{(\emptyset,[1])}(t) = -1$. Using these values we compute

$$\mathrm{tr}(t) = QW_{([1],\emptyset)}(q,Q) - W_{(\emptyset,[1])}(q,Q) = \frac{(Qq^{r_2}+1)(1-q^{r_1})}{(1-q^r)} - 1. \qquad \square$$

In [W1] Lemma 3.5 Wenzl showed that if $l(\mu) > r$ then $s_{\mu,r}(q) = 0$. Also he showed that $s_{\mu,r}(q)$ is well-defined when q is a primitive l-th root of unity with $l > 1$ if $\mu_1 - \mu_r \leq l - r + 1$, and $s_{\mu,r}(q) = 0$ if and only if $\mu_1 - \mu_r = l - k + 1$. In particular, $s_{\mu,r}(q) \neq 0$ for all (r, l) diagrams. Furthermore, he shows that the weight vector for the restriction of tr to $H_n^{(r,l)}(q)$ is given by the vector $(s_{\mu,r}(q))_{\mu \in \Lambda_n^{(r,l)}}$.

Proposition 4.2.6. *The Markov trace defined by the weights in equation in equation (4.2) factors over the quotient of the Hecke algebra of type B,* $H_n^{(r,l)}(q, -q^{r_1+m})$.

Proof. This proposition is a direct consequence of the results cited before the statement of this proposition. We know that the quotient $H_n^{(r,l)}(q, -q^{r_1+m})$ is isomorphic to a quotient of the reduced algebra $p_{t^\lambda}^{(r,l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$ which has weight vectors given by $\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}$. Since $[m^{r_1}]$ is an (r, l) diagram we have that $s_{[m^{r_1}],r}(q) \neq 0$. So the weights are well-defined. Furthermore, they will be zero exactly when $s_{\mu,r}(q) = 0$. □

The results in this chapter have appeared in the paper Weights of Markov traces on Hecke algebras, R.C. Orellana, 1999. The dissertation author was the primary investigator and single author of this paper.

Chapter 5

Subfactors

5.1 C^*-representations of $H_n(q,Q)$

A factor is a C^*-algebra. Thus, we need to find nontrivial C^*-representations of $H_n(q,Q)$. That is, we need to find the values of the parameters q and Q which make the generators e_i self-adjoint.

In what follows we show that there are C^* representations of $H_\infty(q,Q)$ when $Q=-q^k$ and q is an l-th root of unity.

Definition 5.1.1. A representation ρ of $H_n(q,Q)$ or $H_\infty(q,Q)$ on a Hilbert space is called a C^* *representation* if $\rho(e_t)$ and $\rho(e_i)$ for $i=1,2,\cdots,n-1$ or for all $i\in\mathbb{N}$ are self-adjoint projections.

Wenzl in [W1] showed that there are nontrivial C^* representations of $H_\infty(q)$, if q is real and positive or if $q=e^{2\pi i/l}$, where l is a positive integer greater than or equal to 4. Since $H_\infty(q)\subset H_\infty(q,Q)$ it follows that to obtain C^* representations of $H_\infty(q,Q)$ it is necessary for q to be real and positive or an l-th root of unity, unfortunately, this is not sufficient, we will also need a condition for Q.

Proposition 5.1.2. *If q and Q are both real and positive there are faithful C^* representations of $H_n(q,Q)$ for all $n\in\mathbb{N}$. If $q=e^{\pm 2\pi i/l}$ and $Q=-q^{r_1+m}$ for $l,m,r_1,r_2\in\mathbb{N}$ with $l\geq 4$, and $r_1\leq m+r_1\leq l-r_2$ then $\pi^{(r,l)}_{(\alpha,\beta)}$ is a C^* representation* *for all (m, l, r) diagrams (α, β).*

Proof. By Theorem 3.5.1 there exists a surjective homomorphism from the specialized algebra $H_n^{(r,l)}(q, -q^{r_1+m})$ onto $p_{t^\lambda}^{(r,l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$. In [W1] Wenzl showed that there exists a quotient of $H_{n+f}(q)$ which is a C^* algebra when q is an l-th root of unity. Furthermore, this quotient is semisimple and the irreducible modules are indexed by (r, l)-diagrams.

Since $p_{t^\lambda}^{(r,l)}$ is an idempotent in $H_{n+f}^{(r,l)}(q)$ and it is well-defined when q is a root of unity then $p_{t^\lambda}^{(r,l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$ has a C^* representation and the irreducible modules are indexed by (r, l)-diagrams which contain the diagram λ. $\square$

We have obtained a quotient of $H_n(q, -q^{r_1+m})$ for which there are C^* representations.

5.2 Subfactors, Index and Commutants

In the previous section we showed that there are quotients of the specialized Hecke algebra of type B, $H_n(q, -q^{r_1+m})$, which are C^* algebras. We have also shown that these quotients are isomorphic to $p_{t^\lambda}^{(r,l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$, whenever q is an l-th root of unity. We are now in position to construct the subfactors which arise from the inclusion of the Hecke algebra of type A into the Hecke algebra of type B. We will give the index and relative commutants for these subfactors.

In order to construct the subfactors we will use the following two sequences of algebras. Let $l,\ m,\ r_1,\ r_2 \in \mathbb{N}$ and set $r = r_1 + r_2$

(i) Let $B_n = p_{t^\lambda}^{(r,l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$ be the finite dimensional C^* algebras in the previous section. Then we have the sequence given by the proper inclusion of $B_n \subset B_{n+1}$ for all $\mathbb{N}$.

(ii) Let $A_n = p_{t^\lambda}^{(r,l)} H_{f,n+f}^{(r,l)}(q) p_{t^\lambda}^{(r,l)}$, where $H_{f,n+f}^{(r,l)}(q)$ is the finite dimensional C^* algebra generated by $g_{f+1}, \cdots, g_{f+n}$ in $H_{n+f}^{(r,l)}(q)$. Furthermore, we have that $A_n \subset B_n$.

Thus, we have two sequences (A_n) and (B_n) of C^* algebras such that $A_n \subset B_n$. $p^{(r,l)}_{t^\lambda}$ is the identity in A_n and B_n.

From the work of Jimbo [J1] and Drinfel'd [D] we know that the if q is not a root of unity and V is the fundamental module of $U_q(\mathfrak{sl}(r))$, quantum group of $\mathfrak{sl}(r)$. Then there is a representation

$$H_n(q) \to End_{U_q(\mathfrak{sl}(r))}(V^{\otimes n})$$

Moreover, if we restrict this map to the quotient of $H_n(q)$ with simple modules indexed by Young diagrams with r rows, then we get a faithful representation.

By the onto homomorphism $\rho_{f,n} : H_n(q, -q^{r_1+m}) \to p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ we have the following representation

$$H_n(q, -q^{r_1+m}) \to End_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \otimes V^{\otimes n})$$

where $V_{[m^{r_1}]}$ is the $U_q(\mathfrak{sl}(r))$-module of highest weight $[m^{r_1}]$.

Let q be an l-th root of unity. We outline some of the results and definitions about $U_q(\mathfrak{sl}(r))$ modules as necessary for our purpose, see [W3, A] for more details. A *tilting module* of $U_q(\mathfrak{sl}(r))$ is a direct summand of a tensor power of the fundamental module V, or it is a direct sum of such modules.

Tilting modules satisfy the following properties:

(1) Tensor products of tilting modules are tilting modules.

(2) Any tilting module is isomorphic to a direct sum of indecomposable tilting modules.

These two properties imply that any tilting module can be written as a direct sum of indecomposable tilting modules. Each indecomposable tilting module has a q-dimension. If q is a root of unity, this q-dimension can be zero. An indecomposable tilting module with 0 q-dimension will be called *negligible*. The indecomposable negligible modules generate a tensor ideal, which we will denote by $\mathcal{N}eg(T)$.

Thus, let W_1 and W_2 be two tilting modules, we now define a tensor product $\bar{\otimes}$ as follows:

$$W_1 \bar{\otimes} W_2 = (W_1 \otimes W_2)/\mathcal{N}eg(W_1 \otimes W_2)$$

Using this tensor product we have the following representation of the Hecke algebras at roots of unity. Let V be the fundamental module of $U_q(\mathfrak{sl}(r))$

$$H_n^{(r,l)}(q) \to End_{U_q(\mathfrak{sl}(r))}(V^{\bar{\otimes} n})$$

and also

$$H_n^{(r,l)}(q, -q^{r_1+m}) \to End_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \bar{\otimes} V^{\bar{\otimes} n})$$

Thus we have that $A_n \cong End_{U_q(\mathfrak{sl}(r))}(1 \bar{\otimes} V^{\bar{\otimes} n})$ and $B_n \cong End_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \bar{\otimes} V^{\bar{\otimes} n})$.

By Proposition 4.3 in [W3] if we take the above identifications for A_n and B_n then we have that the sequence of algebras (A_n) and (B_n) satisfy the commuting square property. And the sequences (A_n) and (B_n) are periodic.

It is well-known that under the periodicity assumption there exists at most one normalized trace on $B_\infty = \bigcup_{n \geq 0} B_n$, which must be a factor trace, that is, the weak limit of $\pi_{tr}(\bigcup_{n \geq 0} B_n)$ must be a factor. Similarly, for $A_\infty = \bigcup_{n \geq 0} A_n$. Therefore, one obtains a pair of hyperfinite II_1 factors

$$A = \pi^{(r,l)}(A_\infty)'' \subset B = \pi^{(r,l)}(B_\infty)''$$

In [W1] Wenzl showed that if a factor is generated by a ladder of commuting squares and if the Bratteli diagrams are periodic, then the index is given as a quotient of the weight vectors of the unique normalized trace, i.e., if $\vec{s}_n$ is the weight vector on A_n and $\vec{t}_n$ is the weight vector on B_n then the index is given by the following formula whenever n is big enough:

$$[B : A] = \frac{||\vec{s}_n||^2}{||\vec{t}_n||^2}. \tag{5.1}$$

Proposition 5.2.1. *Let r_1, $r_2 \in \mathbb{N}$ and set $r = r_1 + r_2$. For each pair m, $l \in \mathbb{N}$ such that $m \leq l - r$, $Q = -q^{r_1+m}$ and $q = e^{2\pi i/l}$, there is a subfactor of the* *hyperfinite II_1 factor obtained from the inclusion $A \subset B$ with index given by the following formula:*

$$\prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\frac{\sin^2((r_1+m+j-i)\pi/l)}{\sin^2((r_1+j-i)\pi/l)}.$$

Proof. In Theorem 4.2.5 we showed that the weight formula of the Markov trace on $H_n(q,-q^{r_1+m})$ is given by

$$\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}$$

where μ is an (r,l)-diagram with $n+f$ boxes containing $[m^{r_1}]$. Thus the norm of the weight vectors is

$$||\vec{t}_n|| = \sum_{[m^{r_1}]\subset\mu\vdash n+f}\frac{(s_{\mu,r}(q))^2}{(s_{[m^{r_1}],r}(q))^2}.$$

Now, note that $H_{f,n+f}(q)$ commutes with $p^{(l)}_{t^\lambda}$, thus $p^{(l)}_{t^\lambda}H_{f,n+f}(q)p^{(l)}_{t^\lambda} = p^{(l)}_{t^\lambda}H_{f,n+f}(q)$, thus the weight vector is given by

$$||\vec{s}_n|| = \sum_{[m^{r_1}]\subset\nu\vdash n+f}(s_{\nu,r}(q))^2$$

Therefore, we have by Wenzl's index formula in [W1] that the index for this subfactors is

$$\begin{aligned}[B:A] &= (s_{[m^{r_1}],r}(q))^2\\ &= \prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\frac{\sin^2((r_1+m+j-i)\pi/l)}{\sin^2((r_1+j-i)\pi/l)}\end{aligned}$$

The last equality is obtained by the substitution $q = e^{2\pi i/l}$ into this Schur function. □

Remark: This proposition can also be proved in much more generality using the tensor category machinery introduced before the statement of the proposition, see [W3]. It is well-known that the index is not a complete invariant of II_1 factors. A finer invariant is given by the higher relative commutants. Consider the following tower of II_1 factors associated to $A \subset B$

$$A \subset B \subset B^{(1)} = \langle B, e_1\rangle \subset B^{(2)} = \langle B^{(1)}, e_2\rangle \subset B^{(3)} = \langle B^{(2)}, e_3\rangle \cdots$$

where $B^{(1)} = \langle B, e_1 \rangle$ is obtained by the basic construction applied to $A \subset B$ and e_1 is the projection onto the trivial representation. Since $[B^{(i)} : A] = [B : A]^{i+1} < \infty$, $[B^{(i)} : B] = [B : A]^i < \infty$ the higher relative commutants $A' \cap B^{(i)}$ are all finite dimensional algebras.

Observation: It is clear from the relations of the Hecke algebra of type A that the relative commutant, $A' \cap B$ is

$$p_{t^\lambda}^{(l)} H_f(q) p_{t^\lambda}^{(l)} \cong p_{t^\lambda}^{(l)} \mathbb{C}.$$

since $\lambda = [m^{r_1}] \vdash f$, where p_{t^λ} is the identity of these factors.

The subfactors we have obtained are special cases of the subfactors obtained by Wenzl [W3]. The higher relative commutants for the subfactors obtained in this paper are given as follows.

Proposition 5.2.2. *Let $m,\ r_1,\ r_2 \in \mathbb{N}$. Let $A \subset B$ be the pair of factors constructed above with index as described in the previous proposition, then the higher relative commutants are given by*

$$A' \cap B^{(i)} = End_{U_q(\mathfrak{sl}(r))}(\cdots V_{[m^{r_1}]} \bar{\otimes} V_{[m^{r_2}]} \bar{\otimes} V_{[m^{r_1}]}) \quad (i+1 \ factors) \tag{5.2}$$

where $U_q(\mathfrak{sl}(r))$ is the quantum group of $\mathfrak{sl}(r)$ and $V_{[m^{r_2}]} \cong (V_{[m^{r_1}]})^*$.

Proof. The proof of this proposition follows from the proof of Theorem 4.4 in [W3]. We will outline the proof for the readers convenience. In order to compute the higher relative commutants we first compute the i-th extension $B^{(i)}$ via Jones' basic construction.

One obtains $B^{(1)}$ by taking the union of $End_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_2}]} \bar{\otimes} V_{[m^{r_1}]} \bar{\otimes} V^{\bar{\otimes} n})$ since we have the periodicity condition for n sufficiently large. By induction on i one obtains the i-th extension $B^{(i)}$ via Jones' construction by taking the union of $End_{U_q(\mathfrak{sl}(r))}((\cdots V_{[m^{r_1}]} \bar{\otimes} V_{[m^{r_2}]}) \bar{\otimes} V_{[m^{r_1}]} \bar{\otimes} V^{\bar{\otimes} n})$. From this it is obvious that $A' \cap B^{(i)}$ contains an algebra isomorphic to

$$End_{U_q(\mathfrak{sl}(r))}(\cdots V_{[m^{r_2}]} \bar{\otimes} V_{[m^{r_1}]}) \quad (i+1 \ factors).$$

For sufficiently large n we have that a copy of the trivial representation in $V^{\otimes n}$ is in the direct summand of $V_{[m^{r_1}]}\bar{\otimes} V^{\bar{\otimes} n}$. Thus, if p is the minimal projection onto the trivial representation, then we have an isomorphism between $A' \cap B^{(i)}$ and $pB_n^{(i)}p$; in particular they have the same dimensions by Theorem 1.6 in [W1]. $\square$

Remarks: (1) This proposition can also be shown by computing all the iterations of Jones' basic construction by adding generators on the "left" and then reducing by the appropriate projection, for details on this construction see [E]. She obtains all the higher relative commutants as a corollary of this construction for some subfactors of the Hecke algebra of type A.

(2) One can use the generalization of the Littlewood-Richardson [GW] rule to obtain a direct sum decomposition of the tensor product of two simple tilting modules with nonzero q-dimension when q is a root of unity.

In most part this chapter is an adaptation of the material as it appears in the preprint The Hecke algebra of type B and D and subfactors, R.C. Orellana,1999. The dissertation author was the primary investigator and single author of this paper.

Chapter 6

The Hecke algebra of type D

6.1 Weights of the Markov trace

The easiest way to study Markov traces on the Hecke algebras of type D is by embedding these algebras into those of type B, and then applying the results obtained for the Hecke algebra of type B. In this section we will denote the Hecke algebra of type B by $H_n^B(q,Q)$.

To obtain an embedding of the Hecke algebra of type D into $H_n^B(q,Q)$ we have to set the parameter Q equal to 1. Notice that in this case we have $t^2 = 1$. The Hecke algebra of type D, H_n^D is generated by $u = tg_1t,\ g_1, \ldots, g_{n-1}$ satisfying the following relations:

(D1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1, \ldots n-2$;

(D2) $g_i g_j = g_j g_i$ whenever $|i-j| \geq 2$;

(D3) $g_i^2 = (q-1)g_i + q$ for all i;

(D4) $g_i u = u g_i$ for all i;

(D5) $u^2 = (q-1)u + q$.

We have $H_n^D(q) \subset H_n^B(q,1)$ for all n; then we have the following inclusion of inductive limits

$$H_\infty^D(q) = \bigcup_{n>1} H_n^D(q) \subset H_\infty^B(q,1).$$

Geck in [G] shows that the restriction of a Markov trace on $H_\infty^B(q,Q)$ is a Markov trace on $H_\infty^D(q)$ and both have the same parameter. Furthermore, he shows that every Markov trace on $H_\infty^D(q)$ can be obtained in this way.

From Hoefsmit [H] we know that the simple components for $H_n^D(q)$ are indexed by double partitions (α,β). If $\alpha \neq \beta$ we have that the $H_n^B(q,Q)$-modules $V_{(\alpha,\beta)}$ and $V_{(\beta,\alpha)}$ are simple, equivalent $H_n^D(q)$-modules. And if $\alpha = \beta$ we have that the $H_n^B(q,Q)$-module $V_{(\alpha,\alpha)}$ decomposes into two simple nonequivalent $H_n^D(q)$-modules, i.e. $V_{(\alpha,\alpha)_i}$ with $i = 1,2$. Using Bratteli diagrams we have the following relations for simple modules of the $H_n^B(q,1)$ and $H_n^D(q)$

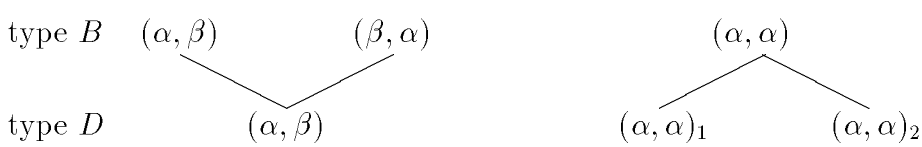

Figure 6.1: Inclusion of H_n^D into H_n^B

Proposition 6.1.1. *Let $r_1, r_2 \in \mathbb{N}$ and set $r = r_1 + r_2$. Then the weight formula for the Markov trace on the Hecke algebra of type D with parameters $z = q^r \frac{(1-q)}{(1-q^r)}$ and $y = \frac{(Qq^{r_2}+1)(1-q^{r_1})}{(1-q^r)} - 1$ is given as follows:*

$$W_{(\alpha,\beta)}^D(q) = W_{(\alpha,\beta)}(q,1) + W_{(\beta,\alpha)}(q,1), \quad \textit{if } \alpha \neq \beta$$

and

$$W_{(\alpha,\alpha)_i}^D(q) = W_{(\alpha,\alpha)}(q,1), \ \textit{for } i = 1,2 \quad \textit{if } \ \alpha = \beta$$

where $W_{(\alpha,\beta)}(q,1)$ denote the weight evaluated at $Q = 1$ of the Hecke algebra of type B.

Proof. The proof of this proposition follows directly from the inclusion matrix of the Hecke algebra of type D into the Hecke algebra of type B. Recall that in order to obtain the weight vector for the Hecke algebra of type D we multiply the inclusion matrix for $H_n^D(q) \subset H_n^B(q, Q)$ with the weight vector for type B. $\square$

6.2 Subfactors via Hecke algebra of type D

Denote the Hecke algebra of type A by $H_n^A(q)$. Now let $r_1, m \in \mathbb{N}$ and assume that q is a primitive $2(r_1+m)$-root of unity. This implies that $Q = -q^{r_1+m} = 1$. Observe that we have the following inclusion of algebras $H_n^A \subset H_n^D \subset H_n^B$. In Chapter 4 we described subfactors obtained from the inclusion $H_n^A(q) \subset H_n^B(q, -q^{r_1+m})$. In what follows we would like to consider the subfactors obtained from the inclusions $H_n^D \subset H_n^B$ and $H_n^A \subset H_n^B$.

We have shown that there exist C^*-representations of $H_n^B(q, -q^{r_1+m})$ with $r_1 + m < l - r_2$ which holds true for $l = 2(r_1 + m)$ and $r_2 < r_1 + m$. Therefore, we have the following inclusion of hyperfinite II_1 factors

$$D = \pi^{(l)}(H_\infty^{(l)})'' \subset B = \pi^{(l)}(H_\infty^B)''.$$

Proposition 6.2.1. *Let $r_1,\ r_2 \in \mathbb{N}$. The index for the inclusion of $D \subset B$ is given as follows:*

$$[B : D] = \begin{cases} 1 & \text{if } r_1 \neq r_2 \\ 2 & \text{if } r_1 = r_2. \end{cases}$$

Proof. Choose $n >> r_1 + r_2$ and assume $r_1 \neq r_2$. Without loss of generality we may assume $r_1 < r_2$, then we have for a pair of Young diagrams that $l(\alpha) = r_1$ and $l(\beta) = r_2$. In this case we have that $W_{(\beta,\alpha)}(q, 1) = 0$. this implies that the weight vectors for the Hecke algebra of type B and type D are equal. Thus by Wenzl's index formula that $[B : D] = 1$.

From equation (4.2) in Chapter 3 we can see that $W^B_{(\alpha,\beta)}(q, 1) = W^B_{(\beta,\alpha)}(q, 1)$ for any pair (α, β). Thus by the previous proposition we have that $W^D_{(\alpha,\beta)}(q) =$

$2W_{(\alpha,\beta)}(q,1)$. Thus we have

$$[B:D] = \frac{\sum_{\alpha\neq\beta}(2W^B_{(\alpha,\beta)}(q,1))^2 + \sum_{\alpha=\beta} W^D_{(\alpha,\alpha)_1}(q)^2 + W^D_{(\alpha,\alpha)_2}(q)^2}{\sum_{\alpha\neq\beta}(W^B_{(\alpha,\beta)}(q,1))^2 + \sum_{\alpha=\beta} W^B_{(\alpha,\alpha)}(q)^2} = 2$$

since $W^D_{(\alpha,\alpha)_i}(q) = W^B_{(\alpha,\alpha)}(q,1)$.□

Corollary 6.2.2. *Let $r_1 = r_2 \in \mathbb{N}$. The index for the inclusion of $A \subset D$ is given as follows:*

$$[D:A] = [B:A]/2$$

Proof. By Proposition 2.18 in [J2] we have that if we have an inclusion of three II_1 factors, $A \subset D \subset B$ then $[B:A] = [D:A][B:D]$. By the previous proposition we have our result. □

The results in this chapter appear in the preprint The Hecke algebra of type B and D and subfactors, R.C. Orellana, 1999. The dissertation author was the primary investigator and single author of this paper.

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