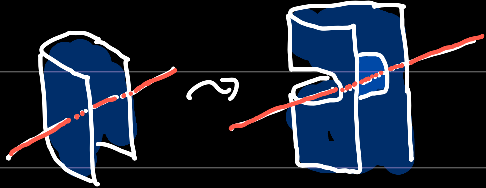
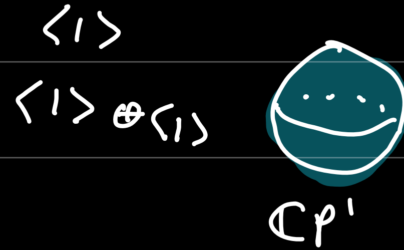
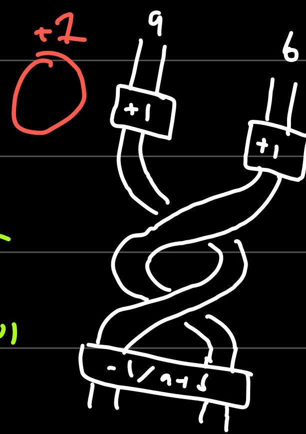


A Note On



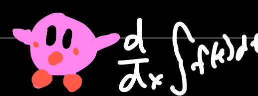
"A Note On Surfaces in $\mathbb{C}P^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$"

Part II

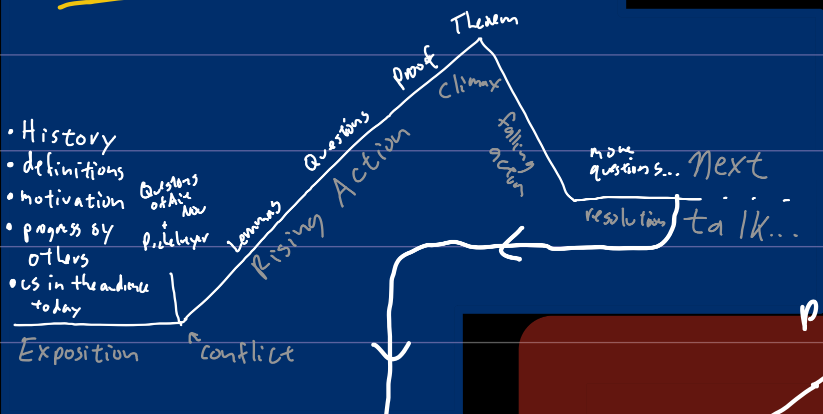


Scotty Tilton
UCSD

April 19th, 2023

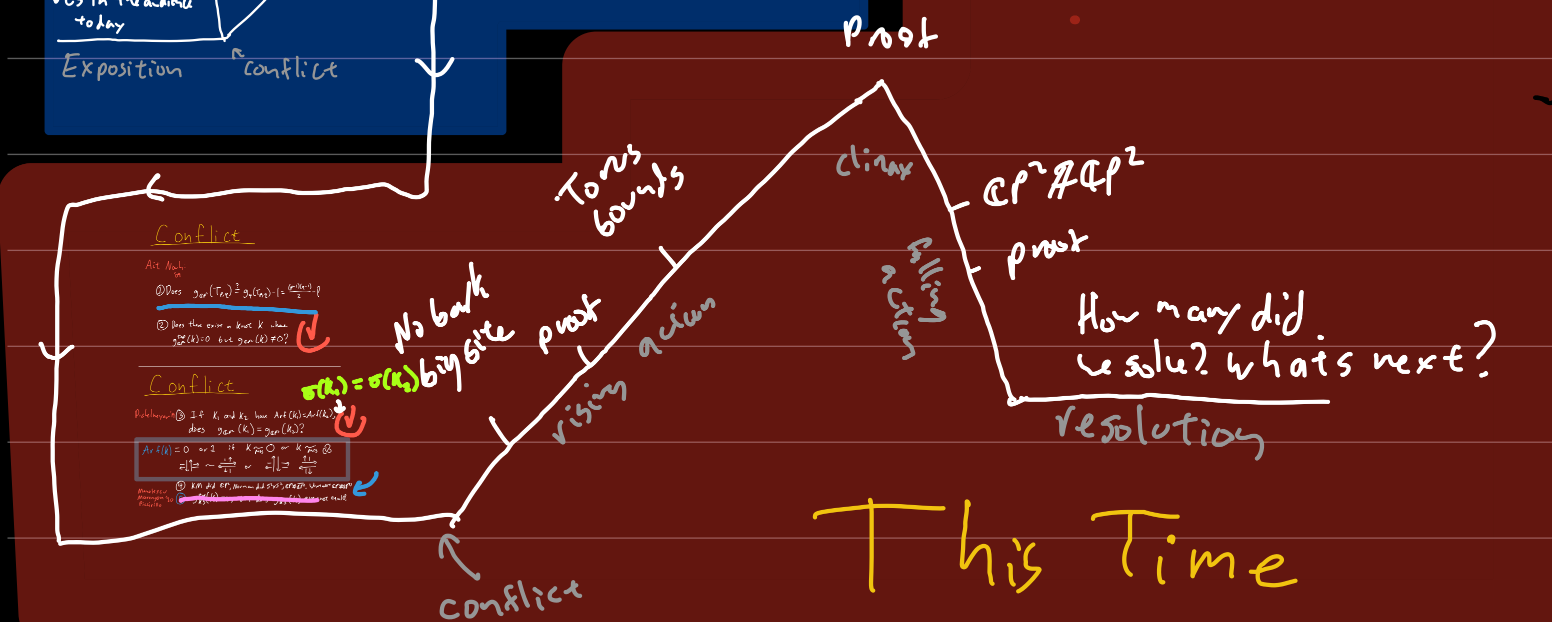


Last time



Big pieces

• Knots w/ arbitrarily large smooth $\mathbb{C}P^2$ genus BUT! are also topologically slice



This Time

arxiv.org/pdf/2210.12496

Things Proved last Time

$\mathbb{C}^\times$ $\quad \mathbb{CP}^2 \setminus \mathring{B}^4$

Proposition (MMRS) There are knots in $(\mathbb{CP}^2)^\times$ with arbitrarily large smooth $\mathbb{CP}^2$ genus.

e.g.

Theorem 1.1 (MMRS) Let $n \geq 0$. There exists a knot with $g^{(top)}_{\mathbb{CP}^2}(K) = 0$ and $g_{\mathbb{CP}^2}(K) \geq n$

Questions We answered

Ait Nouh '09: Does there exist a knot K with $g^{top}_{\mathbb{CP}^2}(K) = 0$ but $g_{\mathbb{CP}^2}(K) \neq 0$?

Answer Yes! The knots from Theorem 1.1

Pichelmeyer '19: If K_1 and K_2 satisfy $Arf(K_1) = Arf(K_2)$, and $\sigma(K_1) = \sigma(K_2)$, ← Important! Missed last time

does $g_{\mathbb{CP}^2}(K_1) = g_{\mathbb{CP}^2}(K_2)$?

Answer No! The knots from theorem 1 have $\sigma_K(-) \equiv 0$ and $Arf = 0$.

Questions left

Natural Question Can we produce a lot of knots with $g^{top}_{\mathbb{CP}^2}(K) = 0$? KPRT22 showed $g^{top}_{\mathbb{CP}^2}(K) \leq 1$ for all K.

Ait Nouh '09 Does

$$g_{\mathbb{CP}^2}(T_{p,q}) = g_4(T_{p,q}) - 1 = \frac{(p-1)(q-1)}{2} - 1 \; ?$$

Natural Question We know $G_{\mathbb{CP}^2}$ because of Kronheimer and Mrowka, we know $G_{S^2 \times S^2}$ and $G_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}}$ because of Norman. Can we glean any info about $G_{\mathbb{CP}^2 \# \mathbb{CP}^2}$?

Answer to first question

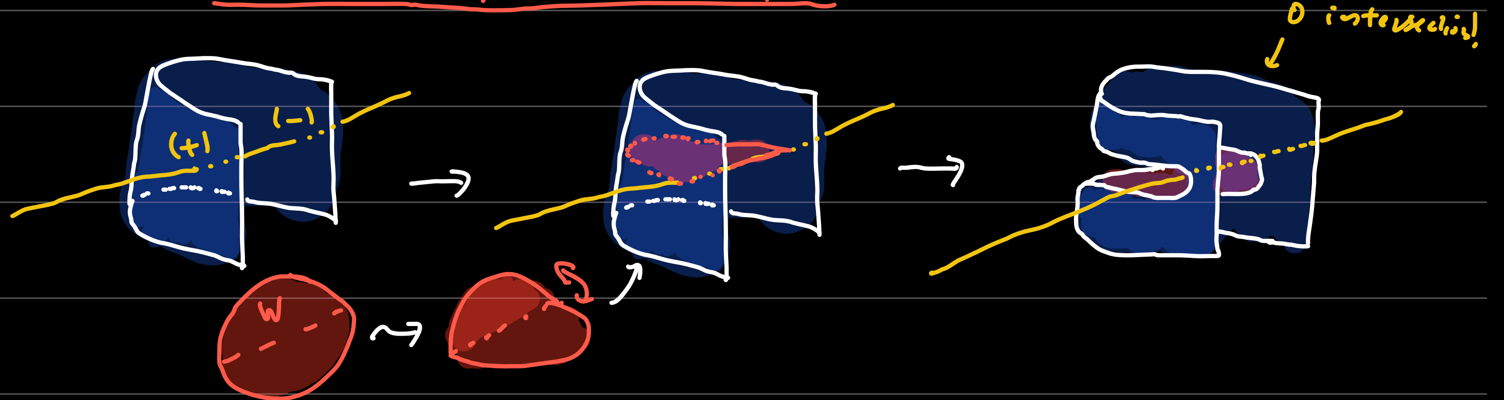
All Bark? Top slice! ← My name for it

MMRS Prop 7.4 Let $K \subset S^3$ be a knot. If $\mathrm{Arf}(K) = 0$ then $g^{top}_{\mathbb{CP}^2}(K) = 0$

Proof • Let Δ be a generically immersed disk in B^4 with $K = \partial\Delta \subset \partial B^4 = S^3$.

- Add trivial, local self-intersections so the double points of Δ algebraically cancel
- Pair these double points with Whitney disks $\{W_i\}$ where $\mathring{W}_i \cap \Delta$ at isolated points

3D whitney disk example



- Matsumoto '78, Freedman Kirby '78, Conant Schneiderman Teichner '14 proved

$$\mathrm{Arf}(K) \equiv \sum_{W_i} \#(\mathring{W}_i \cap \Delta) \bmod 2 = 0$$ ← true!

- Now connect sum $B^4 \# \mathbb{CP}^2 = (\mathbb{CP}^2)^\times$ so that $\Delta, \{W_i\}_i$ are disjoint from $\mathbb{CP}^1$.
- Tube Δ into $\mathbb{CP}^1$, call this Δ' $\quad x: S^2 \hookrightarrow (\mathbb{CP}^2)^\times$
- Δ' is spherically characteristic, i.e. $\Delta' \cdot x = x \cdot x \bmod 2$ for all
- This means FQ '90 + Sto '94 $\Rightarrow \Delta' \simeq_\partial \Delta''$ where Δ'' is an embedding

  Δ' has an algebraically dual sphere C with $\Delta' \cdot C = 1$.
- So Δ'' is a locally flat embedded disk with $\partial\Delta'' = K \Rightarrow g^{top}_{\mathbb{CP}^2}(K) = 0$. ∎

The above theorem gives us a lot of knots with $g^{top}_{\mathbb{CP}^2}(K) = 0$. (No claim it's all of them)

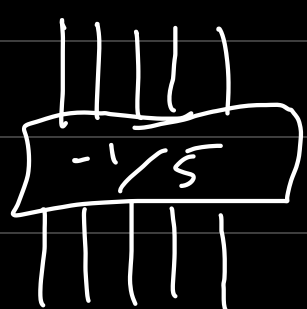
Natural Question 1

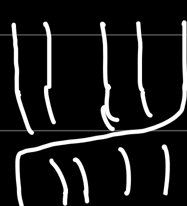
What about the minimal $\mathbb{CP}^2$-genus of torus knots?

Theorem 2.26 MMRS Let $n > d \geq 0$. The torus knot $T_{n,n-1}$ bounds a surface Σ in $(\mathbb{CP}^2)^\times$ with degree d and genus

$$g(\Sigma) = \begin{cases} \frac{1}{4}\left((n+d-2)^2 + (n-d-2)^2\right) & n \equiv d \bmod 2 \\ \frac{1}{4}\left((n+d-1)(n+d-3) + (n-d-1)(n-d-3)\right) & n \not\equiv d \bmod 2 \end{cases}$$

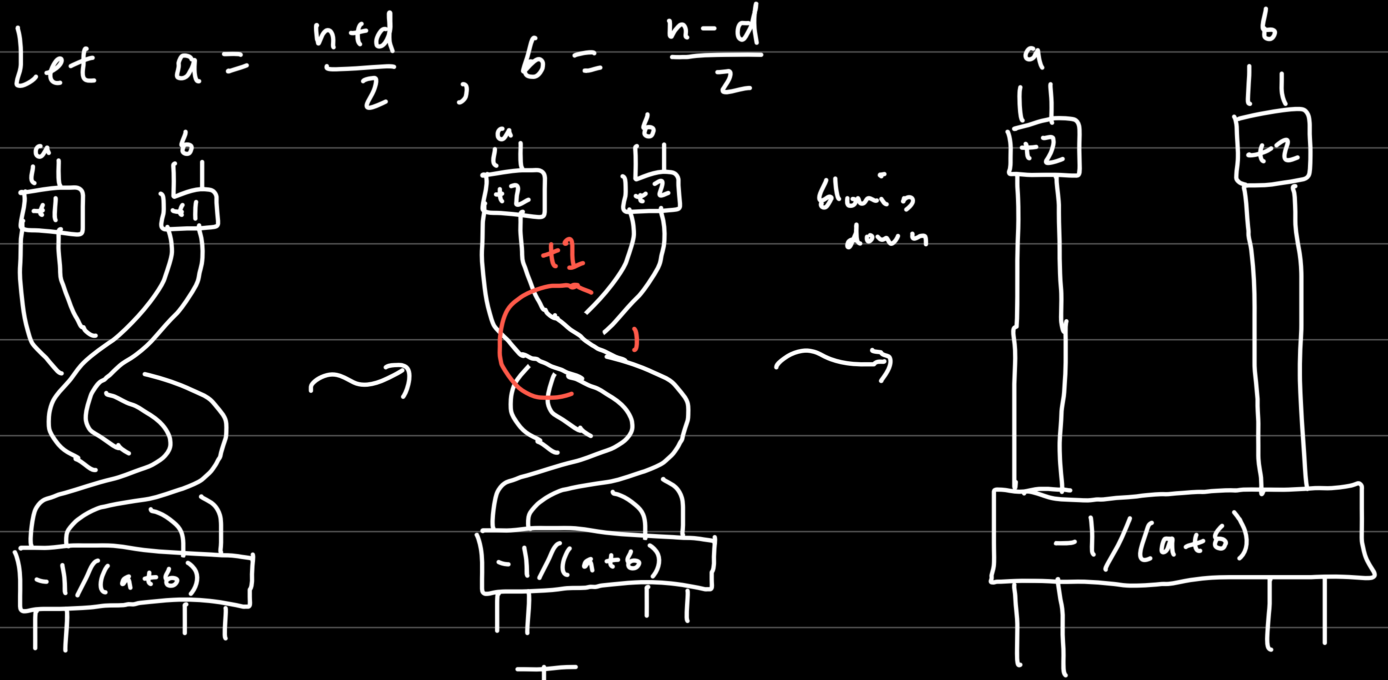
$-1/5$





Idea-o-Proof (same parity case)

Let $a = \frac{n+d}{2}$, $b = \frac{n-d}{2}$

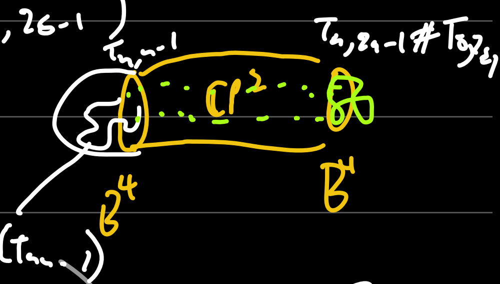


$T_{n,n-1}$ $\qquad$ $T_{n,n-1}$ $\qquad$ $T_{a,2a-1} \# T_{b,2b-1}$

K_0 $\qquad$ exhibiting a concordance in $\mathbb{C}P^2 \setminus (B^4 \sqcup B^4)$ $\qquad$ K_1

- Cap off K_0 with a minimal surface in B^4 to get

$$g(\Sigma) = g_4(T_{a,2a-1}) + g_4(T_{b,2b-1})$$
$$= (a-1)^2 + (b-1)^2$$



$\square$

(Similar story for $n \not\equiv d \mod 2$)

$\Downarrow$ $\quad g_{\mathbb{C}P^2}(K) \leq g_4(T_{n,n-1})$

Corollary 1.2 [MMRS]

$$g_{\mathbb{C}P^2}(T_{n,n-1}) \leq \begin{cases} g_4(T_{n,n-1}) - \frac{n-2}{2} & n \equiv 0 \bmod 2 \\ g_4(T_{n,n-1}) - \frac{n-1}{2} & n \equiv 1 \bmod 2 \end{cases}$$

pf) Set $d = 0$.

This answers a question!

Ait Nouh '09 | Does

$$g_{\mathbb{CP}^2}(T_{p,q}) = g_4(T_{p,q}) - 1 = \frac{(p-1)(q-1)}{2} - 1 \;?$$

Answer | Nope! plug in $n > 2$

$\leq$

Finally, $\mathbb{CP}^2 \# \mathbb{CP}^2$

Note | $H_2(\mathbb{CP}^2 \# \mathbb{CP}^2) \underset{M.V.}{=} H_2(\mathbb{CP}^2) \oplus H_2(\mathbb{CP}^2)$

$G_{\mathbb{CP}^2 \# \mathbb{CP}^2} \leq G_{\mathbb{CP}^2} + G_{\mathbb{CP}^2}$?

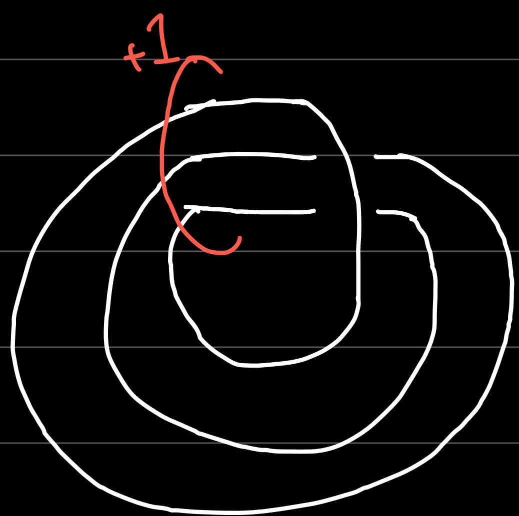
Yeah probably, but how much less?

- Note $\quad g_{\mathbb{CP}^2}(-T_{n,n-1}) = 0$ ☆

- We can choose the slice disk to be

$$[\Sigma] = n\,[\mathbb{CP}^1] \in H_2(\mathbb{CP}^{2*}, \partial\mathbb{CP}^{2*})$$

- $n=3$



← handle decomp for $(\mathbb{CP}^2)^*$

← blowdown gives empty surgery for S^3 and $-T_{n,n-1}$

← image of std. slice gives $n \cdot$ generator of rel H_2

- Take $((\mathbb{CP}^2)^*, T_{n,n-1}) \# ((\mathbb{CP}^2)^*, -T_{n,n-1})$ with the surface from theorem 2.6 and our disk just now, so we get a surface with genus from 2.6
- it's boundary is $T_{n,n-1} \# -T_{n,n-1}$ which is slice in B^4
- Cap off with disk to get a closed surface Σ in $\mathbb{CP}^2 \# \mathbb{CP}^2$ representing (n,d). $\boxtimes$

So what?

Theorem 1.3 MMRS

$$G^{(n,d)}_{\mathbb{CP}^2\#\mathbb{CP}^2} \le G_{\mathbb{CP}^2}(n) + G_{\mathbb{CP}^2}(d) + \begin{cases} \frac{n-3d}{2} & n \equiv d \bmod 2 \\ \frac{n-3d+1}{2} & n \not\equiv d \bmod 2 \end{cases}$$

$\Longrightarrow$ The difference between

$G^{(n,d)}_{\mathbb{CP}^2\#\mathbb{CP}^2}$ and $G_{\mathbb{CP}^2}(n) + G_{\mathbb{CP}^2}(d)$

can be arbitrarily large.

AN'14 improved the naive upper bound by 1.

Recap

Thm 1.1 For any $n \geq 0$, there are knots with $g^{top}_{\mathbb{CP}^2}(K) = 0$ and $g_{\mathbb{CP}^2}(K) \geq n$.

Cor 1.2 For any $n \geq 0$

$$g_{\mathbb{CP}^2}(T_{n,n-1}) \le \begin{cases} g_4(T_{n,n-1}) - \frac{n-2}{2} & n \equiv d \bmod 2 \\ g_4(T_{n,n-1}) - \frac{n-1}{2} & n \not\equiv d \bmod 2 \end{cases}$$

Theorem 1.3 MMRS

$$G^{(n,d)}_{\mathbb{CP}^2\#\mathbb{CP}^2} \le G_{\mathbb{CP}^2}(n) + G_{\mathbb{CP}^2}(d) + \begin{cases} \frac{n-3d}{2} & n \equiv d \bmod 2 \\ \frac{n-3d+1}{2} & n \not\equiv d \bmod 2 \end{cases}$$

What's Next?

- $g_{K3}(K) \neq 0$? Fact: Needs $u(K) > 21$
- All knots with $g^{top}_{\mathbb{CP}^2}(K) = 0$?
- Better bounds for $g_{\mathbb{CP}^2}(T_{p,q})$ or $G_{\mathbb{CP}^2 \# \mathbb{CP}^2}$?

$$g^{top}_{K3}(K) = \min\left\{ \text{genus } \Sigma \;\middle|\; \Sigma \xrightarrow[\text{flat embedding}]{\text{locally}} K3^{\times} \right\}$$

Thank You!