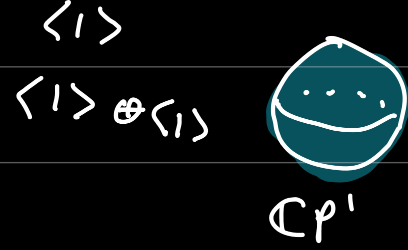
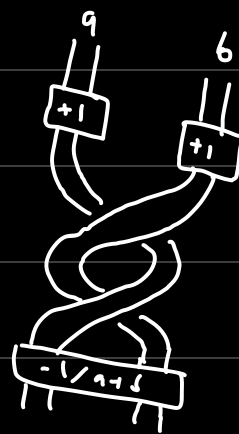
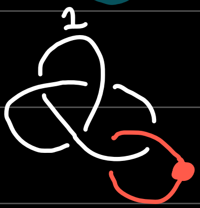
A Note On



"A Note On Surfaces in $\mathbb{CP}^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$"



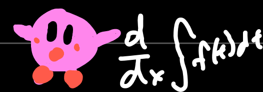


Part I

Scotty Tilton

UCSD

April 12th, 2023



I'm presenting on

"A Note on Surfaces in $\mathbb{CP}^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$",

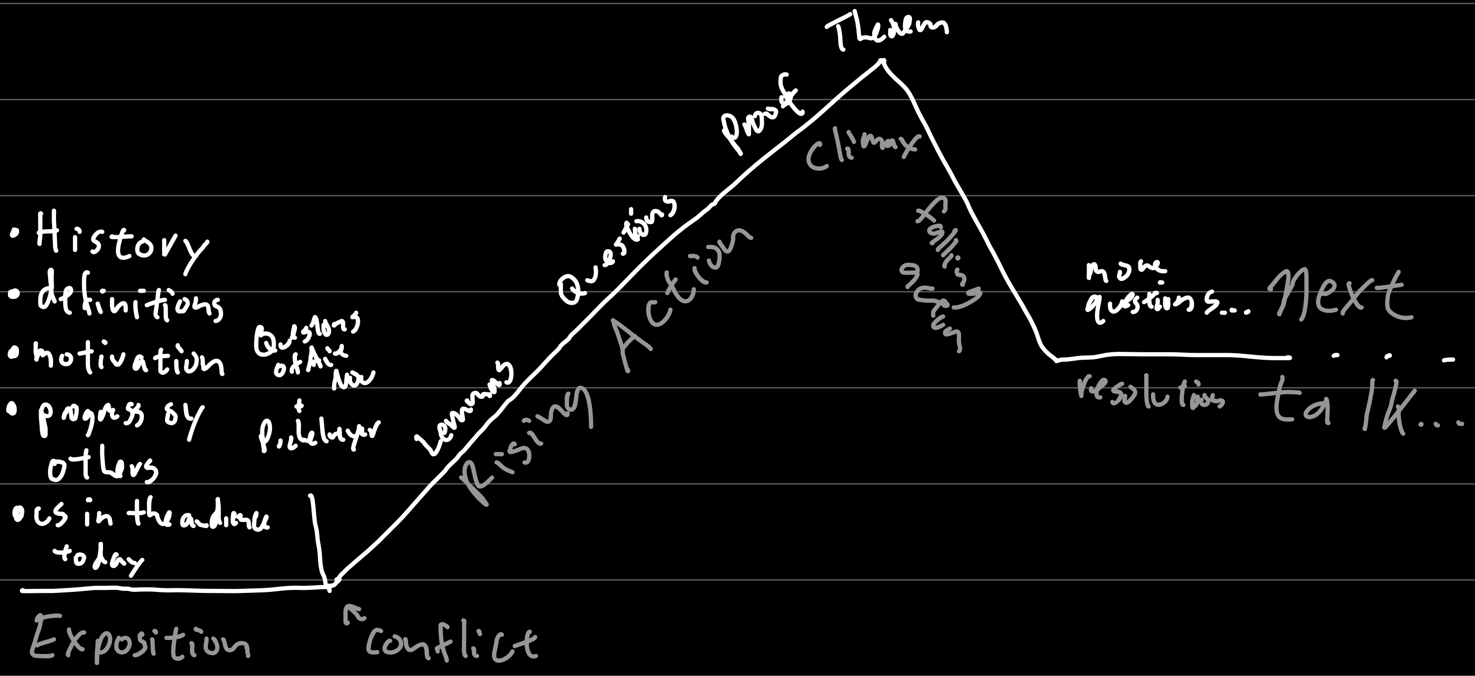
a paper written and added on the arXiV in Oct 2022 by

Marco Marengon
Allison N. Miller
Arunima Ray
András Stipsicz

arxiv.org/abs/2210.12486

I may have added some of my own thoughts and errors.

Outline for this talk



We may go further or shorter on this plot, but we'll have an exciting conclusion next week at the same Bat-time on this same Bat-channel.

Exposition

Mathematicians love to classify things

prime numbers, finite groups, semisimple Lie algebras,
manifolds(???) (let's start with spheres)

$\boxed{\text{Poincaré conjecture}}$ if $M^n \simeq S^n$ is $M^n \cong S^n$?

homeo
diffeo
PL-eo

Topological
Poincare

- 1D ✓
- 2D ✓
- 3D — 2001 ✓ Perelman
- 4D — 1980s ✓ Freedman
- 5D+ — 1960s ✓ Smale

Exposition

Mathematicians love to classify things

Smooth poincare

odd dim	even dim
1, 3, 5, 61	0 ✓
else	2 ✓
	4 +

Moise

Milnor, Smale, Serre

Kervaire-Milnor 60s
Hill-Hopkins-Ravenel '09
Wang-Xu '16

So let's work on 4

Exposition: Classifying smooth 4-mflds

Tools + Techniques

Freedman's theorem

Seiberg-Witten equations

Bauer-Furuta invariants

Kirby Calculus

☆ Minimal genus surfaces

Gompf-Freedman-Morrison-Walker suspect one can solve the smooth poincare conjecture in dim 4 using slice knots in $W \setminus \mathring{B}^4$ where $W \simeq S^4$!

Manolescu-Piccirillo have candidates for counterexamples using 0-surgery.

Minimal genus surface Crash Course

$\circ\circ\{\mathbb{C}^\times = \mathbb{C} \setminus \{0\}\}$

- Let M be a smooth 4-mfld and let $M^\times := M \setminus \mathring{B}^4$
- Note $H_2(M^\times, \partial M^\times) \underset{LES}{\cong} H_2(M^\times) \underset{M.V.}{\cong} H_2(M)$.

- $\partial M^\times \cong S^3$ and knots love living in S^3.
- Every knot bounds an oriented surface called a Seifert surface.
- Since we classified 2-D surfaces, the genus is well-defined.

Definitions - 0 - Min - Genus

Let $\alpha \in H_2(M)$

- $G_M(\alpha) := \min\left\{ \text{genus}(\Sigma) \;\middle|\; \begin{array}{l} i : \Sigma \xrightarrow[\text{embedding}]{\text{smooth}} M \\ i_*([\Sigma]) = \alpha \end{array} \right\}$

Let K be a knot $K \subset \partial M^\times \cong S^3$.

- $g_M^{top}(K) := \min\left\{ \text{genus}(\Sigma) \;\middle|\; \begin{array}{l} \partial\Sigma = K \\ \Sigma \xrightarrow[\text{embedding}]{\text{loc. flat}} M^\times \end{array} \right\}$

- $g_M(K) := \min\left\{ \text{genus}(\Sigma) \;\middle|\; \begin{array}{l} \partial\Sigma = K \\ \Sigma \xrightarrow[\text{embedding}]{\text{smooth}} M^\times \end{array} \right\}$

- $g_M^H(K) := \min\left\{ \text{"} \;\middle|\; \text{"} \quad \text{"} \; [\Sigma] = 0 \in H_2(M^\times, \partial M^\times) \right\}$

" g_4^H

Definitions/Facts about min genus

slice if $g_M(K) = 0$, topologically slice if $g_M^{top}(K) = 0$.

H-slice if $g_M^H(K) = 0$.

FACT | $g_M^{(top)}(K) \leq g_4^{(top)}(K)$ $\qquad g_M(K) \leq g_4(K)$

connect sum S^4 w/ M.

Successes on the minimal genus front for $M=\mathbb{C}P^2$

Thm

Kronheimer - Mrowka '94 (Thom Conjecture)

Let $h \in H_2(\mathbb{C}P^2;\mathbb{Z})$ be a generator. Let $d \neq 0$ be an integer.
Then $G_{\mathbb{C}P^2}(d \cdot h) = \dfrac{(|d|-1)(|d|-2)}{2}$.

(Morgan, Szabó, Taubes 1996 proved for Kähler mflds)
for surfaces w/ nonnegative self intersection

(Later Ozsváth-Szabó proved for symplectic mflds)
1998

Successes on the minimal genus front for $M=\mathbb{C}P^2$

Cor 1.12

Kasprowski
Powell '22
Ray
Teichner

showed $g^{top}_{\mathbb{C}P^2}(h) \leq 1$ for any h. Specifically

① $g^{top}_M(h) = 0$ if M is simply connected and not diffeo. to $\mathbb{C}P^2$ or S^4

② $g^{top}_{\mathbb{C}P^2}(h) \leq 1$ and $g_{\mathbb{C}P^2}((T_{2,3})^{\#3}) = 1$

Successes on the minimal genus front for $M = \mathbb{CP}^2$

Yasuhara '91, Ait Nouh '09, '14, Pichelmeyer '19 studied problems related to smooth minimal genus in $\mathbb{CP}^2$.

Thm Nouh '09 If $3 \leq q \leq 17$ (q odd)

$$g_{\mathbb{CP}^2}(T_{2,q}) = g_4(T_{2,q}) - 1 = \frac{q-3}{2}$$

$$\Longrightarrow g_{\mathbb{CP}^2}(T_{2,17}) = 7 \leftarrow \text{gold standard!}$$

Conflict/Questions

Ait Nouh '09:

① Does $g_{\mathbb{CP}^2}(T_{p,q}) \overset{?}{=} g_4(T_{p,q}) - 1 = \frac{(p-1)(q-1)}{2} - 1$?

② Does there exist a knot K where $g^{top}_{\mathbb{CP}^2}(K) = 0$ but $g_{\mathbb{CP}^2}(K) \neq 0$?

Conflict/Questions

Piccirillo '19 ③ If K_1 and K_2 have $\mathrm{Arf}(K_1) = \mathrm{Arf}(K_2)$, does $g_{\mathbb{CP}^2}(K_1) = g_{\mathbb{CP}^2}(K_2)$?

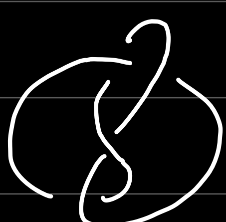
$\mathrm{Arf}(K) = 0$ or 1 if $K \sim_{pass} \bigcirc$ or $K \sim_{pass}$ 

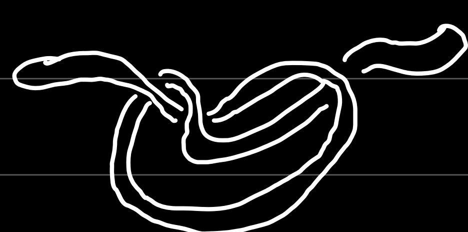
④ KM did $\mathbb{CP}^2$, Norman did $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. What about $\mathbb{CP}^2 \# \mathbb{CP}^2$?

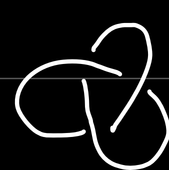
Manolescu Marengon Piccirillo '20 ⑤ $g^{tor}_{K3}(K) = 0$, but does $g_{K3}(K)$ ever not equal 0?

Rising Action

On our journey to find a solution to our problem, we bump into some old theorems lying around.







Some Invariants

chain $CF(S^3)$
$K \subset S^3$ induces a filtration
$F(K,m)$
$\subseteq$ subcomplex generated by intersection pts, Alex $\leq m$.

τ-invariant

Oszváth Szabó '03

$$\min \left\{ m \;\middle|\; H_*(F(K,m)) \xrightarrow{i_*} \widehat{HF}(S^3) = \mathbb{Z} \text{ is nontrivial} \right\}$$

– lower bound

$\sigma(K)$ Knot signature

$\text{signature}\left(V + V^T\right)$: V is Seifert matrix

$V_{ij} = \phi(b_i, b_j)$

$\phi : H_1(S) \times H_1(S) \to \mathbb{Z}$

$a, b \mapsto lk(a^+, b^-)$

$\sigma_K(z)$ Tristram-Levine Signature

$$\text{Signature}\left((1-z)V + (1-\bar{z})V^T\right)$$

See Lickorish Chap 8 for truth.

Let $K \subset S^3$ be a knot bounding a smooth, cpt, connected, properly embedded surface Σ of genus g in $(\mathbb{C}P^2)^*$ s.t

$$[\Sigma] = d[\mathbb{C}P^1] \text{ in } H_2(\mathbb{C}P^{2*}; \mathbb{Z}) \cong H_2(\mathbb{C}P^2; \mathbb{Z})$$

$\boxed{\text{Theorem Oszváth Szabó '03}}$ For the Heegaard-Floer invariant τ,

$$g \geq -\tau(K) + \frac{|d|(1-|d|)}{2}$$

$\boxed{\text{Theorems Gilmer '81, Viro '70}}$

$$2g+1 \geq \begin{cases} \left| \dfrac{d^2}{2} - 1 - \sigma(K) \right|, & d \text{ divisible by even prime} \\[2ex] \left| \dfrac{p^2-1}{2p^2} d^2 - 1 - \sigma_p(K) \right|, & d \text{ divisible by odd prime} \end{cases}$$

where $\sigma_p(K) = \sigma_K\left(e^{\pi i \frac{p-1}{p}}\right)$

With those in mind, can we find a knot with huge genus?

MMRS invite us to try

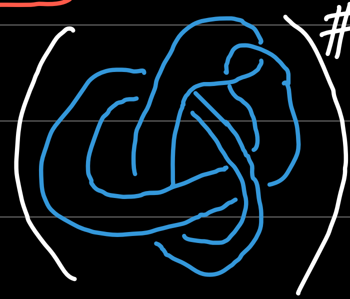
Let $g_0 \geq 0$ be as big as you'd like.

Let $c_0 > \frac{3}{2}\sqrt{2g_0+2} > 1$.

Let K be a knot satisfying

- $\sigma_K(-) \equiv 0$
- $-\tau(K) \geq g_0 - \frac{c_0(1-c_0)}{2}$

ex | $-Wh$(Trefoil knot)



$\#N$

alexander poly is 1

- Let Σ be a genus g surface in $(\mathbb{C}P^2)^\times$ with $\partial\Sigma = K$ and $[\Sigma] = d[\mathbb{C}P^1] \in H_2((\mathbb{C}P^2)^\times)$

Osvath-Szabo '03 $\Rightarrow$

$$g \geq -\tau(K) + \frac{|d|(1-|d|)}{2}$$
$$\geq g_0 - \frac{c_0(1-c_0)}{2} + \frac{|d|(1-|d|)}{2}$$

$\Rightarrow$ $g \geq g_0$ ✓

or

$\Downarrow$ $|d| \geq c_0 > 1$.

$2 \mid |d|$

$\Downarrow$ Gil, Vir

$$2g+1 \geq \left|\frac{d^2}{2} - 1\right|$$
$$\geq \frac{c_0^2}{2} - 1$$
$$\geq \frac{9}{4}(g_0+2) - 1$$

$\Rightarrow$ p odd $\mid |d|$ $\overset{\text{Gil, Vir}}{\Rightarrow}$

$$2g+1 \geq \left|\frac{p^2-1}{2p^2}d^2 - 1\right|$$
$$\geq \frac{p^2-1}{2p^2}c_0^2 - 1$$
$$> \left(\frac{1}{2} - \frac{1}{2p^2}\right)\frac{9}{4}(2g_0+2) \geq 2g+1$$

$$\boxed{g \geq g_0.}$$

∎

What did we prove?

Proposition (proved above) There exist knots with arbitrarily large $\mathbb{CP}^2$ genus.

Question | Can we find knots in $(\mathbb{CP}^2)^\times$ with $g^{top}_{\mathbb{CP}^2}(K)=0$ but $g_{\mathbb{CP}^2}(K)\neq 0$?

Question | Can we find knots in $(\mathbb{CP}^2)^\times$ with $g^{top}_{\mathbb{CP}^2}(K)=0$ but $g_{\mathbb{CP}^2}(K)\neq 0$?

Answer | Thm MMRS | Yes.

pf | • Recall $g^{top}_{M}(K) \leq g^{top}_{4}(K)$

$\#n$

- Knots here are topologically slice (Alexander Poly is 1, same for #, apply FQ'90)

  $g^{top}_{M}(K) \leq g_4(K) = 0$

- they have arbitrarily large genus by proposition!

Recall the conflict!

Conflict

Ait Nouh: '09

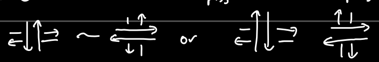
① Does $g_{\mathbb{CP}^2}(T_{p,q}) \stackrel{?}{=} g_4(T_{p,q}) - 1 = \frac{(p-1)(q-1)}{2} - 1$? ← still not done.

② Does there exist a knot K where $g^{top}_{\mathbb{CP}^2}(K) = 0$ but $g_{\mathbb{CP}^2}(K) \neq 0$? ← Heck yeah there does!

Conflict

Pichelmeyer '19 ③ If K_1 and K_2 have $Arf(K_1) = Arf(K_2)$, does $g_{\mathbb{CP}^2}(K_1) = g_{\mathbb{CP}^2}(K_2)$? ← Can we approach this?

$Arf(K) = 0$ or 1 if $K \underset{pass}{\sim} \bigcirc$ or $K \underset{pass}{\sim}$



④ KM did $\mathbb{CP}^2$, Norman did $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. What about $\mathbb{CP}^2 \# \mathbb{CP}^2$? ← Can we do this?

Manolescu, Marengon, Piccirillo '20 ⑤ $g^{top}_{K3}(K) = 0$, but does $g_{K3}(K)$ ever not equal 0? ← Save for another day.

Next time...

- Will the bark (Arf!) be bigger than the bite (slice!)?
- Do we know about torus knots in $\mathbb{CP}^2$?
  $\boxed{8}$ stars point to yes....
- Can we make a good estimate at what $G_{\mathbb{CP}^2 \# \mathbb{CP}^2}$ might be?

Thank

You!