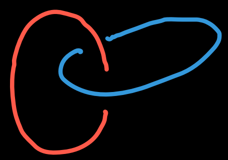
$\mathcal{M}$



Pin(2)-Bauer-Furuta Invariants

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$\not{\partial}^2 = D$

$\pi_*(S^0)$

$$\begin{array}{c} Spin(V) \\ \downarrow \times 2 \\ SO(V) \end{array}$$

Disclaimer

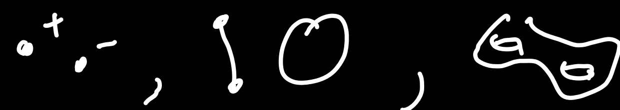
a) I'm going to talk about
Seiberg Witten invariants
$SW(M) \in H_*(M)$ and
Bauer Furuta invariants
$\pi_* S^0$

b) Overview, some sketches of proofs and my problem(s)

Outline

- Disclaimer
- Motivation / Progress
- $(S)Pin^{(c)}(V)$
- SW, (Pin(2)) BF
- dehn-twist
- Kronheimer-Mrowka
- Jianfeng
- Me

Problems in Topology

Classify things: 

S^4, 4, 3 pendulum

$\pi_* S^0 \quad \pi_k(S^n)$

Realm I'm in: Clasifying exotic structures

Geography Problem

given a top manifold is
it smoothable

Botany Problem

how many exotic smooth
structures

Progress on finding Exotic Structures

Donaldson theory - Found exotic $\mathbb{R}^4$, disproved smooth h-cob for 4 mflds

Seiberg-Witten (94) - find differential nonsmooth 4mflds

SW Ruberman (1998) - Find $f \in \text{Diff}(X \# \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2})$ that $f \not\simeq g$

Pin2SW Flor Manolescu (2003) - triangulation conjecture SW-Floer

BF Bauer-Furuta (2002) - $BF \in \pi^*_{\text{ind}\,\ell}(S^0)$

BF SW Baraglia-Konno (2019) - K theory obstruction, recover SW from BF

BF Kronheimer-Mrowka (2020) - $K3 \# K3 \ni \delta \not\simeq id$

Pin(2)-BF Jianfeng Lin (2020) - $K3 \# K3 \# S^2 \times S^2 \ni \delta \not\simeq id$

Pin(2)-BF Lin, Mukherjee (2021) - $\Sigma, \Sigma' \subset K3$ exotic, even after stabilization...

PreReqs for Bauer Furuta

Clifford algebra $\quad T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n}$ $\quad$ V is a rl innerproduct space

$$Cl(V) := T(V) \big/ v \otimes v + \|v\|^2 1$$

even odd

$$= Cl_0(V) \oplus Cl_1(V)$$

Ex $\quad Cl(\mathbb{R}^1) = \mathbb{R} \oplus \mathbb{R} \oplus \cdots \big/ \sim = \mathbb{C}$

$$Cl(\mathbb{R}^2) = \mathbb{H} \qquad Cl(\mathbb{R}^3) = \mathbb{H} \oplus \mathbb{H}$$

Def-0- (S)Pin^c(V)

$\mathrm{Pin}(V) = \langle v \in Cl(V)^\times \cap Cl_1(V) \mid \|v\| = 1 \rangle$ $\quad v \otimes v = -\|v\|^2 1 = -1$

$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap Cl_0(V)$

$\mathrm{Spin}^c(V) \subseteq Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$

generated by $\mathrm{Spin}(V)$ and $U(1)$

If you don't like that

$\mathrm{Spin}(n)$ $n \geq 3$

$\mathrm{Spin}(n) \xrightarrow{\times 2} SO(n)$

$\mathrm{Spin}(n) := \{ \gamma : [0,1] \to SO(n) \mid \gamma(0) = \mathbb{1} \}$

$\mathrm{Spin}^c(n) = \{ (A,B) \in U(n) \times U(n) \mid \det A = \det B \}$

$= \mathrm{Spin}(n) \times_{\{\pm\}} S^1$

$SU(2) \times SU(2) = SO(4)$

Examples

	1	2	3	4	$\mathrm{Spin}(4)$
Pin	$\langle i \rangle \subseteq \mathbb{C}$	$\langle j S^1 \rangle$			
Spin	$\{\pm 1\}$		S^3	$S^3 \times S^3$	

More

Spin structure

$$SO(4) \curvearrowright TM \qquad \begin{tikzcd} TM \arrow[d] \\ M \end{tikzcd}$$

$$\begin{tikzcd} Spin(4) \arrow[r, hook] \arrow[d, "\times 2"] & P \arrow[r, "\tau"] & P \\ SO(4) \arrow[r, hook] & Fr(M) \arrow[d] & \\ & M & \end{tikzcd}$$

spin^c structure

$$\begin{tikzcd} Spin^c(4) \arrow[d, "S^1"] \\ SO(4) \end{tikzcd} \qquad \begin{tikzcd} \tilde{P} \arrow[rr, "\tau"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

Definition of non-equivariant BF

Let X be a 4-mfld w/ $b_1(X) = 0$

X has a spin structure

The Seiberg Witten map: $\qquad S^+ \otimes_{\mathbb{H}} (S^-)^* \cong TM$

$$SW: W^+ \xrightarrow[\gamma \oplus (d^+, d^*) + \gamma(ia, \phi); q(\phi, \bar{\phi})]{\text{Hilbert spaces}} W^-$$

$$\begin{array}{ccc} W^+ & \longrightarrow & W^- \\ \| & & \| \\ \mathcal{V}^+ \oplus \mathcal{U}^+ & \longrightarrow & \mathcal{V}^- \oplus \mathcal{U}^- \\ \text{“}\|\text{”} & & \text{“}\|\text{”} \\ \Gamma(S^+) \oplus \Omega^1(X) & \longrightarrow & \Gamma(S^-) \oplus \Omega^+(X) \oplus \Omega^0(X)/\mathbb{R} \end{array}$$

Bauer-Furuta Invariant

Fredholm map between Hilbert spaces is a linear map

$f: \mathcal{H}' \to \mathcal{H}$ st $\dim(\ker f) < \infty$ $\dim(\operatorname{coker} f) < \infty$

Corollary (Bauer Furuta) Let

$f = l + c : \mathcal{H}' \longrightarrow \mathcal{H}$ be a compact perturbation of the linear fredholm map such that the preimages of bounded sets are bounded. Then f defines an element $[f] \in \pi^{st}_{ind\, l}(S^0)$ $\pm(\dim \ker l - \dim \operatorname{coker} l)$

$Pin(2) \curvearrowright \widetilde{\mathbb{R}}$ $\qquad$ $Pin(2) \curvearrowright \mathbb{H}$ left

$e^{i\theta} \curvearrowright$ trivially

$je^{i\theta} \curvearrowright$ multiply by -1

Approximate

Choose $\qquad U^+ = \widetilde{\mathbb{R}}^m$

$\mathbb{R} \xrightarrow{d^*} \Omega^n \xrightarrow{\beta} \Omega^{n+-1}$ $\qquad V^- = \mathbb{H}^n$

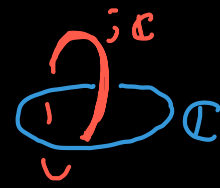
Set $\qquad U^- = H_2^+ \oplus (d^+, d^*)U^+)$

$V^+ = \phi^{-1}(V^-)$

$[sw] \in [W^+_\infty, W^-_\infty] = \pi^{st}_{ind\, l}(S^0)$

Fact

Pin(2)-Bauer Furuta



$Pin(2) \curvearrowright V^{\pm}$: $Pin(2) \curvearrowright U^{\pm}$

Let $\mathcal{U}$ contain the explicit representations

$$\bigoplus_{\infty} \mathbb{R} \oplus \bigoplus_{\infty} \tilde{\mathbb{R}} \oplus \bigoplus_{\infty} \mathbb{H}$$

then $BF^{Pin(2)}(X, \mathfrak{s}) \in \pi_*^{Pin(2)} S^0$

$$\operatorname*{colim}_{V \subset \mathcal{U}} [S^* \wedge S^V, S^0 \wedge S^V]$$

Dehn Twist

$$\begin{tikzcd} & Spin(4) \arrow[d, "\times 2"] \\ {[0,1]} \arrow[r, "\gamma"] \arrow[ur, "\tilde{\gamma}"] & SO(4) \end{tikzcd}$$

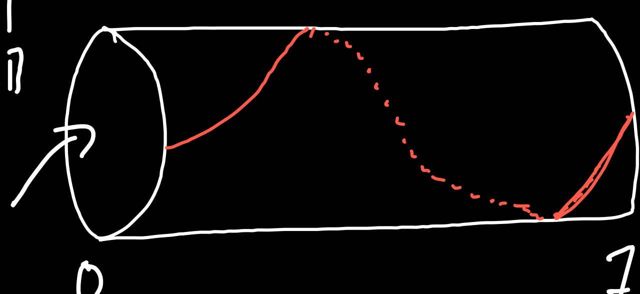
$\pi_1(SO(4)) = \mathbb{Z}/2$

So pick a nontrivial loop

$[\gamma] \in \pi_1(SO(4))$

$\gamma(0) = \mathbb{1}$ $\gamma(\mathbb{1})$

$$\begin{array}{ccc} X \# Y & & X \# Y \to X \# Y \\ \cup & & \cup \qquad \cup \\ S^3 \times [0,1] & & S^3 \times [0,1] \to S^3 \times [0,1] \\ & & (x, t) \mapsto (\gamma(t) x, t) \end{array}$$



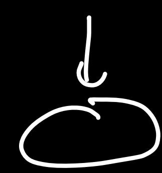
Theorem (Kronheimer–Mrowka 20)

On $K3\#K3$, the dehn twist is not isotopic to the identity.

Use of families is important

$N_g = K3\#K3\times[0,1] \big/ (x,0)\sim(g(x),1)$

$\downarrow$



Prop 5.1 For the family of 4-manifolds $N \to$ over the circle with fiber $K3\#K3$,

$$BF(N, s) = \eta_1 \times BF(X, s_X) \times BF(Y, s_Y)$$

Proof of theorem $BF(K3) = \eta_1$, $\eta_1^3 \in \pi_3 = \mathbb{Z}/24$ is the order 2 element.

Theorem Jianfeng 2020

The dehn twist on $K3\#K3$ is still not isotopic to id even after a stabilization ($K3\#K3\#S^2\times S^2$)

Proof (using some lemmas unwritten for the talk)

Suppose $BF^G(\text{Family}(K3\#K3\#S^2\times S^2), \tilde{s}) = \underbrace{\alpha \cdot e_{\tilde{\mathbb{R}}}}_{S^0 \hookrightarrow S^{\tilde{\mathbb{R}}}} = 0$

Then by a lemma $\Rightarrow$ $\mathrm{Res}^G_{S^1} = 0$. However,

$$\eta^3 = BF(\text{Family}(K3\#K3)) = \mathrm{Res}^{S^1}_e \underbrace{\mathrm{Res}^G_{S^1}}_{0}(BF^G(\text{Family } K3\#K3)) = 0 \quad \lightning$$

η^3 is not 0

My problems

- boundary dehn twist on $E(4)$
  (see monopoles and 3 mflds by Kronheimer Mrowka for details)

- boundary dehn twist on $K3 \# K3$

pick two homogenous deg 3 polynomials in 3 complex variables
$f, g \in \mathbb{C}[x, y, z]$ $\quad \mathbb{C}P^2 \dashrightarrow \mathbb{C}P^1$

where $\quad [x, y, z] \longmapsto [t_0 : t_1]$

$t_0 f + t_1 g\,(x:y:z) = 0$ $\quad$ Bezouts theorem says 9 points of int

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(1) blow up 9 times for f, g
$E(1)$ $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2 \longrightarrow \mathbb{C}P^1$

THANK YOU

Questions