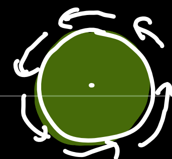
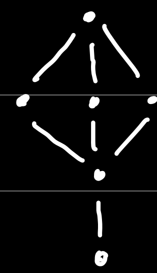
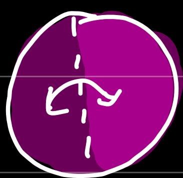
$\mathcal{O}$ $\mathcal{F}$  $G$

# Operads $101_{N_\infty}$



Scotty Tilton
eCHT REU
July 7, 2023

$\mathcal{U}$  $G \curvearrowright X$

$\mathcal{E}_n$ $f * g * h$

$H_0(N^\infty\text{-}Op^G) \simeq Tr(Sub(G))$

---

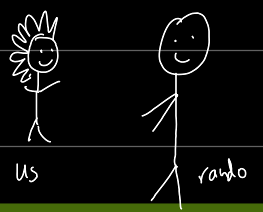
**Purpose**
- Introduction to $N_\infty$-operads
- show you why this REU cares

**Successful Outcome**
- We know what an $A_n, A_\infty = E_1, E_n, E_\infty, N_\infty$ operad is abstractly and with examples
- You see a nice theorem
- You see why we're doing what we're doing

**Plan**
- Recap last time
- Define new things
- See examples
- New adjectives
- Examples
- Theorem

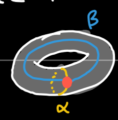
# Recap

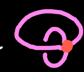
Scenario



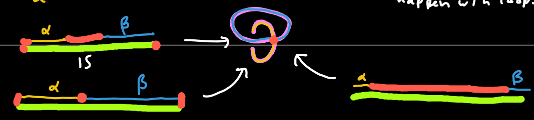

R: Math? Do you love multiplication?
U: Yeah!
R: Impress me.

$\Omega X_* =$ based loops in $X$




$\alpha * \beta =$  but ambiguous

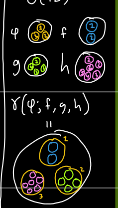
$\mathcal{O}(n) = \mathrm{Emb}((\mathbb{D}^1)^{\sqcup n}, \mathbb{D}^1) =$ all the ways this multiplication can happen w/n loops




**Defn** An operad is
- A sequence $\{\mathcal{O}(n)\}_{n\geq 0}$ of spaces where $\mathcal{O}(0)=*$
- For each $k$, for any $n_1, n_2, \dots, n_k$, maps $\gamma: \mathcal{O}(k)\times \mathcal{O}(n_1)\times\cdots\times\mathcal{O}(n_k) \to \mathcal{O}(\Sigma n_k)$
- A specific element $\mathbb{1}\in\mathcal{O}(1)$
- An action $\Sigma_n \curvearrowright \mathcal{O}(n)$

Satisfying
- $\gamma(\gamma(k;n_1,\dots n_k); m_1 \cdots m_{\Sigma n_i}) = \gamma(k; \gamma(n_1; m_1\dots m_{n_1}), \gamma(n_2; m_{n_1+1}\dots m_{n_1+n_2}), \dots \gamma(n_k;\dots))$

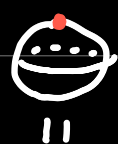
$\mathcal{O}(3)\times\mathcal{O}(2)\times\mathcal{O}(4)\times\mathcal{O}(6)$
$\downarrow$
$\mathcal{O}(12)$




assoc $A_n$ — $\mathcal{O}(k)$ are contractible for $k\leq n$
$E_1 = A_\infty$ — " all $k$
everything $E_n$ — $\Sigma_k \overset{\text{free}}{\curvearrowright}_{\text{action}} \mathcal{O}(k)$ and $\mathcal{O}(k)$ contractible for $k\leq n$
$E_\infty$ — " all $k$

---

# Operad example



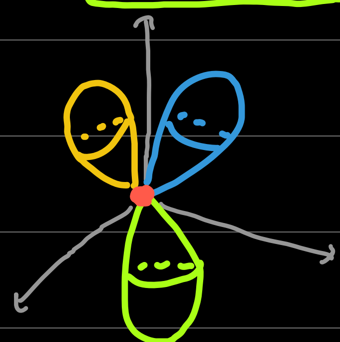
$$\Omega^2 X \cong \mathrm{Maps}^{cts}_*(S^2, X)$$

trying w/ $\mathbb{R}^3$

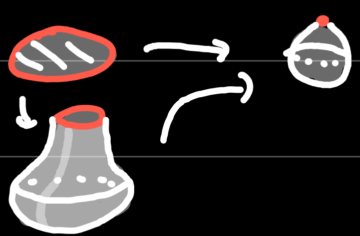
let $s, b, g \in \Omega^2 X$

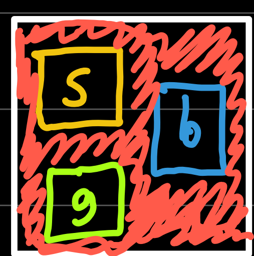
$s * b * g \in \Omega^2 X$

**Nice Fact**
$\Omega^n X$ is an $E_n$-algebra
i.e. an $E_n$ operad acts on its multiplication



**Note** If $\partial\mathbb{D}^2 \to \cdot$, $\mathbb{D}^2 \to X$, this is the same info as $S^2 \to X$






$\mathcal{O}(3)$
"
$\mathrm{Emb}((\mathbb{D}^2)^{\sqcup 3}, \mathbb{D}^2)$

# New things

Note: A group $G$ is
- a set $G$
- a assoc. multiplication $-\cdot-$
- with an identity and inverses

Let $G$ be a finite group

A **G-Space** is
- a topological space $X$
- an action of $G$ on $X$ $\quad (G \overset{\text{acts}}{\curvearrowright} X)$

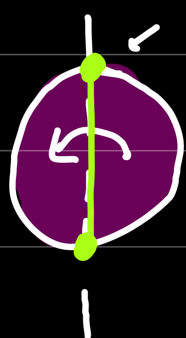
Two ways I think of actions

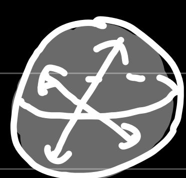
| $G \curvearrowright X$ | $G \curvearrowright X$ |
|---|---|
| $g \cdot x \in X.$ | a homomorphism |
| $g(g^{-1} \cdot x) = e \cdot x = x$ | $G \xrightarrow{\varphi} \text{Homeo}(X)$ |
| $g \cdot (h \cdot x) = (gh) \cdot x$ | $g \cdot x = \underbrace{(\varphi(g))}_{\text{Homeo}(X)}(x)$ |

---

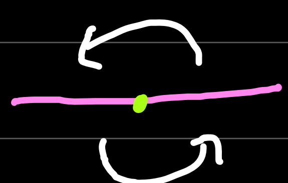
## Examples of G-spaces

$G = C_2 = \mathbb{Z}/2\mathbb{Z} = (\{0,1\}, +_2) = (\{\pm 1\}, \times)$
$= (\{1, a\}, \cdot)\ a^2 = 1$

① $\mathbb{D} =$  $\leftarrow \mathbb{D}^{C_2}$

② 

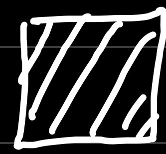
$\vec{x} \longmapsto -\vec{x}$

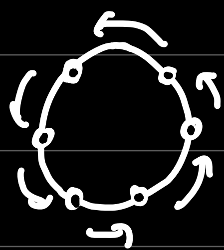
③ $X =$ 

④ $Y =$  no action

$X^{C_2} = \cdot \quad X^e =$ 

# Another Couple G-spaces

⑤ $G = D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ $\quad X =$ 

⑥ $G = C_6 = (\mathbb{Z}/6, +_6)$ $\quad X =$ 

⑦ $G = \Sigma_3$ $\quad X =$ 


$(1\,3) \cdot X \rightsquigarrow$ 


---

**Def** A **G-equivariant map** is a continuous map between G-spaces which is equivariant under the G-action

i.e. $f : X \to Y$ s.t.

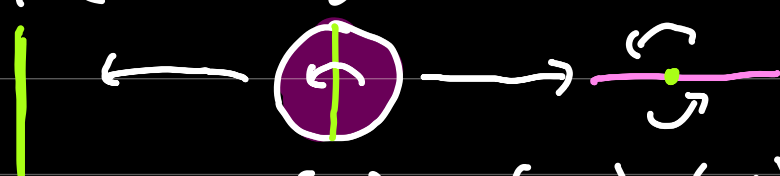
$$f(g \cdot x) = g \cdot f(x).$$

**Ex**

$$y \longleftarrow\!\shortmid (x,y) \longmapsto x$$

$$Y \xleftarrow{p_y} D \xrightarrow{p_x} X$$

trivial action



$$(-1) \cdot p_x(x,y) = (-1) \cdot x = -x \quad ✓$$

$$p_x((-1) \cdot (x,y)) = p_x(-x,y) = -x$$

# New Adjective

Let $G$ be a group.

**Def$^{n}$** A **$G$-operad** is

- a sequence of $G \times \Sigma_n$-spaces $(\mathcal{O}(n))_{n \geq 0}$

such that

- $\mathbb{1} \in \mathcal{O}(1)$
- there are $G$-equivariant maps

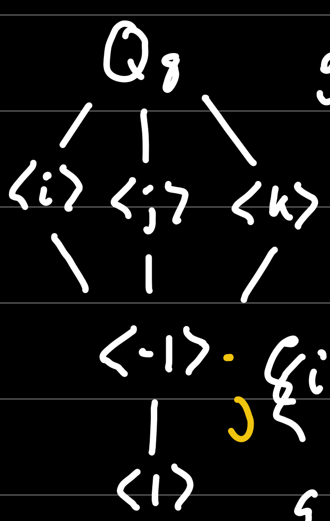
$$\gamma : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \longrightarrow \mathcal{O}\left(\sum_{i=1}^{k} n_i\right)$$

---

**Def$^{n}$** A **family for $G$, $\mathcal{F}$,** is a collection of subgroups which is closed under passage to subgroups and conjugation.

**Ex** $G = Q_8 = \{i,j,k \mid i^2 = j^2 = k^2 = ijk = -1\}$

$gHg^{-1}$

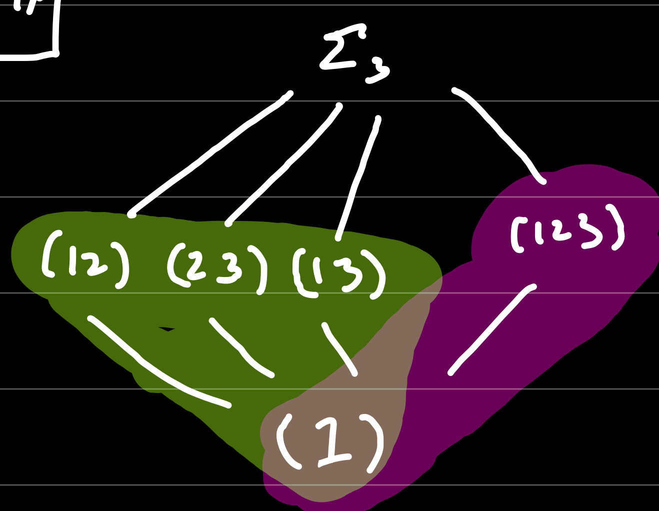



$j\{i,-1,-i,1\}j$

$=$

$\{i,-1,-i,1\}$

**Ex** $G = \Sigma_3$



**Def^n** If $\mathcal{F}$ is a family for $G$, a **universal space for $\mathcal{F}$** is a $G$-space $E\mathcal{F}$ such that for all subgroups $H \leq G$

points fixed by H

$$(E\mathcal{F})^H \simeq \begin{cases} * & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F}. \end{cases}$$

**Ex** $G = C_2$ $\quad \mathcal{F} = \{e\}$ $\quad (S^\infty)^{C_2} = \emptyset$

$$E\mathcal{F} = S^\infty = \left\{ (x_i)_{i=0}^\infty \;\middle|\; \sum x_i^2 = 1, \; \begin{array}{c} \text{finitely many} \\ \text{non zero terms} \\ \text{(infinitely many 0)} \end{array} \right\}$$

$C_2 \curvearrowright S^\infty$ $(-1)\cdot(x_i) \mapsto (-x_i)$

$$(E\mathcal{F})^e = S^\infty \simeq *.$$

---

# FiN∞A∞lly!

**Def^n** An **$N_\infty$-operad** $\mathcal{O}$ is

- a $G$-operad
- a collection of $G \times \Sigma_n$-families $(\mathcal{F}_n(\mathcal{O}))_{n \geq 0}$

such that

- $\mathcal{O}(0) \simeq * \simeq \mathcal{O}(1)$
- $\Sigma_n \underset{\text{freely}}{\curvearrowright} \mathcal{O}(n)$ for all $n$
- $\mathcal{F}_n(\mathcal{O})$ contains all $H \times \{1\} \leq G \times \Sigma_n$
- $\mathcal{O}(n)$ is a universal space for each family $\mathcal{F}_n(\mathcal{O})$

$N_\infty$ operad is a $G$-$E_\infty$ operad

**Examples** Let $G$ be a group and let $\mathcal{U}$ be a universe - a $G$ representation containing all f.d. subrepresentations - An $\infty$-dim $G$-vector space.

Linear isometries operad

① $\mathcal{L}(\mathcal{U})$ where $\mathcal{L}(\mathcal{U})_n = \mathrm{Maps}^G_{\mathrm{lin.}}(\mathcal{U}^n, \mathcal{U})$

$G \times \Sigma_n \curvearrowright \mathcal{L}(\mathcal{U})_n$ by conjugation and diagonal

Little disks

② $D(\mathcal{U})$, $D(\mathcal{U})_n = \operatorname{colim}_{V \subseteq \mathcal{U}} \left(\mathrm{Emb}^G\left(D(V)^{\amalg n}; D(V)\right)\right)$

Kind of Messy

Steiner

③ $K(\mathcal{U})$ - See Blumberg-Hill '15

---

How does our stuff relate?

**Theorem** (Bonventre-Pereira, Gutiérrez-White, Rubin)

both Cor IV; cofibrant Thm 4.7; Explicit, like paper

Prop 4 in BBR

$$\mathrm{Ho}\left(N^\infty\text{-}Op^G\right) \simeq \mathrm{Tr}(\mathrm{Sub}(G))$$

- conjectured or hinted at in Blumberg-Hill 15 and proved by the above.

We're counting homotopy classes of $N_\infty$-operads

History

Conjecture Blumberg Hill

Proof BP
cofibrant + Explicit

Proof Rubin
Explicit

REU was helpful

Proof GW
cofibrant

Takeaway
understanding transfer systems is key to understanding $N_\infty$ operads.

---

THA$_\infty$N$_\infty$K

YOU!

# References


- "Operadic Multiplications ...." Blumberg - Hill 2015
- "$N_\infty$-Operads and Associahedra" Balchin - Barnes - Roitzheim 2021
- "Combinatorial $N_\infty$ Operads" Rubin
- "Encoding Equivariant ..." Gutierrez - White
- "Genuine Equivariant Operads" Bonventre - Pereira