Morava's Orbit Picture and Morava Stabilizer groups

Scotty Tilton
UCSD

Reminders from last time

Lazard Ring \( L \cong \mathbb{Z} \langle x_1, x_2, \ldots \rangle \cong \text{MU}^* \)

Universal group law \( G(x, y) \) over \( L \) s.t. a formal group law

Four \( R, \exists! \theta : L \to R \) s.t. \( F(x, y) = \sum \theta(a_i) x^i y^j \)

where \( a_i, j \) are the coefficients of universal formal group law

\[
\Gamma = \left( \sum b_i \right) \quad \text{where} \quad b_i \in \mathbb{Z}/3, \quad 0
\]

\( \Gamma \approx L \), \( \gamma \mapsto \theta \gamma \) induced by \( \delta^{-1} G(\gamma(x), \gamma(y)) \)

log of a formal group law is a power series s.t.

\[
\log_F(F(x, y)) = \log_F(x) + \log_F(y)
\]
\[ \langle n \rangle (x) = F(x, \langle n-1 \rangle (x)) \text{ with } \langle 1 \rangle (x) = x. \]

height \[ F(x,y) \text{ formal group law has height } h \text{ if } \langle p \rangle (x) = ax^p + (\text{higher terms}) \]

\[ w/a \text{ invertible.} \]

\[ \forall n \text{ given } p \text{, coefficient of } x^p \text{ in } G(x,y) \]

\[ \mathcal{L} \Gamma \text{ category of finitely presented, graded } L \text{-modules} \]

\[ w/a \text{ compatible } \Gamma \text{ action.} \]

Class-o-Formal group laws \[ \underline{\text{Two formal group laws over } } F_p \text{ are isomorphic } \Leftrightarrow \text{ same height.} \]

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\textbf{Invariant prime ideal theorem:}

The only prime ideals in \( L \) which occur in various categories \( \Gamma \approx L \) are \( I_{p,n} := (p, v_1, \ldots, v_{n-1}) \) where \( p \) is prime and \( 0 \leq n \leq \infty \).

\[ I_0 = (0), \quad I_{p,0} = (p, v_1, \ldots) \]

Moreover, in \( L/I_{p,n} \) for \( n > 0 \), the subgroup fixed by \( \Gamma \) is \( \mathbb{Z}/(p) [v_n] \).

In \( L \), the invariant subgroup is \( \mathbb{Z} \).
Every module $M$ in $C \Gamma$ admits a finite filtration by submodules in $C \Gamma$

$$0 = F_0 M \subset F_1 M \subset \cdots \subset F_n M = M$$

such that each $F_i M / F_{i-1} M$ is a suspension of $k/p^n$ for some $p, n$.

**Takeaway:** Can localize at $p$ and study $V_p = Z(p) [v_1, v_2, \ldots ]$ aka BP*.

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**Chapter Four**
4.1 The action of $\Gamma$ on $L$

**Notation**
Let $H_2/L := \text{Hom}_{\text{ring}}(L, \mathbb{Z})$

**Definition**
An automorphism of a formal group law $F$ is a power series $f(x)$ satisfying $f(F(x, y)) = F(f(x), f(y))$.

It is strict if it has the form $x + O(x^2)$

$$f(x) = x + \sum_{i \geq 2} a_ix^i$$

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**Prop 4.1.1** Let $\Gamma \curvearrowright H_2/L$ be the action induced by $\Gamma \curvearrowright L$.

1. $H_2/L \cong \{ \text{Formal group laws over } \mathbb{Z} \}$
   \[ \Theta : L \to L \]
   \[ \Theta(x) \text{ formal} \]

2. $F, G \in H_2/L$ are in same $\Gamma$ orbit iff $F \cong G$ over $\mathbb{Z}$.

3. $\text{stab}_\Gamma(\Theta) = \text{strict automorphisms in group of } \Theta \in H_2/L$.

4. Strict automorphism groups of isomorphic formal group laws are conjugate in $\Gamma$. 
Classification of formal group laws over \( \mathbb{Z} \) is tough, but we classified the over \( k := \text{GF}(p) \).

**Prop 4.1.2** The formal group law over \( k \) corresponding to \( \Theta \in H_k L \) has height \( n \) if and only if \( \Theta(v_i) = 0 \) for \( i \in n \) and \( \Theta(v_n) \neq 0 \).

Moreover, each \( v_n \in L \) is indecomposable, i.e., is a unit multiple (in \( \mathbb{Z}(p) \)) of \( x_p^{n-1} + \text{decomposables} \).
4.2 Morava Stabilizer Groups

The $n$th Morava stabilizer group $S_n$ is the strict automorphism group of a height $n$ formal group law over $K = \mathbb{F}_p$:

$$f(F(x,y)) = F(x^p, fy).$$

It is contained in a division algebra $D_n$ over the $p$-adic numbers $\mathbb{Q}_p$.

We'll get there.

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Recall:

- $F_{p^n} = \mathbb{F}_p[\sqrt[n]{5}]$ where $\sqrt[n]{5}$ is a $(p^n-1)$th root of 1.
- $\frac{5^{p^n-1} - 1}{5 - 1} = 1$
- $\text{Gal}(F_{p^n}/\mathbb{F}_p)$ is cyclic of order $n$ generated by Frobenius automorphism $x \mapsto x^p$.
- There is a degree $n$ extension of the $p$-adic integers $\mathbb{Z}_p$, which we denote $W(F_{p^n})$ by adjoining a $(p^n-1)$st root of $-1$, $5$ where $5 \equiv \sqrt[n]{5} \mod p$.
- The Frobenius automorphism has a lifting $\sigma$, which fixes $\mathbb{Z}_p$, $\sigma(5) = 5^p$ and $\sigma(x) \equiv x^p \mod p$ for all $x \in W(F_{p^n})$. 
The fraction field of $W(F_{p^n})$ is denoted $K_n$.

Let $K_n \langle S \rangle$ be $K_n$ adjoined with a noncommuting power series variable $S$ where

$$Sx = \sigma(x)S.$$

$$S^n x = \sigma^n(x) S^n = x S^n$$

(Note $S$ commutes with $Q_p \subset K_n$ and $S^n$ commutes with everything.)

The division algebra

$$D_n := K_n \langle S \rangle/(S^n - p).$$

Note: This is a rank $n^2$ algebra over $Q_p$ with center $Q_p$. Rav86 6.2.12

$$E_n := W(F_{p^n}) \langle S \rangle/(S^n - p) \subset D_n.$$

$E_n$ is a complete local ring with maximum ideal $\langle S \rangle$ and fraction field $F_{p^n}$. 

Every $a \in E_n$ can be written as
\[
a = \sum_{i=0}^{n-1} a_i S^i, \quad a_i \in W(\mathbb{F}_p^n).
\]

**OR**

\[
a = \sum_{i \geq 0} e_i S^i, \quad \text{with } e_i \in W(\mathbb{F}_p^n).
\]

where $e_i^p - e_i = 0$ or $e_i = 0$ or a root $-1$.

\[
E_n^\times = \left\{ \sum_{i \geq 0} e_i S^i \mid e_0 \neq 0 \right\} \text{ or } \left\{ \sum_{i \geq 0} a_i S^i \mid a_0 \in W(\mathbb{F}_p^n)^\times \right\}.
\]

**Proposition 4.25** The full automorphism group of a formal group law over $\mathbb{K}$ of height $h$ is isomorphic to $E_n^\times$.

The strict automorphism group $S_n$ is isomorphic to the subgroup
\[
\left\{ 1 + \sum_{i \geq 0} e_i S^i \in E_n^\times \mid e_i^p = e_i = 0 \right\} \leq E_n^\times
\]
Consider each \( e_i : S_n \xrightarrow{cts} F_p^n \). The ring of all such functions is

\[
S(n) := F_p^n [e_1, e_2, e_3, \ldots] / (e_i^{p^n} - e_i)
\]

This is a Hopf algebra over \( F_p^n \) with coproduct induced by \( S_n \).

Compare to Morava K-theory

\[
\Sigma(n) = K(n)[t_1, t_2, \ldots] / (t_i^{p^n} - u_i^{p^n} - t_i)
\]

and then

\[
S(n) = \Sigma(n) \otimes K(n_+)^{F_p^n} \quad (v_i \mapsto 1)
\]

Let's see how \( S_n \) (strict automorphisms of formal group laws) acts on a formal group law of height \( n \), \( F_n \).

**Making \( F_n \)**

Let \( F \) be a formal group law over \( \mathbb{Z}[x] \) with

\[
\log_F(x) = \sum_{i=0} \frac{x^{p^{ni}}}{pi}
\]

Then \( F_n \) is obtained by reducing \( F \mod p \) and tensoring with \( F_p^n \).
An automorphism $e$ of $F_n$ is a power series

$$e(x) \in \mathbb{F}_p [[x]]$$

satisfying

$$e(F_n(x,y)) = F_n(e(x), e(y)).$$

So, given $e = 1 + \sum_{i>0} e_i S^i \in S_n$

$$e(x) = \sum_{i\geq 0} F_n e_i x^i = F(e_0 x, F e_1 x^2) \cdots$$

(Notation $x +^F y = F(x, y)$)

See more in Rav 86 Appdx 2.

4.3 Cohomological Properties of $S_n$.

(to be used by Arseniy and Shangjie later on).

(3 big theorems)

Note: $\Pi$ is essentially the multiplication in $\text{MU}$ theory. Similarly, $S_n$ is essentially the multiplication in Morava-K-theory. How?
\[ F_k(u)_* \cdot (x) := k(u)_* \cdot (x) \otimes k(u)_* \rightarrow F_{p^n} \]

Legro: of multiplicative operations here is precisely $S_n$.

**Remark** \( F_{p^n} \) is essential. (Larger or smaller, we lose)

- Topology of $S_n$ matters in computation.
- You can think of $S_n$ as a $p$-adic Lie group.

La 265

Let $H^*(S_n)$ denote the nod $p$ cohomology of $S_n$ and check out Rau 86 Chapter 6 for more.

**Theorem 4.3.2**

a) $H^*(S_n)$ is a finitely generated algebra.

b) If $p \not| n$, then

\[
H^i(S_n) = \begin{cases} 
0 & i > n^2 \\
H^{n^2-i}(S_n) & 0 \leq i \leq n^2
\end{cases}
\]

Poincaré Duality

If $p \not| n$, then $H^*(S_n)$ is periodic. i.e.

There exists $i > 0$ s.t. $H^*(S_n)$ is a free module over $\mathbb{Z}/(p)[x]$. 

c) Every sufficiently small open subgroup is cohomologically abelian i.e., same cohomology as $\mathbb{Z}/(p)^n$.

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E. L.
Theorem 4.3.3 \[ \text{Let } S_{n,i} \subset S_n, i \geq 1 \text{ be the subgroup of } E_n^* \text{ that } s \equiv 1 \mod (S)^i. \]

i) $S_{n,i}$ are cofinal in the set of open subgroups of $S_n$.

ii) The ring of cts. \(p^n\)-valued functions is

$$S(n, i) = S(n) / (e_j)_{j < i}$$

iii) If $i > \frac{pn}{2p-2}$, the cohomology of $S_{n,i}$ is an exterior algebra on $n^2$ germs.

iv) Each $S_{n,i}$ is open and normal and

$$[S_{n,i} : S_{n,\infty}] = p^{ni} \text{ and } S_{n,\infty} / S_{n,i} \text{ is abelian.}$$

Theorem 4.3.4 \[ \text{All finite abelian subgroups of } S_n \text{ are cyclic.} \]

$S_n$ contains an element of order $p^{i+1}$ iff $p^i(p-1) \mid n$. 
I hope you have a picture of these things. Thanks!

\[ \text{Alg} \quad A \otimes A \rightarrow A \]
\[ \text{Coalg} \quad A \leftrightarrow \text{coal} \]
\[ A \rightarrow A \otimes A \]