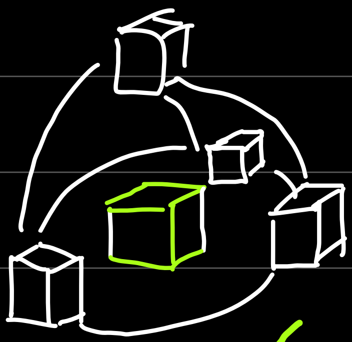
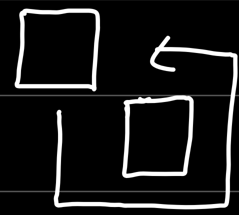


$\mathrm{Chain}_k^{\otimes_k}$ $(\infty,1)$



Factorization Homology 101

$\mathrm{Vect}_k^{\otimes_k}$

Λ^n_k

Scotty Tilton
UCSD

May 30, 2023

$$\begin{tikzcd} \mathrm{Disk}^{\amalg}_{1,fr} \arrow[r] \arrow[d, hook] & \mathcal{C}^{\otimes} \\ \mathrm{Mfld}^{\amalg}_{1,or} \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

Plan for Today

- Define Factorization Homology and unpack it
- See how it works in a small example and how it relates to the rest of the quarter
- Some useful results, some words on efficacy of the invariant, and problems in this direction.

Factorization Homology

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category such that $\mathcal{C}$ has

- sifted colimits and
- $\otimes$ preserves sifted colimits in each variable.

Fix an $\mathbb{E}_n$-algebra A. Factorization homology with coefficients in A is the Left Kan extension

$$\begin{tikzcd} \mathrm{Disk}^{\sqcup}_{n,G} \arrow[r, "A"] \arrow[d, hook] & \mathcal{C}^{\otimes} \\ \mathcal{M}\mathrm{fld}^{\sqcup}_{n,G} \arrow[ur, dashed, "\int_{(\,)} A"'] & \end{tikzcd}$$

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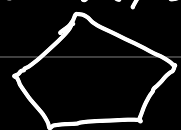
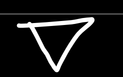
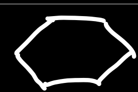
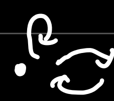
Symmetric Monoidal Category

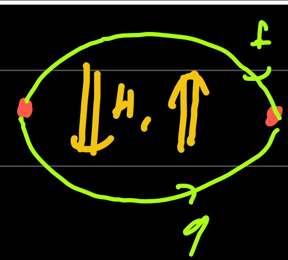
A **symmetric, monoidal category** is a category $\mathcal{C}$ equipped with

- a functor $\mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C}$
- an object $1 \in \mathcal{C}$
- natural isomorphisms
  - $(x\otimes y)\otimes z \to x\otimes(y\otimes z)$
  - $1\otimes x \to x \leftarrow x\otimes 1$
  - $x\otimes y \to y\otimes x$

Satisfying

- $((w\otimes x)\otimes y)\otimes z \to (w\cdot x)\cdot(y\cdot z) \to w\cdot(x\cdot(y\cdot z))$
  $((w\otimes x)\otimes y)\otimes z \to (w\cdot(x\cdot y))\cdot z \to w\cdot((x\cdot y)\cdot z) \to w\cdot(x\cdot(y\cdot z))$
  
- 
- 
- 

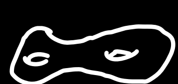
(∞, 1) Category



An **(m, n)-category** is a category with k-morphisms up to m where all morphisms > n are isomorphisms.

An **(∞, 1)-category** is a category with morphisms all the way up where all n-morphisms are invertible.

Ex)



- objects points
- 1-morphisms paths
- 2-morphisms homotopies
- 3-morphisms ...

Slogan for today: They "are" topologically enriched categories.

* sifted colimits $D \xrightarrow{\Delta} D \times D$ is cofinal.

Ex) Δ^{op}, filtered categories.

Functors Between SM Cats

A symmetric monoidal functor $F: A^{\otimes} \longrightarrow B^{\otimes}$

data of
- a functor
- isomorphisms $F(X \otimes Y) \xrightarrow{\simeq} F(X) \otimes F(Y)$

such that - it respects the symmetric monoidal structures in each category, including the swap map.

$\mathrm{Disk}^{\amalg}_{n,G}$, $\mathcal{M}\mathrm{fld}^{\amalg}_{n,G}$

$\mathrm{Disk}_{1,*}$ $\mathbb{R} \amalg \mathbb{R}$

$\mathrm{Disk}^{\amalg}_{n,G}$:
- objects: G-oriented open n-disks
- morphisms: $\hom((\mathbb{R}^n)^{\amalg k}, (\mathbb{R}^n)^{\amalg \ell})$ + weak whitney top is G-preserving embeddings
- monoid mult: $\amalg$

$\mathcal{M}\mathrm{fld}^{\amalg}_{n,G}$: " "

if $G = *$ we get framed manifolds

if $G = SO_n$ we get oriented manifolds

etcetera.

$dG =$

$$\begin{tikzcd}
V^{-1} \arrow[r, "d"] & V^{0} \arrow[r, "d"] \arrow[dl] & V^{1} \arrow[dl] \\
\oplus & \oplus & \oplus \\
W^{-1} \arrow[r, "d"] & W^{0} \arrow[r, "d"] & W^{1}
\end{tikzcd}$$

$\mathrm{Vect}_k^{\otimes_k}$, $\mathrm{Chain}_k^{\otimes_k}$

$f_0, f_1, f_2, \overleftarrow{H_{01}}, \overleftarrow{H_{02}}, \overleftarrow{H_{12}}, G$

$H_{01} + H_{12} - H_{02}$

$\mathrm{Vect}_k^{\otimes_k}$:
- objects: k-vector spaces
- morphisms: $\hom(V,W)$ — linear maps w/ discrete top.
- mult: $\otimes_k$

$\mathrm{Chain}_k^{\otimes_k}$:
- objects: cochain complexes $V^\bullet = (\cdots \to V^{-1} \xrightarrow{d} V^0 \xrightarrow{d} V^1 \to \cdots)$
- morphisms: $\hom(V^\bullet, W^\bullet)$ — 0-cells: chain maps; 1-cells: chain htpys; 2-cells: homotopies of htpys; ⋮

Dold-Kan space!

- monoid: $\otimes_k$

Left Kan Extension

Fix functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "a"] \arrow[d, "F"'] & \mathcal{E} \\ \mathcal{D} & \end{tikzcd}$$

Suppose you are given a pair (G, η) where

$$\begin{tikzcd} \mathcal{C} \arrow[r, "a"] \arrow[d, "F"'] \arrow[dr, phantom, "\eta \Uparrow"] & \mathcal{E} \\ \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}.$$

A Left Kan extension is an initial such pair.

Factorization Homology

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category such that $\mathcal{C}$ has

- sifted colimits and
- $\otimes$ preserves sifted colimits in each variable

Fix an $\mathbb{E}_n$-algebra A. Factorization Homology with coefficients in A is the Left Kan extension

$$\begin{tikzcd} \mathrm{Disk}^{\amalg}_{n,G} \arrow[r, "A"] \arrow[d, hook] & \mathcal{C}^{\otimes} \\ \mathrm{Mfld}^{\amalg}_{n,G} \arrow[ur, dashed, "\int_{( )} A"'] & \end{tikzcd}$$

$\mathrm{Disk}^{\amalg}_{1,or}$ - example

Let $A : \mathrm{Disk}^{\amalg}_{1,or} \longrightarrow \mathrm{Vect}_k^{\otimes_k}$ where if $f \simeq g \in \hom(\mathbb{R}^{\amalg k}, \mathbb{R}^{\amalg n})$, then $A(f) = A(g)$.

$A = A(\mathbb{R})$

Then what is

$$\begin{tikzcd} \mathrm{Disk}^{\amalg}_{1,or} \arrow[r, "A"] \arrow[d, hook] & \mathrm{Vect}_k^{\otimes k} \\ \mathrm{Mfld}^{\amalg}_{1,or} \arrow[ur, dashed, "\int_{( )} A"'] & \end{tikzcd}$$

Specifically, what is

$$\int_{S^1} A.$$

Quick Result

Theorem (Tanaka's book) $F^{\otimes} : \mathrm{Disk}^{\sqcup}_{1,or} \longrightarrow \mathrm{Vect}_k^{\otimes_k}$

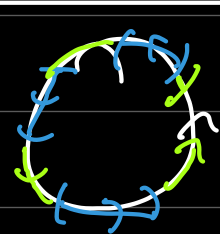
where isotopic embeddings are sent to the same linear map.

Then the data of $F^{\otimes}$ is the same data as a unital, associative k-algebra.

proof sketch ($\Rightarrow$) Define, then check! ($\Leftarrow$) Reverse engineer

---

Need
$$\begin{tikzcd} \mathbb{R}^{\sqcup k} \arrow[rr] \arrow[dr] & \circlearrowleft & S^1 \\ & \mathbb{R} \arrow[ur] & \end{tikzcd}$$
, so



$$\begin{tikzcd} A^{\otimes k} \arrow[rr] \arrow[dr] & & F(S^1) \\ & A \arrow[ur] & \end{tikzcd}$$

needs to commute, and needs to commute after cyclic permutation.

Guess $F(S^1) = A/_{[A,A]}$.

Fact If A is associative, then

$$A/[A,A] \xrightarrow[\cong]{\text{natural}} A \otimes_{A\otimes A^{op}} A.$$

Regular Old Homology

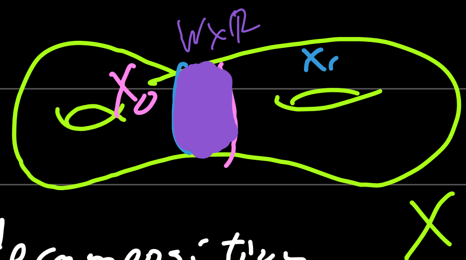
Let $\mathcal{C}^{\otimes} = \mathcal{C}hain^{\oplus}$, and let

A denote the chain $(\cdots \to \underset{-1}{0} \to \underset{0}{A} \to \underset{1}{0} \to \cdots)$.

Then $\int_M A \simeq C_*(M;A)$.

Exercises 78 + 79 in Tanaka's book.

$\otimes$ - Excision



Theorem (Ayala-Francis)-Tanaka

Fix $X \in \mathcal{M}fld^{\sqcup}_{n,fr}$ and a decomposition

$X = X_0 \underset{W\times\mathbb{R}}{\cup} X_1$ where $X_0, X_1 \subseteq X$

and $X_0 \cap X_1 \cong W \times \mathbb{R}$.

Then

$$\int_X A = \int_{X_0} A \underset{\int_{W\times\mathbb{R}} A}{\otimes} \int_{X_1} A.$$

regular instances Mayer-Vietoris!

Free $\mathbb{E}_n$-algebra

Let $\mathcal{C}^{\otimes} = \mathcal{T}op^{\times}$ and $A = \coprod_{\ell \geq 0} \mathrm{Conf}_{\ell}(\mathbb{R}^n)$ the free E_n-algebra.

Then
$$\int_M A \simeq \coprod_{\ell \geq 0} \mathrm{Conf}_{\ell}(M)$$

A-F-T.

Nice Results

Fubini
$$\int_{X \times Y} A \simeq \int_Y \int_{X \times \mathbb{R}^{\dim(Y)}} A$$

Non-Abelian Poincaré Duality: Let $\mathcal{C}^{\otimes} = \mathcal{T}op^{\times}$ and let X be a topological space with $\pi_k(X) = 0$ for $k \leq n-1$, consider the $\mathbb{E}_n$-algebra $\Omega^n X$. Then
$$\int_{M^n} \Omega^n X \simeq \mathrm{Map}_c(M^n, X)$$

Salvatore, Lurie, Ayala-Francis, Ayala-Francis-Tanaka.

Problems

Cobordism Hypothesis

$$\mathrm{Fun}^{\otimes}\left((\mathcal{C}ob_n^{fr})^{\amalg}, \mathcal{C}^{\otimes}\right) \xrightarrow{\simeq} \mathcal{C}^{\text{fully dualizable}}$$

$$Z \longmapsto Z(*^+)$$

Proof strategy laid out by Lurie, but FH inspired proof laid out by Ayala-Francis in "The cobordism hypothesis"

Open Question

Can you classify the $\mathbb{E}_n$-algebras in $\mathcal{M}fld_{n,fr}^{\amalg}$

How Good of An Invariant?

"about as good as the homotopy type of configuration spaces" which has been shown to distinguish manifolds

$M \underset{htpy}{\simeq} N$ but $M \underset{diffeo}{\not\cong} N$ (as in Longoni-Salvatore-05)

$L_{7,1}$ $L_{7,2}$

More Instances

(Ayala - Francis) "Primer" paper

Algebra	Output $\int_M A$
• A - associative algebra	$\int_{S^1} A \simeq HH_\bullet(A)$
• A - abelian group	$\int_M A \simeq H_*(M; A)$
• A - spectrum	$\int_M A \simeq \Sigma^\infty_* M \wedge A$
• A - $\Omega^n K$, K n-connective	$\int_M A \simeq \mathrm{Map}_c(M, K)$

$S^1 \mapsto A(S^1) \in \mathcal{C}^\otimes$

$S^1 \to S^1 \in \mathrm{Mfld}_1^{\sqcup}$

$S^1 \circlearrowleft \mathrm{Diff}^+(S^1)$

THANK YOU

References

- Tanaka: "Lectures on Factorization Homology, ∞-categories, and Topological Field Theories"
- Ayala, Francis: "A Factorization Homology Primer"

Side Lessons (if time)

$\sqcup$ vs $\amalg$

disjoint union coproduct

\sqcup \coprod

$A + B$ $\underbrace{A - B}_{T}$

$$\begin{tikzcd}
TX \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EGL_n(\mathbb{R}) \arrow[d] \\
X^n \arrow[r, "\tau_X"] & BGL_n(\mathbb{R})
\end{tikzcd}$$

$$G \longrightarrow GL_n(\mathbb{R})$$

$$BG \longrightarrow BGL_n(\mathbb{R})$$

$$\begin{tikzcd}
 & BG^* \arrow[dd] & \\
X^n \arrow[ur] \arrow[rr] \arrow[dr] & & Y^n \arrow[ul] \arrow[dl] \\
 & BGL_n(\mathbb{R}) &
\end{tikzcd}$$

3-simplex in Top

X is framed, if $\exists$ $TX \xrightarrow{\cong} X \times \mathbb{R}^n$

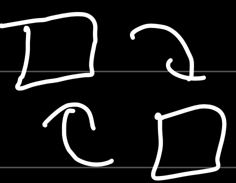
$$G = * \longrightarrow GL_n$$

An $\mathbb{E}_n$-algebra in this context is a functor

$$F^{\otimes} : \mathrm{Disk}_{n,fr}^{\sqcup} \longrightarrow \mathcal{C}^{\otimes}$$

$\mathrm{Disk}_{1,or}^{\sqcup} \longrightarrow \mathrm{Vect}_k^{\otimes_k}$ is an associative algebra.

$$\mathrm{Disk}_{2,fr}^{\sqcup} \longrightarrow \mathcal{C}^{\otimes}$$



$\mathrm{Disk}_{3,fr}^{\sqcup}$