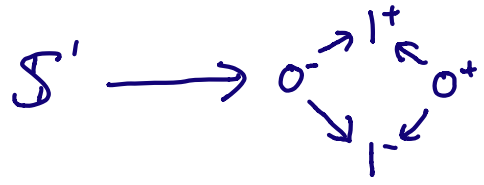
Stratified Spaces and Grassmannians

- What is a stratified space?
- What is a Grassmannian?
- Is a Grassmannian a stratified space?
- Some cool things my collaborator and I found out.

Definition a **stratified space** $(X \xrightarrow{\varphi} P)$ is
- a topological space X
- a partially ordered set (poset) P endowed with the downward closed topology.
  (i.e $F \subseteq P$ is closed if $x \in p \Rightarrow \forall y \leq x, y \in F$)
- a continuous map $X \xrightarrow{\varphi} P$

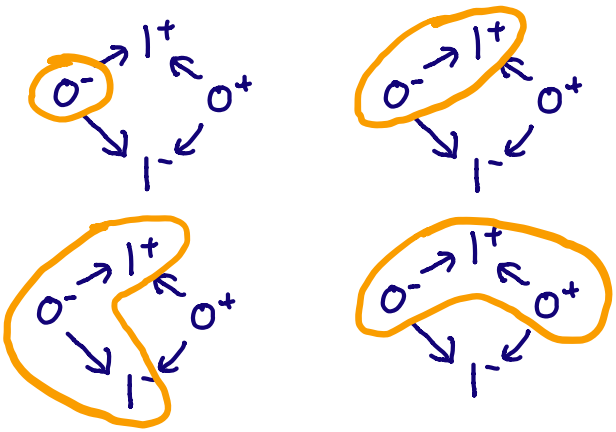
Example 1



arrows point to things that are "bigger" in the poset.

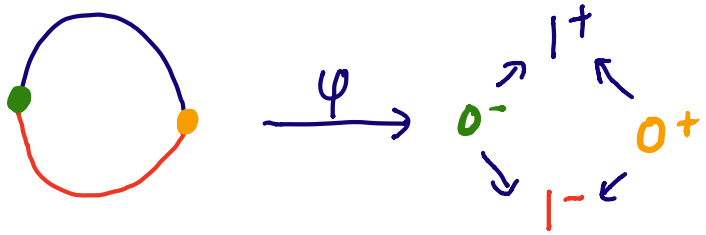
Topology on

$F \subseteq P$ is closed if $x \in p \Rightarrow \forall y \leq x, y \in F$

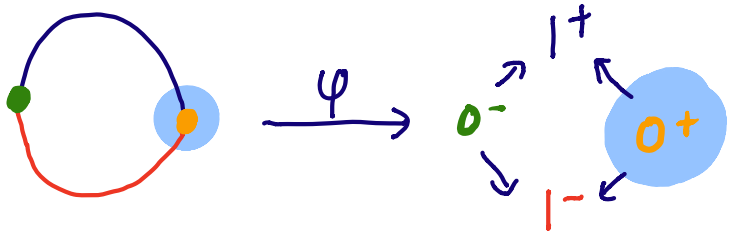


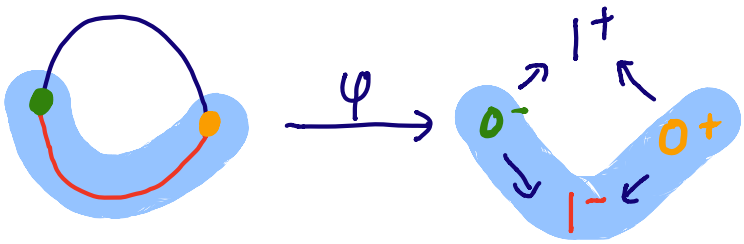
So we have the topology on our poset.

Now, let the continuous map be

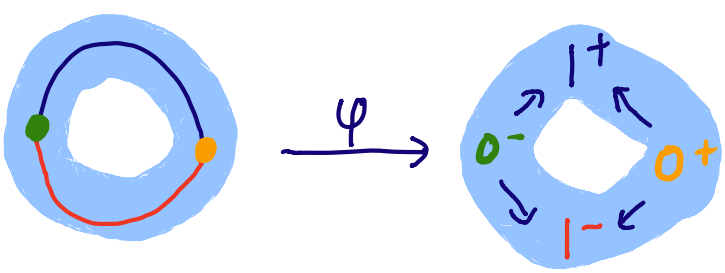


continuous if the preimage of closed is closed

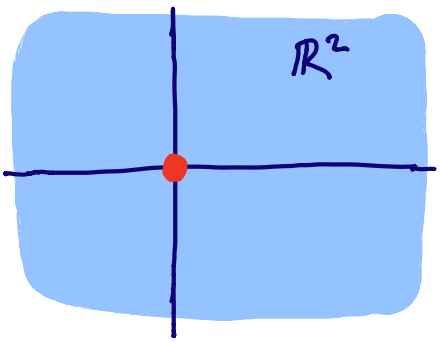


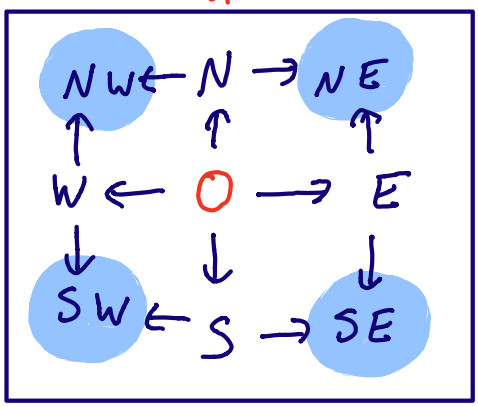


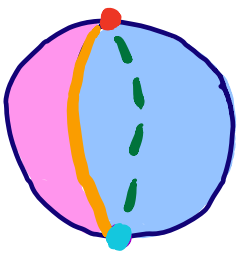
So this is a stratified space!

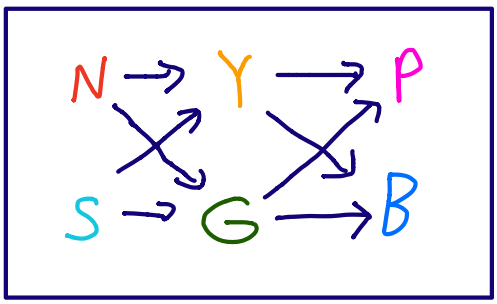


More Examples to keep in mind

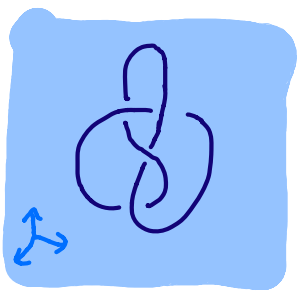




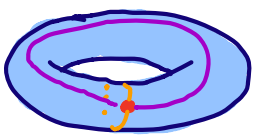


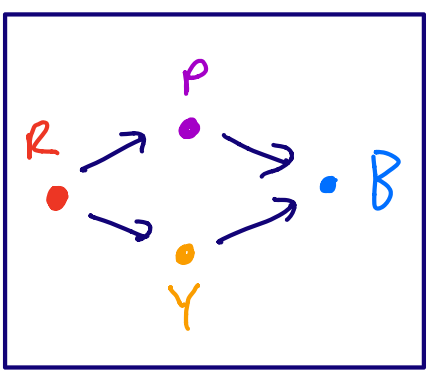


$S^1 \subset \mathbb{R}^3$ ⟶ $0 \to 1$



$* \subset S^n$ ⟶ $0 \to 1$

$T^2 =$  ⟶



$$Gr_k(n) \longrightarrow \left\{ {n \atop k} \right\}$$

What is that???

Definition the poset $\left\{ {n \atop k} \right\}$ is the set of all ordered subsets of $\{1, \ldots, n\}$ with k elements. "n choose k" ordered by $S \leq T$ if $s_i \leq t_i \ \forall i \in \{1, \ldots, k\}$.

Examples

$$\left\{ {3 \atop 1} \right\} = \{1\} \to \{2\} \to \{3\}$$

$$\left\{ {4 \atop 2} \right\} = \begin{array}{ccc} & & \{3,4\} \\ & & \uparrow \\ \{2,3\} & \to & \{2,4\} \\ \uparrow & & \uparrow \\ \{1,2\} & \to & \{1,3\} \to \{1,4\} \end{array}$$

$$\left\{ {6 \atop 3} \right\} = \{1,2,3\} \to \{1,2,4\} \to \begin{array}{l} \{1,3,4\} \to \{2,3,4\} \to \{2,3,5\} \to \{2,4,5\} \to \{3,4,5\} \\ \{1,3,5\} \to \{1,4,5\} \to \{2,3,6\} \to \{2,4,6\} \to \{3,4,6\} \\ \{1,2,5\} \to \{1,2,6\} \to \{1,3,6\} \to \{1,4,6\} \to \{1,5,6\} \end{array} \to \begin{array}{l} \{3,5,6\} \\ \{2,5,6\} \end{array} \to \{4,5,6\}$$

Notes

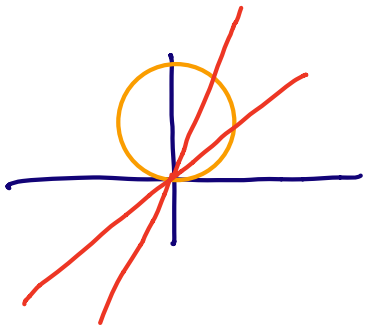
- $\left\{ {n \atop k} \right\}$ has $\binom{n}{k}$ elements.
- $\left\{ {n \atop k} \right\}$ as sets $= \{S \in \mathcal{P}(\{1, \ldots, n\}) : |S| = k\}$

Definition The real Grassmannian of k-planes in $\mathbb{R}^n$, denoted $Gr_k(n)$, is defined to be

$$Gr_k(n) := \left\{ V \overset{\text{vector subspace}}{\subset} \mathbb{R}^n \mid \dim(V) = k \right\}$$

endowed w/ a suitable topology.

Ex $Gr_1(2) = \{ \underbrace{\text{lines through the origin in } \mathbb{R}^2}_{\text{1-dimensional subvectorspaces}} \}$



$\cong S^1$

This is the last Grassmannian you can see with your eyes. (except trivial ones like $Gr_n(n)$)

$$Gr_2(3) = \{ \underbrace{\text{all planes through the origin in } \mathbb{R}^3}_{\text{2-dim subvector spaces}} \}$$

$$Gr_2(4) = \{ \text{all planes through the origin in } \mathbb{R}^4 \}$$

Nice Fact $Gr_1(n+1) = \mathbb{RP}^n$ the real projective space!

So tell me about this stratified space now, buddy!

$$Gr_k(n) \longrightarrow \left\{ {n \atop k} \right\}$$

$$V \longmapsto S_V$$

where S is the maximal element in the poset such that $\text{proj}_{\mathbb{R}^S}(V) \cong V$.

$$\mathbb{R}^S = \text{span}\{e_{s_i} \mid s_i \in S\}$$

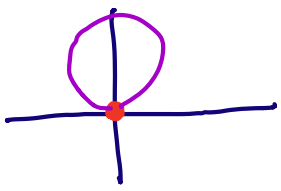
Ex Let me talk about the preimages of this continuous map so we have an idea of how it works.

in $Gr_1(2) \xrightarrow{\varphi}$ $\left\{ {2 \atop 1} \right\} = \{1\} \to \{2\}$

Note: preimages of elements in your posee are called strata

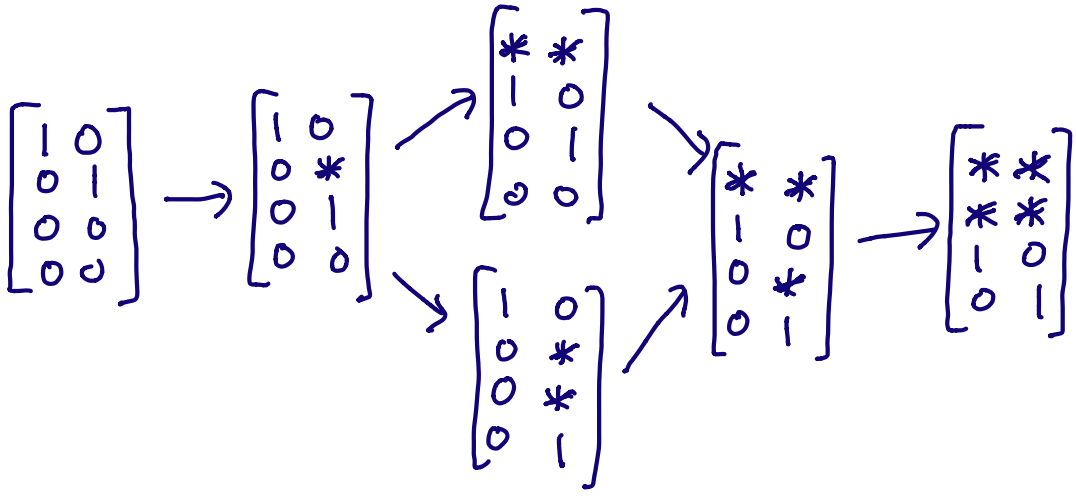
$$\varphi^{-1}(\{1\}) = \text{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\varphi^{-1}(\{2\}) = \left\{ \text{span}\begin{bmatrix} * \\ 1 \end{bmatrix} \mid * \in \mathbb{R} \right\}$$

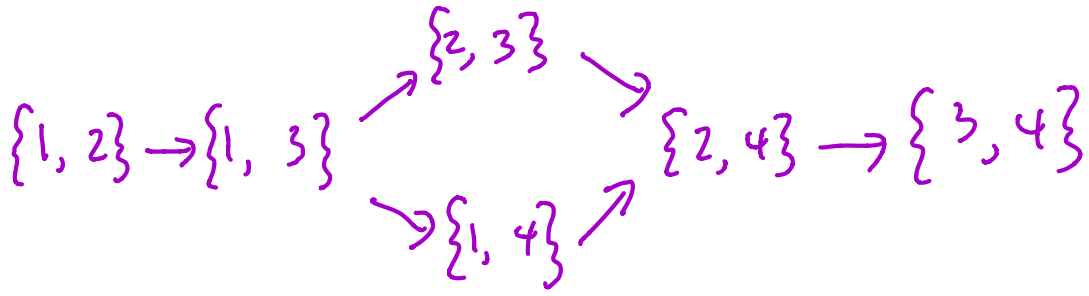


$\xrightarrow{\varphi}$ $\{1\} \to \{2\}$

In $Gr_2(4) \to \begin{Bmatrix} 4 \\ 2 \end{Bmatrix}$, we can see the strata as



each of the $*$'s can be any value in $\mathbb{R}$



Facts • In this stratified space $\left(Gr_k(n) \to \begin{Bmatrix} n \\ k \end{Bmatrix}\right)$ each stratum is homeomorphic to $\mathbb{R}^m$ for some $m \in \{0, \dots, k(n-k)\}$

- This stratification is "the same" as the CW complex in Milnor-Stasheff's Characteristic classes.
- They found the $\mathbb{F}_2$ cohomology of Grassmannians, but the $\mathbb{Z}$ coefficient cohomology has remained unknown

Using the structure of stratifications, my collaborator and I were able to work this out, not just for $\mathbb{Z}$, but any ring R.

Cool pictures and results w/out diving into how to compute cohomology or what cohomology is.

Think of cohomology as

- an amazing tool to tell topological spaces apart
- an amazing way to uncover important facts about maps between topological spaces (like fixed points, retractions, existence of nice structure, etc.)

And one way to compute cohomology is to measure how "twisty" the gluing of these cells (in our case strata) is.

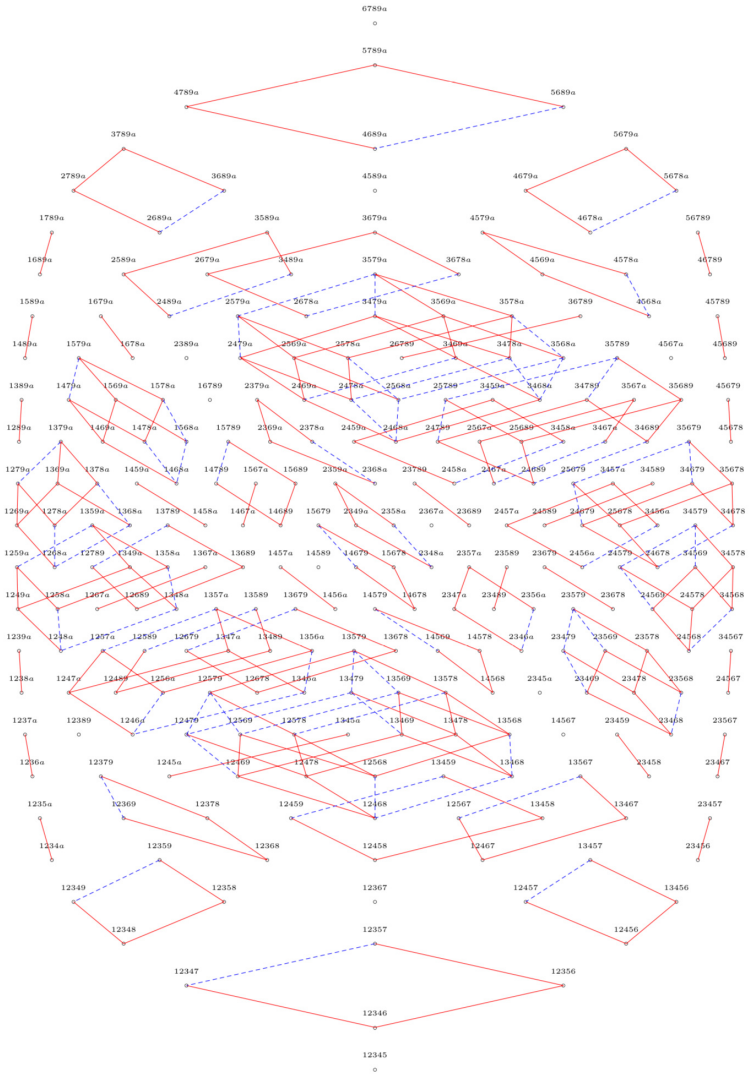
Here are some calculations my collaborator and I did.

Boundary maps for $Gr_5(10)$

—— means +2

- - - - means −2

$\emptyset$ means 0

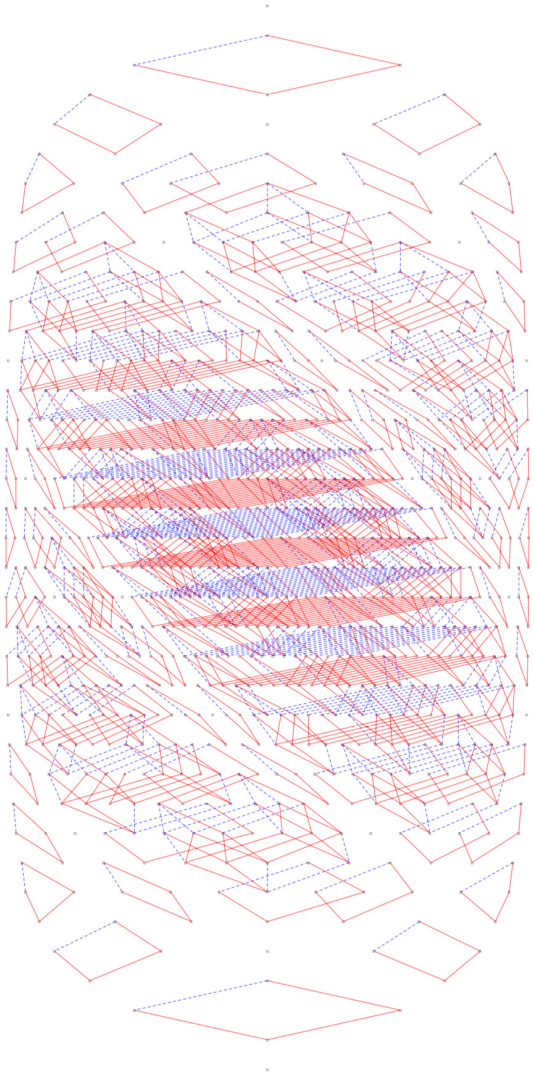


Boundary maps for $Gr_6(12)$

— means +2

- - - means -2

$\emptyset$ means 0



cohomology groups for $Gr_{12}(24)$

$Gr_{12}(24)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
H^0	1	0
H^1	0	0
H^2	0	1
H^3	0	1
H^4	1	2
H^5	0	2
H^6	0	5
H^7	0	6
H^8	2	9
H^9	0	11
H^{10}	0	19
H^{11}	0	23
H^{12}	3	33
H^{13}	0	41
H^{14}	0	58
H^{15}	0	73
H^{16}	5	95
H^{17}	0	117
H^{18}	0	156
H^{19}	0	191
H^{20}	7	239
H^{21}	0	291
H^{22}	0	367
H^{23}	0	441
H^{24}	11	536
H^{25}	0	638
H^{26}	0	777
H^{27}	0	916
H^{28}	13	1087
H^{29}	0	1268
H^{30}	0	1503
H^{31}	0	1740
H^{32}	18	2017
H^{33}	0	2316
H^{34}	0	2680
H^{35}	0	3048
H^{36}	22	3470
H^{37}	0	3917
H^{38}	0	4440
H^{39}	0	4974
H^{40}	28	5562

H^{41}	0	6179
H^{42}	0	6886
H^{43}	0	7594
H^{44}	32	8356
H^{45}	0	9151
H^{46}	0	10029
H^{47}	0	10901
H^{48}	39	11821
H^{49}	0	12760
H^{50}	0	13776
H^{51}	0	14770
H^{52}	42	15791
H^{53}	0	16814
H^{54}	0	17898
H^{55}	0	18936
H^{56}	48	19967
H^{57}	0	20990
H^{58}	0	22040
H^{59}	0	23015
H^{60}	51	23959
H^{61}	0	24864
H^{62}	0	25764
H^{63}	0	26573
H^{64}	55	27307
H^{65}	0	27984
H^{66}	0	28635
H^{67}	0	29166
H^{68}	55	29596
H^{69}	0	29963
H^{70}	0	30272
H^{71}	0	30456
H^{72}	58	30525
H^{73}	0	30525
H^{74}	0	30456
H^{75}	0	30272
H^{76}	55	29963
H^{77}	0	29596
H^{78}	0	29166
H^{79}	0	28635
H^{80}	55	27984
H^{81}	0	27307
H^{82}	0	26573

H^{83}	0	25764
H^{84}	51	24864
H^{85}	0	23959
H^{86}	0	23015
H^{87}	0	22040
H^{88}	48	20990
H^{89}	0	19967
H^{90}	0	18936
H^{91}	0	17898
H^{92}	42	16814
H^{93}	0	15791
H^{94}	0	14770
H^{95}	0	13776
H^{96}	39	12760
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H^{131}	0	58
H^{132}	3	41
H^{133}	0	33
H^{134}	0	23
H^{135}	0	19
H^{136}	2	11
H^{137}	0	9
H^{138}	0	6
H^{139}	0	5
H^{140}	1	2
H^{141}	0	2
H^{142}	0	1
H^{143}	0	1
H^{144}	1	0

Thank you!

Questions?

Corollary 5.7. *There is an isomorphism of graded R-modules*

$$H^*(\mathsf{Gr}_k(n); R) \cong \bigoplus_{S \in \left\{ {n \atop k} \right\}} V_S[-d(S)] \ ,$$

where

$$V_S := \begin{cases} R, & \text{if } \mathsf{In}(S) = \emptyset = \mathsf{Out}(S) \\ \ker(R \xrightarrow{2} R), & \text{if } \mathsf{Min}(\mathsf{In}(S) \cup \mathsf{Out}(S)) \in \mathsf{Out}(S) \\ \mathsf{coker}(R \xrightarrow{2} R), & \text{if } \mathsf{Min}(\mathsf{In}(S) \cup \mathsf{Out}(S)) \in \mathsf{In}(S) \end{cases} .$$