Some extensions of the Ananyan–Hochster principle

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(joint work with Daniel Erman, Andrew Snowden)

In response to a question posed by Stillman, Ananyan and Hochster proved the following theorem [1]:

Theorem 1 (Ananyan–Hochster)**.** *Let $d_1, \dots, d_r$ be positive integers. There exists a bound $N(d_1, \dots, d_r)$ such that any ideal in a polynomial ring over a field generated by homogeneous polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$ has projective dimension at most $N(d_1, \dots, d_r)$.*

The main content is that the bound $N(d_1, \dots, d_r)$ does not depend on the number of variables in the polynomial ring. The main idea behind the proof is that once the degrees $d_1, \dots, d_r$ are fixed, polynomials behave as if they are defined in a bounded number of variables.

More precisely, define the **strength** of a homogeneous polynomial f to be the minimal k such that there is an expression $f = g_1 h_1 + \cdots + g_k h_k$ where $\deg(g_i) < \deg(f)$ and $\deg(h_i) < \deg(f)$. The **collective strength** of a vector space of polynomials is the minimum strength of a nonzero homogeneous polynomial contained in it. Using this, one gets the existence of "small subalgebras":

Theorem 2 (Ananyan–Hochster)**.** *Let $d_1, \dots, d_r$ be positive integers. There exists a bound $M(d_1, \dots, d_r)$ such that any sequence of homogeneous polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$ is contained in a subring generated by a regular sequence of length at most $M(d_1, \dots, d_r)$.*

This immediately implies the first theorem: Hilbert syzygy theorem implies that the projective dimension of $(f_1, \dots, f_r)$ over the subring is at most $M(d_1, \dots, d_r)$; by flatness, the projective dimension can be computed either over the subring or the original ring.

We extend this idea to two other settings. The first concerns Hartshorne's conjecture, which states that any smooth subvariety X of $\mathbf{P}^n$ of codimension $< n/3$ is a complete intersection. One can show using the ideas above that if X is equidimensional, then there exists a bound depending only on its codimension and degree such that if its singular locus has codimension greater than this bound, then X must be a complete intersection. The outline is as follows:

- When the codimension and degree are fixed, there is a finite list of possibilities $(d_1, \dots, d_r)$ such that the ideal of X is generated by homogeneous polynomials of these degrees [2].
- Fixing the degrees $(d_1, \dots, d_r)$ now, an elementary argument with Jacobian matrices shows that forcing the singular locus to have high codimension forces the strength of the polynomials to be high as well.
- Rearranging the generators if necessary, we may assume that the strength is weakly decreasing and that the strength of f_k is the collective strength of the span of $f_1, \dots, f_k$. If c is codimension of X, this implies that $f_1, \dots, f_c$ generates a regular sequence if the strength is sufficiently large.

- If the locus cut out by $f_1, \ldots, f_c$ has multiple irreducible components, then its singular locus has codimension at most $2c$ since the intersection of two components gives singular points. Hence, if we assume that the codimension is larger than $2c$, the ideal generated by $f_1, \ldots, f_c$ must be prime. However, if $k > c$, f_{c+1} is a nonzerodivisor on this complete intersection which contradicts that X has codimension exactly c. Hence X is a complete intersection defined by $f_1, \ldots, f_c$.

An explicit bound can be found in [4].

Our second extension concerns the cohomology tables of coherent sheaves $\mathcal{E}$ on projective space. The bound on projective dimension in the Ananyan–Hochster theorem in fact implies that once the initial degrees are fixed, there are only a finite set of possibilities for the graded Betti numbers of the ideal. Cohomology tables are similar to graded Betti tables in many ways, and record the numbers $\dim \mathrm{H}^i(\mathcal{E}(j))$. More precisely, these dimensions are the graded Betti numbers of the associated Tate resolution of $\mathcal{E}$ (up to a re-indexing), which is a doubly-infinite minimal complex over an exterior algebra that complete encodes the data of $\mathcal{E}$. Define the kth column of a cohomology table to be the ranks of the graded components of the kth term in this complex. We prove the following analogue of finiteness of Betti tables in the polynomial ring case:

Theorem 3. *Fix the values of the kth and $(k+1)$st columns for some k. Then there are only finitely many ways to complete this data to the cohomology table of a coherent sheaf on some projective space.*

Again, the main content is that this finiteness does not depend on a fixed projective space. This follows from an analogue of the existence of small subalgebras. More precisely:

Theorem 4. *Fix the values of the kth and $(k+1)$st columns for some k. There exist bounds k_0 and n_0 such that, for any coherent sheaf $\mathcal{E}$ on $\mathbf{P}^n$ (for any n) whose cohomology table has these columns, we have:*

- *The regularity of $\mathcal{E}$ is at most k_0, and*
- *$\mathcal{E}$ is the pushforward along some linear map $\mathbf{P}^{n_0} \to \mathbf{P}^n$.*

The main input is a recent noetherianity result of Draisma [3] for polynomial functors (we just need the case of finite sums of exterior powers).

References

[1] Tigran Ananyan and Melvin Hochster, Small subalgebras of polynomial rings and Stillman's conjecture (2016). arXiv:1610.09268v1.

[2] Giulio Caviglia, Marc Chardin, Jason McCullough, Irena Peeva, Matteo Varbaro, Regularity of prime ideals, *Math. Z.*, to appear.

[3] Jan Draisma, Topological noetherianity for polynomial functors (2017). arXiv:1705.01419v1.

[4] Craig Huneke, Criteria for complete intersections, *J. London Math. Soc.* **32** (1985), 19–30.