# Big polynomial rings and Stillman's conjecture

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(joint work with Daniel Erman and Andrew Snowden)

Throughout, $\mathbf{k}$ will refer to an algebraically closed field (not fixed). Stillman's conjecture (now a theorem of Ananyan–Hochster [AH]) is the following statement:

**Theorem 1** (Ananyan–Hochster)**.** *Fix integers $d_1, \dots, d_r \geq 1$. There is a constant $C$ such that any ideal in $\mathbf{k}[x_1, \dots, x_n]$ generated by homogeneous polynomials of degrees $d_1, \dots, d_r$ has projective dimension $\leq C$ (independent of $n$ and $\mathbf{k}$).*

The motivation for the work in this talk was to show this as a consequence of a more basic structural result. Our first idea involved "**GL**-noetherianity":

- Remove independence on $n$ by working in $\mathbf{k}[x_1, x_2, \dots] = \mathrm{Sym}(\mathbf{k}^\infty)$.
- The relevant parameter space for choices of polynomials is
$$X = \mathrm{Sym}^{d_1}(\mathbf{k}^\infty) \times \cdots \times \mathrm{Sym}^{d_r}(\mathbf{k}^\infty).$$
- For each $d$, we define a $\mathbf{GL}_\infty(\mathbf{k})$-equivariant subset
$$X_{\geq d} = \{(f_1, \dots, f_r) \in X \mid \mathrm{pdim}(f_1, \dots, f_r) \geq d\},$$
which gives a decreasing chain $X_{\geq 1} \supseteq X_{\geq 2} \supseteq \cdots$.
- If each $X_{\geq d}$ is closed and $X$ is $\mathbf{GL}_\infty(\mathbf{k})$-noetherian (i.e., decreasing chains of closed $\mathbf{GL}_\infty(\mathbf{k})$-invariant subsets always stabilize), then we would have $X_{\geq d} = X_{\geq d+1}$ for $d \gg 0$ and get an upper bound for projective dimension.
- Noetherianity follows from work of Draisma [Dr]; closedness of $X_{\geq d}$ is less clear. A priori, it is only an infinite union of closed sets given by vanishing conditions on graded Betti numbers: $\beta_{d,d} = \beta_{d,d+1} = \cdots = 0$.

This idea can be developed to give a proof (see [ESS1, §5]), but we ended up finding a simpler proof which we explain.

A key step to proving this is the notion of strength [AH]: a homogeneous element $f$ in a graded ring has **strength** $\leq s$ if we can write $f = g_1 h_1 + \cdots + g_s h_s$ with $g_i, h_i$ homogeneous and $0 < \deg g_i < \deg f$ for all $i$. The strength is $s$ if it has strength $\leq s$ but not strength $\leq s - 1$. The strength is $\infty$ if there is no such decomposition. The strength of a linear space of elements is the minimal strength of a nonzero homogeneous element in it. Then:

**Theorem 2** (Ananyan–Hochster)**.** *Fix integers $d_1, \dots, d_r \geq 1$. Given polynomials $f_1, \dots, f_r$ with $\deg(f_i) = d_i$, if the strength of $\langle f_1, \dots, f_r \rangle$ is sufficiently large (with respect to $d_1, \dots, d_r$), then $f_1, \dots, f_r$ is a regular sequence.*

*In particular, there is a constant $C$ such that any $f_1, \dots, f_r \in \mathbf{k}[x_1, \dots, x_n]$ with $\deg(f_i) = d_i$ belong to a subalgebra generated by a regular sequence with $\leq C$ homogeneous elements.*

The $C$ in the theorem gives a bound for the original statement: a minimal free resolution for $(f_1, \ldots, f_r)$ can first be computed in the subalgebra generated by the regular sequence; by flatness, its base change to $\mathbf{k}[x_1, \ldots, x_n]$ remains a resolution.

Ultraproducts give a context for working with the notion of "sufficiently large" without having to explicitly identify bounds.

Let $\mathcal{I}$ be an infinite set (typically the positive integers). We fix a **non-principal ultrafilter** $\mathcal{F}$ on $\mathcal{I}$, which is a collection of subsets of $\mathcal{I}$ satisfying the following properties:

(1) $\mathcal{F}$ contains no finite sets,
(2) if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$,
(3) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$,
(4) for all $A \subseteq \mathcal{I}$, either $A \in \mathcal{F}$ or $\mathcal{I} \setminus A \in \mathcal{F}$ (but not both).

Intuition: the sets in $\mathcal{F}$ are neighborhoods of some hypothetical (and non-existent) point $*$ of $\mathcal{I}$. We say that some condition holds near $*$ if it holds in some neighborhood of $*$.

Given a family of sets $\{X_i\}_{i \in \mathcal{I}}$, their ultraproduct $\operatorname{ulim}_{i \in \mathcal{I}} X_i$ is the quotient of the usual product $\prod_{i \in \mathcal{I}} X_i$ in which two sequences $(x_i)$ and $(y_i)$ are identified if the equality $x_i = y_i$ holds near $*$.

Suppose that each $X_i$ is a graded abelian group. We define the **graded ultraproduct** of the $X_i$'s to be the subgroup of the usual ultraproduct consisting of elements $x$ such that $\deg(x_i)$ is bounded near $*$. The graded ultraproduct is a graded abelian group; the degree $d$ piece of the graded ultraproduct is the usual ultraproduct of the degree $d$ pieces of the $X_i$'s. We again denote this by $\operatorname{ulim}_i X_i$.

The following 3 statements about ultraproducts and regular sequences give a proof of Stillman's conjecture:

**Lemma 1.** *For $i \geq 1$, let $V_i$ be linear subspaces of polynomial rings $R_i$ with strength tending to $\infty$. Then $\operatorname{ulim}_i V_i \subset \operatorname{ulim}_i R_i$ has infinite strength.*

This follows from the definitions of ultraproducts.

**Theorem 3.** *For $i \geq 1$, let $R_i = \mathbf{k}_i[x_1, x_2, \ldots]$ with $\deg(x_j) = 1$. Let $\mathbf{S} = \operatorname{ulim}_i R_i$ and let $\mathfrak{m}$ be its homogeneous maximal ideal. Let $\mathcal{E} \subset \mathfrak{m}$ be a subset of homogeneous elements whose image in $\mathfrak{m}/\mathfrak{m}^2$ is a basis (over $\mathbf{K} = \operatorname{ulim}_i \mathbf{k}_i$). Then $\mathbf{S}$ is a polynomial ring over $\mathbf{K}$ with generators $\mathcal{E}$.*

Note that $\mathfrak{m}^2$ is precisely the set of polynomials of finite strength, so a linear subspace of $\mathfrak{m}$ has infinite strength if and only if its image in $\mathfrak{m}/\mathfrak{m}^2$ is linearly independent.

**Lemma 2.** *For $i \geq 1$, let $f_{i,1}, \ldots, f_{i,r} \in R_i$ be homogeneous polynomials of degrees $d_1, \ldots, d_r$. Then $\operatorname{ulim}_i f_{i,1}, \ldots, \operatorname{ulim}_i f_{i,r} \in \operatorname{ulim}_i R_i$ is a regular sequence if and only if $f_{i,1}, \ldots, f_{i,r}$ is a regular sequence for $i$ near $*$.*

Here is the proof that sufficiently large strength (relative to degrees) implies regular sequence: if not, then we can find a sequence of polynomials $(f_{i,1}, \ldots, f_{i,r})$

whose strength goes to $\infty$ for $i \gg 0$ but which do not form a regular sequence for any $i$. The ultralimit has infinite strength in $\mathbf{S}$ (Lemma 1), which is then part of an algebraically independent generating set for $\mathbf{S}$ (Theorem 3), and hence form a regular sequence. But this contradicts Lemma 2.

Lemma 1 follows from the definitions.

Theorem 3 is proven with the following criterion for polynomiality:

**Theorem 4.** *Let $R$ be a $\mathbf{Z}_{\geq 0}$-graded ring over $\mathbf{k}$ (characteristic $0$ and not necessarily algebraically closed) with $R_0 = \mathbf{k}$. Assume that $R$ has "enough derivations", i.e., for every positive degree element $f$, there exists a negative degree derivation $\partial$ of $R$ such that $\partial(f) \neq 0$. Then $R$ is a polynomial ring over $\mathbf{k}$, and a generating set can be obtained by taking any lift of a $\mathbf{k}$-basis for $R_{>0}/R_{>0}^2$.*

For fields of positive characteristic, this doesn't work ($p$th powers are killed by any derivation and $\mathbf{F}_p[x]/(x^p)$ has enough derivations) but we can give slightly different criteria using Hasse derivatives ($\partial_k x^n = \binom{n}{k} x^{n-k}$) when $\mathbf{k}$ is perfect (the imperfect case is handled in [ESS2]).

The idea is that any lift of a basis for $R_{>0}/R_{>0}^2$ generates $R$, and the derivations are used to show that these elements don't satisfy any nontrivial algebraic relations since any relation can always be used to produce one of lower degree. To get the derivations on $\mathbf{S}$, we can take ultraproducts of usual partial derivatives.

Finally, here is a sketch of the proof of Lemma 2:

- Let $\mathbf{S} = \operatorname{ulim}_i R_i$ and let $I = \operatorname{ulim}_i I_i$ with $I_i \subseteq R_i$ ideals generated in the same degrees. We will show that $\operatorname{codim} I = \operatorname{codim} I_i$ for $i$ near $*$ and apply this to the ideals generated by $f_{i,1}, \dots, f_{i,r}$.

  Also, set $R_i' = \mathbf{k}_i[x_2, x_3, \dots]$ and $\mathbf{S}' = \operatorname{ulim}_i R_i'$. Then $\mathbf{S}'[x_1] \cong \mathbf{S}$.
- First, $\operatorname{codim}(I) < \infty$ since $I$ is finitely generated and defined by finitely many of the variables in $\mathcal{E}$. Let $c = \operatorname{codim}(I)$.
- Now do induction on $c$. If $c = 0$, then $I = 0$ and $I_i = 0$, so there is nothing to show. Otherwise, pick nonzero $f \in I$.
- Pick change of variables $\gamma$ so that $\gamma(f)$ is monic as a polynomial in $x_1$. Then $\operatorname{codim}_{\mathbf{S}}(I) - 1 = \operatorname{codim}_{\mathbf{S}'}(\mathbf{S}' \cap I)$ since $\mathbf{S}' \to \mathbf{S}/(f)$ is finite and flat. Now use the induction hypothesis.

## References


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[ESS2] Daniel Erman, Steven V Sam, Andrew Snowden, Big polynomial rings with imperfect coefficient fields, arXiv:1806.04208v1.