Mini-course on representation stability
Texel island, Netherlands, August 28 – September 1, 2017[1]

1. Introduction

The goal of these lectures is to describe two different aspects of representation stability:

- A generalization of homological stability in the presence of additional symmetry.
- A framework for proving uniformity / boundedness results in commutative algebra / algebraic geometry, again in the presence of symmetries.

Here are two examples of such theorems.

1.1. **Cohomology of configuration spaces.** Let X be a connected oriented manifold of dimension ≥ 2 such that $\dim_{\mathbf{Q}} \mathrm{H}^*(X; \mathbf{Q}) < \infty$ and define

$$\mathrm{Conf}_n(X) = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

It carries an action of the symmetric group Σ_n.

Theorem 1.1 (McDuff, Segal, Church, ...)**.** *Fix an integer $i \geq 0$.*

(a) *The function $n \mapsto \dim_{\mathbf{Q}} \mathrm{H}^i(\mathrm{Conf}_n(X)/\Sigma_n; \mathbf{Q})$ is constant for $n \gg 0$.*
(b) *The function $n \mapsto \dim_{\mathbf{Q}} \mathrm{H}^i(\mathrm{Conf}_n(X); \mathbf{Q})$ agrees with a polynomial for $n \gg 0$.*

The first is an example of homological stability: a family of (co)homology groups of a sequence of objects which are eventually constant. More precisely, one wants naturally defined transition maps which are eventually isomorphisms. The second is an example where homological stability fails: the polynomial in (b) often has positive degree, but can be understood with the help of the symmetric group actions.

Example 1.2. About the dimension requirement: if $X = [0, 1]$ is the unit interval, then $\mathrm{Conf}_n(X)$ has $n!$ connected components corresponding to the relative ordering of the points. Hence $\dim_{\mathbf{Q}} \mathrm{H}^0(\mathrm{Conf}_n(X); \mathbf{Q}) = n!$ does not have polynomial growth. □

1.2. **Equations defining border rank.** Let $V_1, \dots, V_n$ be vector spaces over an algebraically closed field and $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$. An element of $\mathbf{V}$ of the form $v_1 \otimes \cdots \otimes v_n$ with $v_i \in V_i$ is a simple tensor, and a general element has rank $\leq r$ if it can be expressed as a sum of r simple tensors. Finally, an element in the Zariski closure of rank $\leq r$ tensors has border rank $\leq r$. They form an algebraic variety.

Example 1.3. The following example illustrates why the notion of border rank is needed, i.e., why tensor rank is not semicontinuous.

Let $n = 3$ and $\dim V_i = 2$. Pick bases $\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}$ of V_1, V_2, V_3, respectively. The element

$$\begin{aligned} v &= a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 \\ &= a_1 \otimes b_1 \otimes (c_1 + c_2) + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 \end{aligned}$$

has rank ≤ 3, and it can be shown that it does not have rank ≤ 2, so the rank is exactly 3. However,

$$v = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}((\varepsilon - 1) a_1 \otimes b_1 \otimes c_1 + (a_1 + \varepsilon a_2) \otimes (b_1 + \varepsilon b_2) \otimes (c_1 + \varepsilon c_2))$$

shows that v has border rank ≤ 2. □

[1]Last updated August 31, 2017.

Remark 1.4. This variety is also the affine cone over the rth secant variety of the Segre embedding of $\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n)$ in $\mathbf{P}(\mathbf{V})$. $\square$

Theorem 1.5 (Draisma–Kuttler)**.** *For each r, there is a constant $C(r)$ such that the variety of border rank $\le r$ tensors is cut out by polynomials of degree $\le C(r)$. The constant is independent of n and also the V_i.*

Example 1.6. A familiar case is when $n = 2$. Then $\mathbf{V}$ can be thought of as the space of $a \times b$ matrices where $a = \dim V_1$ and $b = \dim V_2$. In this case, a simple tensor is a rank 1 matrix, and rank r in the tensor sense corresponds to rank r in the matrix sense. Border rank $\le r$ is equivalent to rank $\le r$. The polynomials in this case are the determinants of all $(r+1) \times (r+1)$ submatrices, and so they are of degree $r+1$. When $n \ge 3$, the situation is more complicated. $\square$

The general idea is to show that there is a way to take the limit as the parameters n and $\dim V_i$ go to ∞ and then to study the resulting limit space. Border rank is preserved upon change of basis (with respect to each V_i) so there is a symmetry that can be used in this problem. The existence of the constant can be then deduced from an equivariant noetherianity property of the limit space.

2. First example: configuration spaces and FI-modules

2.1. **FI-modules: definitions and basic properties.** Theorem 1.1 can be understood from the theory of **FI**-modules. Define $[n] = \{1, \dots, n\}$. To motivate the definition, consider the symmetries of $\bigoplus_n \mathrm{H}^i(\mathrm{Conf}_n(X))$. First, for any permutation $\sigma \in \Sigma_n$, we can act on $\mathrm{Conf}_n(X)$, and hence get a linear action on its cohomology. More generally, given an injection $f\colon [m] \to [n]$, we have a forgetful map

$$\begin{aligned} \mathrm{Conf}_n(X) &\to \mathrm{Conf}_m(X) \\ (x_1, \dots, x_n) &\mapsto (x_{f(1)}, \dots, x_{f(m)}), \end{aligned}$$

and hence a map on cohomology $f_*\colon \mathrm{H}^i(\mathrm{Conf}_m(X)) \to \mathrm{H}^i(\mathrm{Conf}_n(X))$. Permutations are a special case when $m = n$, and they behave "functorially": $(f \circ g)_* = f_* \circ g_*$.

We can encode this algebraic structure as follows. Define **FI** to be the category whose objects are the finite sets $[n]$ for $n = 0, 1, 2, \dots$ and whose morphisms are injective functions. The discussion above shows that we can define a functor from **FI** to the category of abelian groups by sending $[n]$ to $\mathrm{H}^i(\mathrm{Conf}_n(X))$ and f to f_*. Alternatively, for any commutative ring R, we get a functor to the category of R-modules by using $\mathrm{H}^i(\mathrm{Conf}_n(X); R)$. We will call a functor from **FI** to R-modules an **FI-module over** R.

Given an **FI**-module M, we can also consider the direct sum $\bigoplus_n M_n$ together with all of the operations f_*, one for each injective function f. Clearly, these are equivalent data, so we will often think of them interchangably. A sequence of subspaces $N_n \subset M_n$ is an **FI-submodule** if it is closed under all of the f_*. A morphism between **FI**-modules $M \to M'$ is a natural transformation. Alternatively, it is a degree-preserving map $\bigoplus_n M_n \to \bigoplus_n M'_n$ that are linear with respect to the operations f_*.

We want to understand what constraints are put on a sequence of spaces M_n if they come from an **FI**-module. Each M_n is a representation of the symmetric group Σ_n. However, little else can be said. For example, take an arbitrary sequence $\{M_n\}$ where M_n is a representation of Σ_n. We can define an **FI**-module M by sending $[n]$ to M_n, each permutation to its corresponding action on M_n, and and all non-bijective injections to 0.

Call an **FI**-module M **finitely generated** if $\bigoplus_n M_n$ can be generated by finitely many elements under addition and the operations f_*. Many things can be deduced about $\bigoplus_n M_n$ but we will limit ourselves to three properties:

Theorem 2.1. *Let* $\mathbf{k}$ *be a field (for simplicity) and let* M *be a finitely generated* **FI**-*module over* $\mathbf{k}$.

(a) *The function* $n \mapsto \dim_{\mathbf{k}} M_n$ *agrees with a polynomial for* $n \gg 0$.
(b) *Any submodule* $N \subset M$ *is also finitely generated.*
(c) *The sequence of coinvariants* $(M_n)_{\Sigma_n}$ *has the structure of a finitely generated* $\mathbf{k}[t]$-*module.*

Proof of (3). The action of Σ_n on the set of injections $[m] \to [n]$ is transitive, so given any two injections f, g, the induced maps $(M_m)_{\Sigma_m} \to (M_n)_{\Sigma_n}$ are the same. We define a $\mathbf{k}[t]$-module structure by defining multiplication by t^{n-m} on $(M_m)_{\Sigma_m}$ to be the map just described. Finite generation by $\mathbf{k}[t]$ is inherited from finite generation of M as an **FI**-module. $\square$

2.2. **Proof of Theorem 1.1.**

(1) Given our assumptions on X (which can be weakened substantially), show that the **FI**-module $[n] \mapsto \mathrm{H}^i(\mathrm{Conf}_n(X); \mathbf{k})$ is a finitely generated **FI**-module over $\mathbf{k}$. Given Theorem 2.1(a), this already proves (a).
(2) For $\mathbf{k} = \mathbf{Q}$, we have $\mathrm{H}^i(\mathrm{Conf}_n(X)/\Sigma_n; \mathbf{Q}) = \mathrm{H}^i(\mathrm{Conf}_n(X); \mathbf{Q})^{\Sigma_n}$. Furthermore, coinvariants and invariants can be identified for linear representations of finite groups over a field of characteristic 0, so $\bigoplus_n \mathrm{H}^i(\mathrm{Conf}_n(X)/\Sigma_n; \mathbf{Q})$ can be equipped with the structure of a finitely generated $\mathbf{Q}[t]$-module.
(3) A finitely generated $\mathbf{Q}[t]$-module is a direct sum of a free module and a torsion module. In particular, the dimension is eventually constant (equal to the rank of the free module). This proves (b).

Step 1 uses a spectral sequence argument due to Totaro and Theorem 2.1(b). The basic idea comes from [Ch] and has been improved upon in a series of followup papers. First, define a $\mathbf{Z}^2$-graded skew-commutative algebra $E = \mathrm{H}^*(X^n; \mathbf{k})[G_{a,b}]$ where $1 \le a < b \le n$ where $\mathrm{H}^i(X^n; \mathbf{k})$ has degree $(i, 0)$ and $G_{a,b}$ has degree $(0, \dim X - 1)$. The $G_{a,b}$ satisfy some relations including $G_{a,b}^2 = 0$. Then E is the E_2-page of a spectral sequence that converges to $\mathrm{H}^*(\mathrm{Conf}_n(X); \mathbf{k})$. The upshot is:

- The bigraded components of E naturally carry an **FI**-module structure compatible with the spectral sequence.
- $\mathrm{H}^i(\mathrm{Conf}_n(X); \mathbf{k})$ has a filtration whose associated graded is a subquotient of $\bigoplus_{p+q=i} E^{p,q}$.

Then the assumption on X guarantees that the initial terms are finitely generated as **FI**-modules, and Theorem 2.1(b) guarantees that any subquotient of E is also finitely generated. This gives the desired property. We will not go any further into those details here.

Example 2.2. For $X = \mathbf{R}^2$, we have

$$\mathrm{H}^*(\mathrm{Conf}_n(\mathbf{R}^2); \mathbf{Q}) = \mathbf{Q}[w_{i,j} \mid 1 \le i, j \le n,\ i \ne j]/I$$

where $w_{i,j} \in \mathrm{H}^1$ (so we are taking an exterior algebra generated by the $w_{i,j}$) and I is the ideal generated by

$$w_{ij} = w_{ji}, \qquad w_{jk} \wedge w_{k\ell} + w_{k\ell} \wedge w_{\ell j} + w_{\ell j} \wedge w_{jk} = 0.$$

One can show directly that each H^i is a finitely generated **FI**-module (the morphisms act on the indices of the w). $\square$

2.3. **FI-modules: proofs.** The remaining question is how to prove Theorem 2.1. We can make a few reductions. First, for each n, define an **FI**-module P_n by

$$P_n([m]) = \mathbf{k}[\mathrm{Hom}_{\mathbf{FI}}([m],[n])],$$

i.e., the formal linear span of all injections $[m] \to [n]$. This is a functor by having morphisms act as post-composition. One can readily verify that a morphism $P_n \to M$ is determined by where the identity map in $P_n([n])$ goes to in M_n, and that this choice can be made arbitrarily. Hence, if M is generated by $x_1, \dots, x_d$ where $x_i \in M_{n_i}$, then we have a surjection $\bigoplus_i P_{n_i} \to M$.

Theorem 2.1(b) says that finitely generated modules are noetherian. Noetherianity satisfies a few properties for formal reasons:

- Quotients of noetherian objects are noetherian.
- Direct sums of noetherian objects are noetherian.

In particular, it suffices to show that each P_n is noetherian. As for Theorem 2.1(b), we can see by direct inspection that P_n has polynomial growth: $\dim P_n([m]) = m(m-1)\cdots(m-n+1)$. In general, if M is generated by $x_1, \dots, x_n$, consider the filtration of M by submodules M^j which is generated by $x_1, \dots, x_j$. Each quotient is generated by a single element, and if they have polynomial growth, so does M. So it actually suffices to check that submodules of P_n have polynomial growth.

There are several approaches, but we follow the one that generalizes more easily. Define a subcategory $\mathbf{OI} \subset \mathbf{FI}$ to have the same objects, but only order-preserving injections are taken as morphisms. Define an **OI**-module Q_n by

$$Q_n([m]) = \mathbf{k}[\mathrm{Hom}_{\mathbf{OI}}([n],[m])].$$

Then the restriction of P_n to **OI** is a direct sum of $n!$ copies of Q_n: the copies are indexed by permutations of $[n]$, and an injective function f belongs to the copy indexed by σ if and only if $f \circ \sigma$ is order-preserving. The advantage of working with the category **OI** is that we have removed the symmetries, i.e., all automorphisms are trivial. This allows us to degenerate submodules of Q_n to those generated by monomials, i.e., elements of $\mathrm{Hom}_{\mathbf{OI}}([n],[m])$ (as opposed to linear combinations of them). Let e_f be the element corresponding to $f \in \mathrm{Hom}_{\mathbf{OI}}([n],[m])$.

To do this, first identify an ordered injection $f \colon [n] \to [m]$ with a sequence $w(f)$ of 0's and 1's of length m, where the ith letter is 0 if it is in the image of f, and 1 otherwise. We compare these lexicographically, and given a linear combination $x = \sum_f \alpha_f e_f$, define $\mathrm{init}(x) = e_f$ where f is the largest element such that $\alpha_f \neq 0$. Finally, given a submodule $M \subseteq Q_n$, define a graded subspace $\bigoplus_n \mathrm{init}(M)_n$ where $\mathrm{init}(M)_n$ is the $\mathbf{k}$-linear span of $\mathrm{init}(x)$ for all $x \in M_n$. This satisfies some important properties which are easily verified (for details, see [SS, §§4, 7.1]):

- $\mathrm{init}(M)$ is an **OI**-submodule of Q_n,
- $\dim_{\mathbf{k}} \mathrm{init}(M)_n = \dim_{\mathbf{k}} M_n$,
- If $M \subseteq N \subseteq Q_n$ and $\mathrm{init}(M) = \mathrm{init}(N)$, then $M = N$,
- $\mathrm{init}(M)$ is finitely generated (this also follows from the usual Hilbert basis theorem subject to our identification of injections with monomials in $n+1$ variables below).

In particular, if $M \subseteq Q_n$ is not finitely generated, we can pick $x_1, x_2, \dots \in M$ such that x_n is not generated by $x_1, \dots, x_{n-1}$. Let M^n be the submodule generated by $x_1, \dots, x_n$. This gives a strictly increasing chain

$$M^1 \subsetneqq M^2 \subsetneqq \cdots \subsetneqq M$$

and hence a strictly increasing chain

$$\mathrm{init}(M^1) \subsetneqq \mathrm{init}(M^2) \subsetneqq \cdots \subsetneqq \mathrm{init}(M).$$

Since $\mathrm{init}(M)$ is finitely generated, the finite generating set must live in some $\mathrm{init}(M^k)$, which is a contradiction to the existence of M.

As for the polynomial growth statement, we are reduced to understanding monomial submodules of Q_n. These can be handled in a general framework using formal language theory, but we will omit the details. Alternatively, one can identify $w(f)$ with a monomial $m(f)$ in the polynomial ring $\mathbf{k}[x_0, \dots, x_n]$ by taking $m(f) = x_0^{d_0} \cdots x_n^{d_n}$ where d_i is the number of 1's between the $(i-1)$th and ith instances of 0. Then submodules correspond to ideals and one can use classical results from commutative algebra.

2.4. Other examples.

- Let $\mathcal{M}_{g,n}$ be the moduli space of genus g curves with n marked points. Again, each injection $f \colon [n] \to [m]$ induces a forgetful map $\mathcal{M}_{g,m} \to \mathcal{M}_{g,n}$ and hence a map on cohomology $\mathrm{H}^i(\mathcal{M}_{g,n}) \to \mathrm{H}^i(\mathcal{M}_{g,m})$. Work of Church–Ellenberg–Farb [CEF] and Jimenez Rolland [JR] implies that for fixed i and g, the **FI**-module $[n] \mapsto \mathrm{H}^i(\mathcal{M}_{g,n}; \mathbf{Q})$ is finitely generated.
- Congruence subgroups: given a positive integer ℓ, let $\Gamma_n(\ell) \subset \mathbf{GL}_n(\mathbf{Z})$ be the subgroup of matrices which are the identity modulo ℓ. There is also a functor $[n] \mapsto \mathrm{H}_i(\Gamma_n(\ell); R)$ and work of Putman [Pu] and Church–Ellenberg–Farb–Nagpal [CEFN] shows it is finitely generated.

Compactifications of moduli spaces (e.g., Deligne–Mumford compactification of $\mathcal{M}_{g,n}$ and Fulton–MacPherson compactification of $\mathrm{Conf}_n(X)$ when X is a projective variety) tend to have exponentially growing cohomology as n varies, so they cannot be finitely generated **FI**-modules.

There is a variant: instead of using injective functions, use surjective functions. Also, use the opposite of the category so that morphisms still go up in degree. Call this $\mathbf{FS}^{\mathrm{op}}$. This is still well-behaved in that finite generation is preserved under taking submodules, but now finitely generated objects can have exponential growth (consider the analogue of P_n which sends $[m]$ to the space of surjections $[m] \to [n]$). Using an explicit presentation of the cohomology of $\overline{\mathcal{M}}_{0,n}$, one can show it is a finitely generated $\mathbf{FS}^{\mathrm{op}}$-module. Unfortunately, I do not know if it has any natural geometric interpretation or the action makes sense for higher genus (ignoring questions of finite generation). See [PY] for an example of $\mathbf{FS}^{\mathrm{op}}$ acting on a partial compactification of $\mathrm{Conf}_n(\mathbf{R}^2)$.

Linear variants of **FI** (using vector spaces over finite fields instead of finite sets) have been studied in [PS].

3. Spaces of tensors

3.1. **Equations for bounded rank tensors.** We continue with the setup from the introduction.

Let $V_1, \dots, V_n$ be vector spaces and $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$. An element of $\mathbf{V}$ of the form $v_1 \otimes \cdots \otimes v_n$ with $v_i \in V_i$ is a simple tensor, and a general element has rank $\le r$ if it can be expressed as a sum of r simple tensors.

Recall that on a vector space, such as $\mathbf{V}$, the closed subsets of the Zariski topology are those subsets which are the common solution set of a collection of polynomials. Without loss of generality, we may always assume that the collection of polynomials forms an ideal. We will only be interested in those closed sets which are closed under scalar multiplication, and these are always defined by a collection of homogeneous polynomials. In this case, the notion of a minimal generating set of polynomials for the ideal is well-defined.

An element in the Zariski closure of rank $\le r$ tensors is said to have border rank $\le r$. The following is in [DrK]:

Theorem 3.1 (Draisma–Kuttler)**.** *For each r, there is a constant $C(r)$ such that the variety of border rank $\le r$ tensors is cut out by polynomials of degree $\le C(r)$. The constant is independent of n and also the V_i.*

The outline of the proof is as follows:

(1) Reduce to the case that $\dim V_i = r+1$ for all i. Then construct an infinite limit space $V^{\otimes\infty}$ together with an action of a group G_∞. Also construct a limit of the border rank $\le r$ elements $X_{\le r}$.
(2) Show that G-equivariant closed subsets of an auxiliary space $Y_{\le r}$, which contains $X_{\le r}$, satisfy the descending chain condition.
(3) Translate the above properties into the desired theorem.

We will explain (1) and (3) and skip most of (2). First, we need the notion of a flattening: given a subset $S \subseteq [n]$, set $U = \bigotimes_{i\in S} V_i$ and $U' = \bigotimes_{i \notin S} V_i$. Then $\mathbf{V} = U \otimes U'$, but now we can identify it with the space of matrices. Given $\omega \in \mathbf{V}$, the corresponding matrix is called a flattening of ω (it depends on S). If ω has rank r, then its flattening is a sum of r rank 1 matrices and hence has rank $\le r$.

Lemma 3.2. *It suffices to prove Theorem 3.1 when* $\dim V_i = r + 1$.

Proof. We claim that for any $\omega \in \mathbf{V}$, its rank is at most r if and only if for each set of linear maps $\varphi_i \colon V_i \to \mathbf{k}^r$, the rank of its image is also at most r. The "if" direction is clear: if ω has rank $\le r$, then so are all of its images.

To prove the other direction, suppose that ω has rank $\ge r+1$. We show that for each i, there exists a linear map $\varphi \colon V_i \to \mathbf{k}^{r+1}$ such that the image of ω has rank $\ge r+1$. Without loss of generality, we may assume $i = 1$. We can reinterpret ω as a linear map

$$V_2^* \otimes \cdots \otimes V_n^* \to V_1,$$

and we let W be the image. If $\dim W \le r+1$, let $\mathbf{k}^{r+1}$ be any subspace of V_1 containing W and let $\varphi \colon V_1 \to \mathbf{k}^{r+1}$ be a projection. Then the image of ω has the same rank as ω. Otherwise, if $\dim W > r+1$, let $\varphi \colon V \to \mathbf{k}^{r+1}$ be any linear map such that φ maps W surjectively onto $\mathbf{k}^{r+1}$. The flattening of the image of ω is then $V_2^* \otimes \cdots \otimes V_n^* \to \mathbf{k}^{r+1}$ and has rank $r+1$, so the rank of the image of ω is $\ge r+1$. $\square$

In particular, we may as well fix a single space V of dimension $r+1$ to replace all V_i. We also fix a linear functional $x_0 \in V^*$ and use this to define maps

$$V^{\otimes(p+1)} \to V^{\otimes p}$$
$$v_1 \otimes \cdots \otimes v_p \otimes v_{p+1} \mapsto x_0(v_{p+1}) v_1 \otimes \cdots \otimes v_p.$$

If $\omega \in V^{\otimes(p+1)}$ has rank $\le r$, then so does its image in $V^{\otimes p}$. Now set

$$V^{\otimes\infty} := \varprojlim_p V^{\otimes p}$$

and let $X_{\le r}$ be the inverse limit of the spaces of border rank $\le r$ tensors.

The space $V^{\otimes p}$ carries an action of Σ_p by permuting tensor factors. It also has an action of $\mathbf{GL}(V)^p$ by linear change of coordinates. So the semidirect product $\Sigma_p \ltimes \mathbf{GL}(V)^p$ acts. We have inclusions $G_p \subset G_{p+1}$ compatible with the projection with respect to x_0, so the union

$$G_\infty = \bigcup_p G_p$$

acts on $V^{\otimes\infty}$.

The space $Y_{\le r}$ is the inverse limit of the spaces of elements whose flattenings all have rank $\le r$. This certainly contains $X_{\le r}$. Furthermore, it is defined by taking the determinants of all $(r+1)\times(r+1)$ submatrices of all flattenings, which are in particular polynomials of degree $r+1$. The spaces $X_{\le r}$ and $Y_{\le r}$ are both closed under the action of G_∞. We skip the rest of (2), which says that any infinite descending chain of closed G_∞-subsets in $Y_{\le r}$ must be eventually constant.

Note that G_∞ also acts on the polynomial functions of all of the spaces involved.

Lemma 3.3. *If $Z \subset Y_{\le r}$ is a closed G_∞-subset, then it is the common solution set of finitely many G_∞-orbits of polynomials. In particular, it is defined by bounded degree polynomials.*

Proof. If not, we can find an infinite sequence of polynomials $f_1, f_2, \dots$ that all vanish on Z, but such that Z_i, the common solution set of $G_\infty f_1, \dots, G_\infty f_i$, satisfies $Z_1 \supsetneqq Z_2 \supsetneqq \cdots$ in direct contradiction to (2). $\square$

Taking $Z = X_{\le r}$, we get that it is defined by bounded degree polynomials. To finish, pick $e_0 \in V$ such that $x_0(e_0) = 1$. Define

$$\begin{aligned} \tau \colon V^{\otimes p} &\to V^{\otimes(p+1)} \\ \omega &\mapsto \omega \otimes e_0. \end{aligned}$$

Given $\omega \in V^{\otimes p}$, we can define $\omega_\infty = \omega \otimes e_0^{\otimes\infty} \in V^{\otimes\infty}$, and their ranks are the same. So pulling back all of the equations vanishing on $X_{\le r}$ gives equations that define the border rank $\le r$ locus in $V^{\otimes p}$.

3.2. **Variants.** A similar situation can be considered with both exterior and symmetric powers of vector spaces instead of tensor powers. Given V, an element of $\bigwedge^n V$ of the form $v_1 \wedge \cdots \wedge v_n$ is said to have rank 1. Rank and border rank are defined similarly. For $\mathrm{Sym}^n V$, elements of the form v^n are said to have rank 1 (technically we should be working with the divided power instead of the symmetric power). We note some theorems:

Theorem 3.4 (Draisma–Eggermont [DE])**.** *For each r, there is a constant $C(r)$ such that the variety of border rank $\le r$ anti-symmetric tensors is cut out by polynomials of degree $\le C(r)$. The constant is independent of n and also V.*

Theorem 3.5 (Sam [Sa])**.** *Fix a field of characteristic 0. For each r, there is a constant $C(r)$ such that the full ideal of polynomials vanishing on the variety of border rank $\le r$ symmetric tensors is generated by polynomials of degree $\le C(r)$. The constant is independent of n and also V.*

The last theorem gives an ideal-theoretic statement which improves the set-theoretic statement (though is limited by its restriction on characteristic).

4. Twisted commutative algebras

In this last part, we discuss some algebraic structures related to both of the examples we have discussed. Let **FB** be the category whose objects are the finite sets $[n]$ and whose morphisms are bijective functions. A functor $\mathbf{FB} \to \mathrm{Vec}_{\mathbf{k}}$ is the same as a sequence of Σ_n representations over $\mathbf{k}$. Given functors $M, N\colon \mathbf{FB} \to \mathrm{Vec}_{\mathbf{k}}$, we define their tensor product by

$$(M \otimes N)(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n}(M_i \otimes N_j).$$

This product is naturally associative, and we can define an isomorphism

$$\tau_{M,N}\colon M \otimes N \cong N \otimes M$$

as follows: for each i, j, there is an element $\tau_{i,j} \in \Sigma_{i+j}$ that conjugates $(x, y) \in \Sigma_i \times \Sigma_j$ to $(y, x) \in \Sigma_j \times \Sigma_i$. Applying these elements gives the desired isomorphism. The unit of the product is the sequence of trivial representations.

In particular, one can define commutative algebra objects in the category of functors $\mathbf{FB} \to \mathrm{Vec}_{\mathbf{k}}$, which we call **twisted commutative algebras** (tca's). Concretely, this is a functor $A\colon \mathbf{FB} \to \mathrm{Vec}_{\mathbf{k}}$ together with a map

$$A \otimes A \to A$$

which satisfies associativity and is invariant under the transpose map $\tau_{A,A}$.

For an example, let V be a vector space and define $A_V(n) = V^{\otimes n}$. By Frobenius reciprocity, an Σ_{i+j}-equivariant map

$$\mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_{i+j}} V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes (i+j)}$$

is the same as the data of a $\Sigma_i \times \Sigma_j$-equivariant map

$$V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes (i+j)}$$

which we take as the canonical map.

One can also define modules over tca's. Given a tca A, an A-module M is a functor $M\colon \mathbf{FB} \to \mathrm{Vec}_{\mathbf{k}}$ together with a map

$$A \otimes M \to M$$

which satisfies associativity.

A module M over the tca $A_{\mathbf{k}}$ is a sequence of symmetric group representations M_n together with maps

$$\mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_{i+j}} M_j \to M_{i+j}.$$

This can be shown to be the same as an **FI**-module, so in fact, **FI**-modules are a special case of modules over a tca.

The tca A_V is an example of a finitely generated tca (in fact, it is generated in degree 1), and following classical commutative algebra, one can ask if finitely generated modules over finitely generated tca's are noetherian. In general, this is unknown, but it is known for the algebras A_V.

If $\mathbf{k}$ has characteristic 0, there is an alternative way to define tca's which opens up the possibility of using geometric intuition. The definition makes sense in general but will define different algebraic structures in positive characteristic.

A **polynomial functor** is a functor $F\colon \mathrm{Vec}_{\mathbf{k}} \to \mathrm{Vec}_{\mathbf{k}}$ such that the maps on hom-sets

$$\mathrm{Hom}_{\mathbf{k}}(V, W) \to \mathrm{Hom}_{\mathbf{k}}(F(V), F(W))$$

are given by polynomial functions. We are ignoring certain issues of finiteness for simplicity. An example are symmetric powers, exterior powers, tensor powers. These also have a product given by

$$(F \otimes G)(V) = F(V) \otimes G(V)$$

together with natural isomorphisms $F \otimes G \cong G \otimes F$. So again, we can define commutative algebras and their modules. In characteristic 0, there is an equivalence between polynomial functors and functors $\mathbf{FB} \to \mathrm{Vec}_{\mathbf{k}}$ which is compatible with the products. The functor A_V becomes the polynomial functor $W \mapsto \mathrm{Sym}(W \otimes V)$. This is closer to commutative algebra. It turns out that if we take the colimit as $\dim W \to \infty$, we get an algebra which remembers the entire information of the functor, i.e., $\mathrm{Sym}(\mathbf{k}^\infty \otimes V)$, as long as remember the data of the $\mathbf{GL}_\infty(\mathbf{k})$-action. In particular, when considering what a set of elements generate, we are allowed to use the $\mathbf{GL}_\infty(\mathbf{k})$ action.

The noetherianity question has several variants. Let A be a finitely generated algebra in the category of polynomial functors.

- Are submodules of finitely generated A-modules also finitely generated?
- Are all ideals of A finitely generated?
- Are all ideals of A finitely generated up to radical?

Classically, a positive answer to the second question implies a positive answer to the first question, but no such equivalence (or counterexample) is known in this setting. The last question has a geometric reformulation. Recall that on a vector space over an algebraically closed field, Hilbert's nullstellensatz implies that closed subsets in the Zariski topology are in bijective correspondence with radical ideals. So the last question is equivalent to asking if the "prime spectrum" of A is a noetherian topological space. This is known to be true by recent work of Draisma [Dr].

A particularly interesting example is when F is the symmetric algebra of a sum of symmetric powers:

$$F(V) = \mathrm{Sym}(\mathrm{Sym}^{d_1}(V) \oplus \cdots \oplus \mathrm{Sym}^{d_r}(V)).$$

In that case, points of $\mathrm{Spec}(F(V))$ parametrize generating sets of ideals in $\mathrm{Sym}(V)$ (with degrees $d_1, \ldots, d_r$) and we can consider a consider quotient "space" which is the set of ideals which can be generated by elements of degrees $d_1, \ldots, d_r$.

Let ν be a function that takes as input a homogeneous ideal in a polynomial ring and returns a value in $\mathbf{Z}_{\geq 0} \cup \{\infty\}$. Say it is an ideal invariant if it is unaffected by a linear change of variables, and that it is cone-stable if it is also unaffected by adjoining a variable, i.e., considering the ideal in a polynomial ring with one new variable. An example is the projective dimension of an ideal. Finally, say that ν is degreewise bounded if it only takes on finitely many values when restricted to ideals generated by polynomials of a fixed sequence of degrees (but is independent of the number of variables). The following was shown by Ananyan–Hochster:

Theorem 4.1 (Ananyan–Hochster)**.** *Projective dimension is degreewise bounded.*

Projective dimension has one more property: it is upper semicontinuous in flat families. Using Draisma's theorem and the previous theorem, one can show the following:

Theorem 4.2 (Erman–Sam–Snowden)**.** *If ν is a cone-stable invariant which is upper semicontinuous in flat families, then ν is degreewise bounded.*

The idea is to use Ananyan–Hochster to build a flattening stratification for the universal family on $\mathrm{Sym}^{d_1}(\mathbf{k}^\infty) \oplus \cdots \oplus \mathrm{Sym}^{d_r}(\mathbf{k}^\infty)$ (the space parametrizing tuples of polynomials) which shows that the loci where ν jumps gives a decreasing chain of $\mathbf{GL}$-stable Zariski closed subsets.

References

[Ch] Thomas Church, Homological stability for configuration spaces of manifolds, *Invent. Math.* **188** (2012), no. 2, 465–504, arXiv:1103.2441v3.

[CEF] Thomas Church, Jordan Ellenberg, Benson Farb, FI-modules and stability for representations of symmetric groups, *Duke Math. J.* **164** (2015), no. 9, 1833–1910, arXiv:1204.4533v3.

[CEFN] Thomas Church, Jordan S. Ellenberg, Benson Farb, Rohit Nagpal, FI-modules over Noetherian rings, *Geom. Topol.* **18** (2014), no. 5, 2951–2984, arXiv:1210.1854v2.

[Dr] Jan Draisma, Topological Noetherianity of polynomial functors, arXiv:1705.01419v1.

[DE] Jan Draisma, Rob H. Eggermont, Plücker varieties and higher secants of Sato's Grassmannian, *J. Reine Angew. Math.*, to appear, arXiv:1402.1667v2.

[DrK] Jan Draisma, Jochen Kuttler, Bounded-rank tensors are defined in bounded degree, *Duke Math. J.* **163** (2014), no. 1, 35–63, arXiv:1103.5336v2.

[JR] Rita Jimenez Rolland, Representation stability for the cohomology of the moduli space $\mathscr{M}_g^n$, *Algebr. Geom. Topol.* **11** (2011), no. 5, 3011–3041, arXiv:1106.0947v2.

[PY] Nicholas Proudfoot, Ben Young, Configuration spaces, $\mathrm{FS}^{\mathrm{op}}$-modules, and Kazhdan-Lusztig polynomials of braid matroids, arXiv:1704.04510v2.

[Pu] Andrew Putman, Stability in the homology of congruence subgroups, *Invent. Math.* **202** (2015), no. 3, 987–1027, arXiv:1201.4876v5.

[PS] Andrew Putman, Steven V Sam, Representation stability and finite linear groups, *Duke Math. J.*, to appear, arXiv:1408.3694v3.

[Sa] Steven V Sam, Ideals of bounded rank symmetric tensors are generated in bounded degree, *Invent. Math.* **207** (2017), no. 1, 1–21, arXiv:1510.04904v2.

[SS] Steven V Sam, Andrew Snowden, Gröbner methods for representations of combinatorial categories, *J. Amer. Math. Soc.* **30** (2017), 159–203, arXiv:1409.1670v3.