Twisted homological stability for groups via functor categories

Steven V Sam

(joint work with Andrew Putman)

A sequence of groups and maps $G_1 \to G_2 \to \cdots$ satisfies **homological stability** if, for each $i \geq 0$, the induced map on homology $\mathrm{H}_i(G_n) \to \mathrm{H}_i(G_{n+1})$ is an isomorphism for $n \gg i$. Some sequences of groups that satisfy homological stability (the maps are the usual ones):

- Symmetric groups $G_n = S_n$ (Nakaoka [Nak]);
- For any group Γ, the wreath products $G_n = S_n \ltimes \Gamma^n$ (this seems to have been well-known – it is stated explicitly in [HW, Prop. 1.6]);
- For well-behaved rings R (such as commutative noetherian rings of finite Krull dimension), $G_n = \mathrm{GL}_n(R)$ (van der Kallen [Va]), and
- the symplectic groups $G_n = \mathrm{Sp}_{2n}(R)$ (Mirzaii–van der Kallen [MV]).

More generally, G_n-representations M_n equipped with G_n-equivariant maps $M_n \to M_{n+1}$ satisfy **twisted homological stability** if, for each $i \geq 0$, the induced map $\mathrm{H}_i(G_n; M_n) \to \mathrm{H}_i(G_{n+1}; M_{n+1})$ is an isomorphism for $n \gg i$.

The problem we consider is to determine which kinds of sequences satisfy twisted homological stability. Wahl [W] gave a general setup using the notion of *homogeneous categories* (they are monoidal categories; we omit the definition since we use a special case below). If $(\mathcal{G}, \oplus, 0)$ is a symmetric monoidal groupoid such that $\mathrm{Aut}(0) = \{1\}$ and such that the map $\mathrm{Aut}(A) \to \mathrm{Aut}(A \oplus B)$ given by $f \mapsto f \oplus 1_B$ is injective for all A, B, then there is a minimal homogeneous symmetric monoidal category $\mathcal{UG}$ containing $\mathcal{G}$ as its underlying groupoid [W, 1.4, 1.5].

Corresponding to the previous examples, we give a few cases of $\mathcal{G}$ and $\mathcal{UG}$:

- The groupoid of finite sets under disjoint union gives the category FI, whose objects are finite sets and whose morphisms are injections;
- The groupoid of free Γ-sets under disjoint union gives the category FI_Γ, whose objects are finite sets and whose morphisms are Γ-injections: an injective function $f \colon R \to S$ and a function $\rho \colon R \to \Gamma$; the composition with $(g \colon S \to T, \sigma)$ is given by (gf, τ) where $\tau(x) = \sigma(f(x)) \cdot \rho(x)$);
- The groupoid of finite rank free R-modules under direct sum gives the category $\mathrm{VIC}(R)$, whose objects are finite rank free R-modules and whose morphisms $V \to W$ are pairs of maps $V \to W \to V$ composing to 1_V;
- The groupoid of finite rank free symplectic R-modules under direct sum gives the category $\mathrm{SI}(R)$, whose objects are finite rank free symplectic R-modules and whose morphisms are linear maps preserving the form (and hence must be injective).

The above examples of $\mathcal{UG}$ are in fact complemented categories. A symmetric monoidal category is **complemented** if it satisfies the following properties:

- Every morphism is a monomorphism;
- 0 is an initial object, and so we have canonical maps $V \to V \oplus V'$ and $V' \to V \oplus V'$;

- The map $\mathrm{Hom}(V \oplus V', W) \to \mathrm{Hom}(V, W) \times \mathrm{Hom}(V', W)$ is injective;
- Every subobject $C \subset V$ has a complement, i.e., another subobject $D \subset V$ so that $V \cong C \oplus D$ and where the isomorphism identifies the inclusion $C \subset V$ with the canonical map $C \to C \oplus D$, and similarly for D.

Each one has a **generator** X, i.e., every object is isomorphic to $X^{\oplus n}$.

Fix a commutative ring $\mathbf{k}$. Given a complemented category $\mathcal{C}$ with generator X, and a functor $F \colon \mathcal{C} \to \mathbf{k}\text{-Mod}$, define $\Sigma F \colon \mathcal{C} \to \mathbf{k}\text{-Mod}$ to be the precomposition with the functor $Y \mapsto Y \oplus X$. There is a natural transformation $F \to \Sigma F$, and its kernel and cokernel are denoted $\ker F$ and $\operatorname{coker} F$. We can use this to define the **degree** of a functor:

- If $F = 0$, then its degree is -1;
- If $\ker F$ and $\operatorname{coker} F$ have degree $\leq r - 1$, then F has degree $\leq r$.

Otherwise F has infinite degree. Also, for each n, define a semisimplicial set $W_n(X)$ whose p-simplices are $\mathrm{Hom}(X^{\oplus p+1}, X^{\oplus n})$.

Let $\mathcal{C}$ be a complemented category with generator X. Suppose that there is an integer $k \geq 2$ so that for all $n \geq 1$, $W_n(X)$ is $(n-2)/k$-connected. Then a special case of [W, Theorem 5.6] is that for any functor of finite degree $\leq r$, the map

$$\mathrm{H}_i(\mathrm{Aut}(X^{\oplus n}); F(X^{\oplus n})) \to \mathrm{H}_i(\mathrm{Aut}(X^{\oplus n+1}); F(X^{\oplus n+1}))$$

is an isomorphism when $i \leq (n-r)/k$. Implicitly, we always use the morphisms $X^{\oplus n} \to X^{\oplus n+1}$ as inclusion via the first n factors to define all structure maps. We will say that the functor F satisfies homological stability.

For some purposes, having finite degree is too restrictive of a condition. For example, if $\mathbf{k}$ is a field and F takes finite-dimensional values, then it implies that the function $n \mapsto \dim_{\mathbf{k}} F(X^{\oplus n})$ is a polynomial for $n \gg 0$. A basic property of complemented categories $\mathcal{C}$ with generator X is that for $n \geq r$, the permutation representation $\mathbf{k}[\mathrm{Hom}(X^{\oplus r}, X^{\oplus n})]$ is isomorphic to the induced representation $\mathrm{Ind}_{\mathrm{Aut}(X^{\oplus n-r})}^{\mathrm{Aut}(X^{\oplus n})} \mathbf{k}$. So by Shapiro's lemma, the functor $P_r \colon \mathcal{C} \to \mathbf{k}\text{-Mod}$ defined by $Y \mapsto \mathbf{k}[\mathrm{Hom}(X^{\oplus r}, Y)]$ satisfies homological stability if the same is true for the constant functor, i.e., the groups $\mathrm{Aut}(X^{\oplus n})$ satisfy homological stability. *From now on, we will make this assumption about* $\mathrm{Aut}(X^{\oplus n})$.

By Yoneda's lemma, the set of natural transformations $P_r \to F$ identifies with $F(X^{\oplus r})$, and so the P_r are a set of projective generators for the functor category $[\mathcal{C}, \mathbf{k}\text{-Mod}]$. In particular, any functor F admits a projective resolution of the form

$$\cdots \to \mathbf{P}_d \to \mathbf{P}_{d-1} \to \cdots \to \mathbf{P}_1 \to \mathbf{P}_0 \to F \to 0$$

where $\mathbf{P}_d$ is a direct sum of P_r. If we assume that each $\mathbf{P}_d$ has a decomposition as $\bigoplus_{r \leq D} P_r$ (D depending on d), then $\mathbf{P}_d$ also satisfies homological stability. Note that for each n, there is a spectral sequence

$$\mathrm{E}^1_{p,q}(n) = \mathrm{H}_p(\mathrm{Aut}(X^{\oplus n}); \mathbf{P}_q(X^{\oplus n})) \Longrightarrow \mathrm{H}_{p+q}(\mathrm{Aut}(X^{\oplus n}); F(X^{\oplus n})),$$

and spectral sequence morphisms $\mathrm{E}^1_{*,*}(n) \to \mathrm{E}^1_{*,*}(n+1)$. So with the assumption on $\mathbf{P}_d$ above, we see that for a given diagonal $p+q$, the map of spectral sequences on all relevant terms to calculate H_{p+q} is an isomorphism for $n \gg 0$, and hence F satisfies homological stability.

This motivates the following definitions. Say that F is **finitely generated** if it is a quotient of a finite direct sum $P_{r_1} \oplus \cdots \oplus P_{r_n}$, and say that F is **noetherian** if every subfunctor of F is finitely generated; $[\mathcal{C}, \mathbf{k}\text{-Mod}]$ is (locally) noetherian if every finitely generated functor is noetherian. This implies that $\mathbf{k}$ is a noetherian ring. If $[\mathcal{C}, \mathbf{k}\text{-Mod}]$ is noetherian, then every finitely generated functor has a projective resolution where each $\mathbf{P}_d$ is a finite direct sum of P_r, and hence satisfies homological stability. This is formalized in [PS, Theorem 4.2].

Some examples of when $[\mathcal{C}, \mathbf{k}\text{-Mod}]$ is noetherian (take $\mathbf{k}$ to be any noetherian ring) corresponding to the running examples:

- FI (Church–Ellenberg–Farb–Nagpal [CEFN, Theorem A])
- When Γ is virtually polycyclic, FI_Γ (Sam–Snowden [SS, Cor. 1.2.2])
- When R is a finite commutative ring, $\mathrm{VIC}(R)$ and $\mathrm{SI}(R)$ (Putman–Sam [PS, Theorems C, D])

Finally, a word about cohomology versus homology. Let $\mathbf{k}$ be a field of characteristic $p > 0$ and let $\mathfrak{h}(n) = \{(x_1, \dots, x_n) \in \mathbf{k}^n \mid \sum_i x_i = 0\}$ be the reflection representation of S_n; note that $\{1, \dots, n\} \mapsto \mathfrak{h}(n)$ defines a finitely generated functor $\mathrm{FI} \to \mathbf{k}\text{-Mod}$. For $n \geq 3$ we have $\mathrm{H}_0(S_n; \mathfrak{h}(n)) = 0$, whereas

$$\mathrm{H}^0(S_n; \mathfrak{h}(n)) = \mathfrak{h}^{S_n} = \begin{cases} 0 & \text{if } p \nmid n \\ \mathbf{k} & \text{if } p \mid n \end{cases}.$$

In fact, this periodic behavior is typical: Nagpal shows that if F is a finitely generated FI-module, then for each i, the function $n \mapsto \dim_{\mathbf{k}} \mathrm{H}^i(S_n; F(\{1, \dots, n\}))$ is a periodic function of n for $n \gg 0$ with period a power of p [Nag, Theorem D].

References

[CEFN] Thomas Church, Jordan S. Ellenberg, Benson Farb, Rohit Nagpal, FI-modules over Noetherian rings, *Geom. Top.,* to appear, arXiv:1210.1854v2.

[HW] Allen Hatcher, Nathalie Wahl, Stabilization for mapping class groups of 3-manifolds, *Duke Math. J.* **155** (2010), no. 2, 205–269, arXiv:0709.2173v4.

[MV] B. Mirzaii, W. van der Kallen, Homology stability for unitary groups, *Doc. Math.* **7** (2002), 143–166, arXiv:math/0111117v1.

[Nag] Rohit Nagpal, FI-modules and the cohomology of modular representations of symmetric groups, arXiv:1505.04294v1.

[Nak] Minoru Nakaoka, Decomposition theorem for homology groups of symmetric groups, *Ann. of Math. (2)* **71** (1960), 16–42.

[PS] Andrew Putman, Steven V Sam, Representation stability and finite linear groups, arXiv:1408.3694v2.

[SS] Steven V Sam, Andrew Snowden, Representations of categories of G-maps, arXiv:1410.6054v2.

[Va] Wilberd van der Kallen, Homology stability for linear groups, *Invent. Math.* **60** (1980), no. 3, 269–295.

[W] Nathalie Wahl, Homological stability for automorphism groups, arXiv:1409.3541v1.