"Gröbner bases for twisted commutative algebras"
University of Michigan combinatorics seminar, April 4, 2014
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1. Introduction: some motivating results

General theme: "Finding finiteness in new places"
(Sample results:)

Theorem 1.1 (Church–Ellenberg–Farb–Nagpal)**.** *Let X be a compact connected orientable manifold. Define $C_n(X) = \{(p_1,\dots,p_n) \in X^n \mid p_i \neq p_j\}$. For any $i \ge 0$ and field $\mathbf{k}$, the function*

$$n \mapsto \dim_{\mathbf{k}} \mathrm{H}^i(C_n(M);\mathbf{k})$$

agrees with a polynomial for $n \gg 0$. (Will realize as Hilbert polynomial.)

(Applications of comm. alg. ideas to other areas of math)

2. FI-modules

(To explain, need some formalism.)
Define category FI_d:

- objects are finite sets
- morphism $S \to T$ is an injection $f: S \to T$ and a function $T \setminus f(S) \to \{1,\dots,d\}$.

Definition 2.1. FI_d**-module** M is a functor FI_d to $\mathbf{k}$-vector spaces. This is an Abelian category (so usual notions of algebra make sense)

Morphism $\varphi: S \to T$ gives linear map $M(\varphi): M(S) \to M(T)$. M is **finitely generated** if there are $m_1 \in M(S_1),\dots,m_r \in M(S_r)$ so that every element is

$$c_1 M(\varphi_1)(m_1) + \cdots + c_r M(\varphi_r)(m_r) \qquad (c_i \in \mathbf{k})$$

Hilbert function of M is $h_M(n) := \dim_{\mathbf{k}} M(\{1,\dots,n\})$.
(Note that $\dim_{\mathbf{k}} M(S) = \dim_{\mathbf{k}} M(S')$ whenever $|S| = |S'|$) □

Proposition 2.2. *For $d = 1$, if M is f.g., then $h_M(n)$ is a polynomial function for $n \gg 0$.*

Example 2.3. S finite set, define $C_S(X)$ to be space of injective maps $S \to X$. Given $S \subseteq T$, get forgetful map $C_T(X) \to C_S(X)$ and hence $\mathrm{H}^i(C_S(X);\mathbf{k}) \to \mathrm{H}^i(C_T(X);\mathbf{k})$ which forms FI_1-module. Note $C_S(X) \cong C_{|S|}(X)$. (Is it f.g.?) □

Definition 2.4. Submodule $N \subseteq M$ is collection $N(S) \subseteq M(S)$ closed under all $M(\varphi)$. □

To approach conf. space, can use spectral sequence for inclusion $C_n(X) \subset X^n$. The E_2 page was described by Totaro and has structure of f.g. FI_1-modules. Going to E_∞ page requires taking submodules and quotient modules.

Question: If M is f.g. FI_d-module, is the same true for $N \subseteq M$?
Yes. (S.–Snowden; CEFN for $d = 1$)

For each n, define FI_d-module P_n by $P_n(S) = \mathbf{k}[\mathrm{Hom}_{FI_d}(\{1,\dots,n\}, S)]$.

Lemma 2.5. *Being f.g. is equivalent to being a quotient of a finite direct sum $P_{n_1} \oplus \cdots \oplus P_{n_r}$.*

(So just need to answer question for the modules P_n)
Note that $h_{P_r}(n) = \binom{n}{r} r! = n(n-1)\cdots(n-r+1)$.

3. Gröbner methods

Idea: monomials easier than polynomials. (Usual reduction is taking initial terms)
Want to use leading terms of elements in FI_d. One desirable property:

$$(*) \qquad \text{if } f < f' \text{ (morphisms) then } gf < gf' \text{ for all morphisms } g$$

But consider $f, f' : [2] \to [3]$ with $f(1) = 1, f(2) = 2$ and $f'(1) = 2, f'(2) = 1$ and $g, g' : [3] \to [4]$ with $g(1) = 1, g(2) = 2, g(3) = 3$ and $g'(1) = 2, g'(2) = 1, g'(3) = 3$. Then $gf = g'f'$ and $g'f = gf'$ so both $f > f'$ and $f' > f$ leads to a contradiction.

Fix: Remove the symmetric group actions.
Define category OI_d:

- objects are ordered finite sets
- morphism $S \to T$ is increasing injection $f : S \to T$ and function $T \setminus f(S) \to [d]$.

Define OI_d-modules, submodules, f.g. modules in same way. P_n are replaced by Q_n: $Q_n(S) = \mathbf{k}[\mathrm{Hom}_{OI_d}(\{1,\dots,n\}, S)]$.

Have forgetful functor $\Phi : OI_d \to FI_d$
So given FI_d-module M, have pullback $\Phi^*(M)$ which is an OI_d-module.
Φ^* is exact, so $M \subseteq N$ implies $\Phi^*(M) \subseteq \Phi^*(N)$.

Proposition 3.1. *M is f.g. FI_d-module if and only if $\Phi^*(M)$ is a f.g. OI_d-module.*

Proof. M f.g. implies M is quotient of $P_{n_1} \oplus \cdots \oplus P_{n_r}$.
So $\Phi^*(M)$ is quotient of $\Phi^*(P_{n_1}) \oplus \cdots \oplus \Phi^*(P_{n_r})$. Now use that $\Phi^*(P_n) \cong Q_n^{\oplus n!}$.
Other direction: FI_d has more operators than OI_d. ☐

(So it suffices to prove Q_n are Noetherian)

Definition 3.2. A **monomial** in Q_n is basis vector given by morphism $[n] \to S$.
OI_d-morphism $\{1,\dots,n\} \to \{1,\dots,r\}$ is identified with word in $\{*, 1, \dots, d\}$ of length r where $*$ appears n times.
Monomial submodule of Q_n is one generated by monomials. ☐

Proposition 3.3. *Monomial submodules of Q_n are f.g.*

Proof. Submodule in Q_n generated by word w (basis element) is all other words that contain w as a subword. Get partial order on words: **Higman's lemma** implies there are no infinite antichains. For any monomial submodule, build sequence of monomials $x_1, x_2, \dots$ where x_i is minimal degree such that it is not generated by $x_1, \dots, x_{i-1}$. This must be finite! ☐

Ordering words lexicographically gives total ordering with crucial property (*) and allows us to define initial terms

$$in_<(f) = \max\{m \mid m \text{ monomial with nonzero coeff in } f\}.$$

and initial submodules

$$in_<(M) = \mathbf{k}\{in_<(f) \mid f \in M\}.$$

Standard arguments imply submodule N f.g. iff its initial submodule is f.g.:

Lemma 3.4. *If $N \subseteq M$ and $in_<(N) = in_<(M)$, then $N = M$.*

Proof. If not, pick $f \in M \setminus N$ with $in_<(f)$ minimal. There exists $g \in N$ with $in_<(g) = in_<(f)$. Then $f - g \in M \setminus N$ and has smaller initial term. □

Other uses of Gröbner methods:

- Dotsenko–Khoroshkin: shuffle operads
- Aschenbrenner–Hillar, Hillar–Sullivant: monoidal equivariant ideals in poly. rings

4. Further directions

- Analogy with commutative algebra further developed in S.–Snowden, "GL-equivariant modules over polynomial rings in infinitely many variables", arXiv:1206.2233.
- Can generalize by replacing FI by linear version $VI(\mathbf{F}_q)$ or even $VI(R)$ — applications to cohomology of congruence subgroups of arithmetic groups, mapping class groups, etc.
- What is behavior of Hilbert function of f.g. FI_d-modules? Know that it is bounded by $p(n)d^n$ for polynomial p. Examples of FI_d-modules? one example: configuration spaces of disconnected manifolds (with d components).

  Other approximate candidates: cohomology of (Fulton–MacPherson) compactifications of conf. spaces; cohomology of Deligne–Mumford compactification of $\mathscr{M}_{g,n}$
- Define category $FI^{(2)}$:
  - Objects are finite sets
  - morphism $S \to T$ is an injection $f : S \to T$ and a perfect matching on $T \setminus f(S)$.

  Do $FI^{(2)}$-modules have Noetherian property? This fails for $OI^{(2)}$-modules!

  When $\mathbf{Q} \subseteq \mathbf{k}$, $FI^{(2)}$-modules is a model for $\mathbf{O}(\infty)$-modules where $\mathbf{O}(\infty) = \bigcup_n \mathbf{O}(n)$.
- Use Kruskal tree theorem? (Homeomorphic embeddings of labeled rooted trees)
- Can use category of surjections to prove Δ-modules (in the sense of Snowden) are Noetherian (strengthens his previous results).