Representations of Lie superalgebras and determinantal varieties
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1. $\mathbf{Z}$-gradings on classical Lie superalgebras

Let $(R, \mathfrak{m}, \mathbf{k})$ be a (graded) local ring.
$\mathrm{Ext}^\bullet_R(\mathbf{k},\mathbf{k})$ is the enveloping algebra of a (positively) graded Lie algebra $\pi(R)_\bullet$.
If M is a (graded) R-module, then $\mathrm{Ext}^\bullet_R(M,\mathbf{k})$ is a left-module over $\pi(R)_*$.
If a Lie algebra $\mathfrak{g}$ acts on R, form $\widetilde{\pi}(R)_\bullet = \mathfrak{g} \oplus \pi(R)_\bullet$ with $\deg(\mathfrak{g}) = 0$. $\widetilde{\pi}(R)_*$ acts on M if there is compatible $\mathfrak{g}$-action.

Examples:

(1) $R = \mathrm{Sym}(E \otimes F)$ and $\mathfrak{g} = \mathfrak{gl}(E) \times \mathfrak{gl}(F)$.
$\pi(R) = E^* \otimes F^*$ (odd degree, with trivial bracket)
Note: $\mathfrak{gl}(E|F) = (E \otimes F) \oplus \widetilde{\pi}(R)$ ($\mathbf{Z}$-grading of general linear Lie superalgebra)

(2) $R = \mathrm{Sym}(\mathrm{Sym}^2 E)$ and $\mathfrak{g} = \mathfrak{gl}(E)$
$\pi(R) = \mathrm{Sym}^2 E^*$ (odd degree, with trivial bracket)
Note: $\mathfrak{pe}(E) = \bigwedge^2(E) \oplus \widetilde{\pi}(R)$ ($\mathbf{Z}$-grading of periplectic Lie superalgebra)

(3) Equip V with symplectic form and pick E with $2\dim(E) \le \dim(V)$.
$R = \mathrm{Sym}(E \otimes V)/(\bigwedge^2 E)$ (ideal of positive degree $\mathfrak{sp}(V)$-invariants: Given $\varphi \colon E^* \to V$, take entries of composition $E^* \xrightarrow{\varphi} V \cong V^* \xrightarrow{\varphi^*} E$) and $\mathfrak{g} = \mathfrak{gl}(E) \otimes \mathfrak{sp}(V)$.
$\pi(R) = (E^* \otimes V^*) \oplus (\bigwedge^2 E^*)$
Note: Put orthogonal form on $E \oplus E^*$. Then $\mathfrak{osp}(E \oplus E^* | V) = (\bigwedge^2 E) \oplus (E \otimes V) \oplus \widetilde{\pi}(R)$. ($\mathbf{Z}$-grading on orthosymplectic Lie superalgebra)

Question: When does action of $\widetilde{\pi}(R)$ on $\mathrm{Ext}^\bullet_R(M,\mathbf{k})$ extend to whole Lie superalgebra? (possibly after twisting by character of $\mathfrak{g}$)

Remarks: second case is asymmetric, third case uses longer grading; each algebra R is Koszul

In each case, R is functions on a space of matrices. Let M be an ideal generated by $r \times r$ minors (pick a size $r > 1$ that you like).
M has internal grading, so $\mathrm{Ext}^\bullet_R(M,\mathbf{k})$ is a direct sum of its linear strands since $\mathrm{Ext}^\bullet_R(\mathbf{k},\mathbf{k})$ acts linearly.
Assume $\mathbf{k}$ has char. 0.

Theorem 1.1 (Akin–Weyman)**.** *In case 1, each linear strand of* $\mathrm{Ext}^\bullet_R(M,\mathbf{k})$ *is a h.w. irreducible representation of* $\mathfrak{gl}(E|F)$.

(bottom representation of sth linear strand is $\mathbf{S}_{s^{s+r-1}} E^* \otimes \mathbf{S}_{s^{s+r-1}} F^*$)

Theorem 1.2 (Sam)**.** *In case 2, each linear strand of* $\mathrm{Ext}^\bullet_R(M,\mathbf{k})$ *is a h.w. irreducible representation of* $\mathfrak{pe}(E)$.

(bottom representation of sth linear strand is $\mathbf{S}_{(2s)^{2s+r-2}}E^*$)
(orthosymplectic case explained later)

Proof idea: Pragacz–Weyman construct linear strands of minimal free resolution in case 1 using tensor operations on basic representation $F \otimes R(-1) \to E^* \otimes R$

Approach 2: Use linear acyclicity of linear strands to build a map from appropriate Kac module. Show the map is surjective and kernel closed under Lie algebra action

2. Example: Gulliksen–Negard complex

Case 1: let $\dim(E) = \dim(F) = 3$ and let M be 2×2 minors. Minimal free resolution of M:

$$\begin{array}{ccccc} \bigwedge^2(E) \otimes \bigwedge^2(F) \otimes R(-2) & \longleftarrow & \begin{array}{c} \det(E) \otimes \det(F) \otimes \mathfrak{sl}(F) \\ \det(E) \otimes \mathfrak{sl}(E) \otimes \det(F) \end{array} \otimes R(-3) & \longleftarrow & \det(E) \otimes E \otimes \det(F) \otimes F \otimes R(-4) \\ & & & & \uparrow \\ & & & & (\det E)^2 \otimes (\det F)^2 \otimes R(-6) \end{array}$$

Apply $\mathrm{Hom}_R(-,\mathbf{k})^*$ to get $\mathrm{Ext}_R^\bullet(M,\mathbf{k})$.

Top row can be built as follows: take tensor product of 2-term complexes $E \otimes R(-2) \to F^* \otimes R(-1)$ and $F \otimes R(-2) \to E^* \otimes R(-1)$ (vector and covector representations) and take homology in middle (and twist by $\det E \otimes \det F$ which is unimportant):

$$\begin{array}{ccccc} & & R(-3) & & \\ & & \uparrow & & \\ E^* \otimes F^* \otimes R(-2) & \longleftarrow & \begin{array}{c} E \otimes E^* \\ F \otimes F^* \end{array} \otimes R(-3) & \longleftarrow & E \otimes F \otimes R(-4) \\ & & \uparrow & & \\ & & R(-3) & & \end{array}$$

(explain out loud that vector and covector representations have obvious actions and that tensoring and taking homology preserves this property; or appeal to Koszul duality)

3. Commutative analogy

Let $W = \mathrm{Sym}(E \otimes F)/I_r$ where I_r is ideal of $r \times r$ minors. Then W has action of Lie algebra

$$(\mathfrak{gl}(E) \times \mathfrak{gl}(F)) \oplus (E \otimes F).$$

This is part of a $\mathbf{Z}$-grading for $\mathfrak{gl}(E \oplus F)$, action can be extended after suitable twist of center; W is a h.w. (weight $-r\omega_k$ where $k = \min(\dim E, \dim F)$) irreducible representation.

4. Orthosymplectic results

Theorem 4.1 (Sam). *Let I be the ideal of maximal minors of R. Then $\mathrm{Ext}_R^\bullet(I,\mathbf{k})$ is a l.w. irreducible representation of $\mathfrak{osp}(E \oplus E^*|V)$. In particular, it has a linear resolution.*

(bottom representation is $\bigwedge^e E \otimes \bigwedge^e_0 V$ of $\mathfrak{gl}(E) \times \mathfrak{sp}(V)$)

(Can handle more general class of modules like powers of I, but more complicated to state carefully)

Proof. (Idea: assuming statement is true, the Koszul dual of I is a single module with a linear resolution and we guess where it comes from.)

- Start with $\mathfrak{osp}(E \oplus E^* | V)$-module N which is proposed Koszul dual and which appears in a super Howe dual pair.

  (In case someone asks: let $U = \mathbf{C}^2$ with alternating form — strangely enough, if want I^k, need to use $U = \mathbf{C}^{2k}$ — pick Lagrangian $F \subset V$, and take oscillator representation $\mathrm{Sym}(U \otimes (E^* | F))$ of $\mathfrak{spo}(U \otimes (E \oplus E^* | V)) \supset \mathbf{Sp}(U) \times \mathfrak{osp}(E \oplus E^* | V)$)

  Use this and Hochschild–Serre spectral sequence (and Kostant–BWB to calculate homology of super polynomial functors) coming from natural filtration on $\pi(R)$ to show it has linear resolution over $\pi(R)$.
- So know presentation of $N^!$ (at least know generators and relations as representation of $\mathfrak{gl}(E) \times \mathfrak{sp}(V)$). But in this case, the representations determine the map uniquely up to scalar, so just verify it matches generators and relations of I. $\square$