Growth of ideals in subword posets
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## 1. Combinatorics

Let $X$ be a finite set.
$X^{\star}=\left\{w=w_{1} w_{2} \cdots w_{n} \mid w_{i} \in X\right\}$ words on $X$.
Poset structures $\leq$ on $X^{\star}$ :
(1) $w \leq_{I} w^{\prime}$ if $w$ is a subword of $w^{\prime}$, i.e., there exists a function $f:[\ell(w)] \rightarrow\left[\ell\left(w^{\prime}\right)\right]$ with $f(1)<f(2)<\cdots<f(\ell(w))$ and $w_{i}=w_{f(i)}^{\prime}$.
(2) $w \leq_{I I} w^{\prime}$ if $w \leq_{I} w^{\prime}$ and furthermore, we can choose $f$ as above such that for all $j$, there exists $f(i) \leq j$ such that $w_{f(i)}^{\prime}=w_{j}^{\prime}$.
Example 1.1. $X=[2] .112 \leq_{I} 1212$ but $112 \not \not_{I I} 1212$.
$I \subseteq X^{\star}$ is an ideal if $x \in I$ and $y \geq x$ implies $y \in I$.
Hilbert series of $I$ :

$$
\mathrm{H}_{I}(t)=\sum_{w \in I} t^{\ell(w)}
$$

Proposition 1.2. In both cases, $\mathrm{H}_{I}(t)$ is a rational function $f(t) / g(t)$ where $g(t)=\prod_{i}\left(1-a_{i} t\right)$ and $a_{i} \in\{1,2, \ldots,|X|\}$.

Proof idea:

- Generating function of regular language is rational: translate the elements in an ideal into walks in directed graph and use transfer-matrix method (roots of denominator are inverses of eigenvalues of adjacency matrix)
- For principal ideal $I_{w}=\left\{w^{\prime} \geq w\right\}$, use graph which "tracks progress."

Example: case 1, $X=[3]$ and $w=1213$ :


The labeled walks from $\varnothing$ to 1213 are exactly those words that contain 1213 as a subword under $\leq_{I}$.

For case 2, we introduce a sink "bad" which shows the difference between the two definitions:


Still, labeled walks from $\varnothing$ to 1213 give all words that contain 1213 as a subword under $\leq_{I I}$.

Eigenvalues are integers of desired form.
For finite unions of principal ideals, can use similar ideas (but notation is more cumbersome).

- $X^{\star}$ has no infinite antichains, so every ideal is a finite union of principal ideals.

Case 1: Higman's lemma
Case 2: we prove

## 2. Questions

Poset $\Pi=\bigcup_{n \geq 0} S_{n}$ ( $S_{n}$ is $n$th symmetric group) of permutations: $\sigma \leq \sigma^{\prime}$ if $\sigma$ is a subword of $\sigma^{\prime}$ (when rewritten in relative order). This is definition used in pattern avoidance ( $\sigma^{\prime}$ contains $\sigma$ as a subpattern).

Poset $\mathscr{M} \subset \Pi$ of perfect matchings (i.e., fixed-point free involutions).
What can be said about Hilbert series of ideals in these posets? There are infinite antichains, so might want to restrict to the finitely generated ones (i.e., union of finitely many principal ideals). Might expect them to be D-finite, but should have more refined statements.

## 3. Algebra

Motivation: (algebraic structures for) infinite-dimensional combinatorial commutative algebra (Segre, Veronese varieties, etc.)

A surj-module is graded vector space $M=\bigoplus_{n \geq 0} M_{n}$ such that for every surjective function $f:[n] \rightarrow[m]$ have operator $M(f): M_{m} \rightarrow M_{n}$ (opposite!) and $M(f \circ g)=M(g) \circ M(f)$.

Submodule is graded subspace closed under all operations.
$M$ is finitely generated if there exists finite $S \subset M$ such that smallest submodule containing $S$ is $M$.

Hilbert series of $M$ :

$$
\mathrm{H}_{M}(t)=\sum_{n \geq 0} \operatorname{dim}_{\mathbf{k}}\left(M_{n}\right) t^{n}
$$

Proposition 3.1. If $M$ is finitely generated in degree $\leq D$, then $\mathrm{H}_{M}(t)$ is a rational function $f(t) / g(t)$ where $g(t)=\prod_{i}\left(1-a_{i} t\right)$ and $a_{i} \in\{1,2, \ldots, D\}$.

Proof idea:

- Define Gröbner basis theory for surj-modules to reduce to "monomial modules"
- Monomial modules are identified with ideals in ( $X^{\star}, \leq_{I I}$ ) with $|X| \leq D$

Remark 3.2. - Analogous theory for $\left(X^{\star}, \leq_{I}\right)$ and (decorated) injective functions

- $q$-analogues of surj-modules with sets replaced by $\mathbf{F}_{q}$-vector spaces (related to study of unstable modules over Steenrod algebra)

