

## 1. Motivation

Two main motivations:

The first is *representation stability*: a generalization of homological stability in the presence of group actions which was introduced by Church and Farb. For symmetric group actions, this is formalized in the notion of *FI-modules*, introduced by Church, Ellenberg, and Farb. FI-modules turn out to be modules over a twisted commutative algebra (the free tca generated by 1 element of degree 1). The study of this tca reveals deeper algebraic properties of FI-modules (we will spend a lot of time on this tca). The study of general tca's is relevant for generalized versions of FI-modules.

The second motivation comes from studying equivariant constructions in commutative algebra. We will not discuss this aspect. Some examples are the *pure free resolutions* constructed by Eisenbud, Fløystad, and Weyman. Another is the theory of  $\Delta$ -modules, introduced by Snowden, and used in the study of the structural property of syzygies of Segre embeddings of projective space and related varieties.

## 2. Background

Reference: [arXiv:1209.5122](https://arxiv.org/abs/1209.5122)

**2.1. Definitions.** Let  $\mathbf{FB}$  be the category of finite sets and bijections.

Let  $\mathbf{k}$  be a commutative ring and  $\mathbf{Mod}_{\mathbf{k}}$  the category of  $\mathbf{k}$ -modules.

Set  $\mathcal{V}_{\mathbf{k}} := \mathbf{Fun}(\mathbf{FB}, \mathbf{Mod}_{\mathbf{k}})$ . Objects are just sequences of representations of symmetric groups.

Given a  $\mathbf{k}$ -module  $M$ , define  $M[i] \in \mathbf{Fun}(\mathbf{FB}, \mathbf{Mod}_{\mathbf{k}})$  by

$$S \mapsto \begin{cases} M & \text{if } |S| = i \\ 0 & \text{if } |S| \neq i \end{cases}$$

and all morphisms act by the identity.

Put a monoidal structure on  $\mathcal{V}_{\mathbf{k}}$ :

$$(V \otimes W)(S) = \bigoplus_{S=S_1 \sqcup S_2} V(S_1) \otimes_{\mathbf{k}} W(S_2).$$

This has a symmetry  $\tau$  given by interchanging factors. The unit is the constant functor  $\mathbf{k}[0]$ .

So we can define tensor powers  $V^{\otimes n}$ , and  $\tau$  allows one to define symmetric powers  $\mathrm{Sym}^n(V)$  and exterior powers  $\bigwedge^n(V)$  as quotients of  $V^{\otimes n}$ .

**Example 2.1.**

$$U[1]^{\otimes n}(S) = \begin{cases} \bigoplus_{\sigma \in \text{Aut}(S)} U^{\otimes S} \cong (U^{\otimes n})^{\oplus n!} & \text{if } |S| = n \\ 0 & \text{else} \end{cases}$$

$$\text{Sym}^n(U[1])(S) = \begin{cases} U^{\otimes S} & \text{if } |S| = n \\ 0 & \text{else} \end{cases}$$

The action of  $\text{Aut}(S)$  on  $U^{\otimes S}$  is by permuting tensor factors. □

**Definition 2.2.** A **twisted commutative algebra (tca)** (over  $\mathbf{k}$ ) is a commutative, associative, unital algebra in  $(\mathcal{V}_{\mathbf{k}}, \otimes, \tau)$ .

A morphism of tca's is a natural transformation  $A \rightarrow B$  that intertwines multiplication. □

**Remark 2.3.** The terminology is due to M. Barratt, “Twisted Lie algebras”. □

Explicitly, a tca is a functor  $A: \mathbf{FB} \rightarrow \text{Mod}_{\mathbf{k}}$  with maps

$$\mu: A \otimes A \rightarrow A, \quad \mathbf{k}[0] \rightarrow A$$

such that the diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu_{23}} & A \otimes A \\ \mu_{12} \downarrow & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \quad \begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ \tau \downarrow & \nearrow \mu & \\ A \otimes A & & \end{array}$$

(and unital condition).

**Definition 2.4.** Given a tca  $A$ , an  $A$ -module is a functor  $M: \mathbf{FB} \rightarrow \text{Mod}_{\mathbf{k}}$  together with a map  $\mu: A \otimes M \rightarrow M$  such that the diagram

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mu_{23}} & A \otimes M \\ \mu_{12} \downarrow & & \downarrow \mu \\ A \otimes M & \xrightarrow{\mu} & M \end{array}$$

commutes.

$A$ -modules form an abelian category where the morphisms are natural transformations that intertwine multiplication.

Given  $V: \mathbf{FB} \rightarrow \text{Mod}_{\mathbf{k}}$ ,  $A \otimes V$  is naturally an  $A$ -module. □

**Definition 2.5.** A functor  $V: \mathbf{FB} \rightarrow \text{Mod}_{\mathbf{k}}$  is **finitely generated** if  $V(S) = 0$  for all but finitely many values of  $|S|$  and each  $V(S)$  is a finitely generated  $\mathbf{k}$ -module.

A tca  $A$  is **finitely generated** (in degree  $\leq d$ ) if it is a quotient of  $\text{Sym}(V)$  for finitely generated  $V$  (such that  $V(S) = 0$  for  $|S| > d$ ).

An  $A$ -module  $M$  is **finitely generated** if it is a quotient of  $A \otimes V$  for finitely generated  $V$ .

$M$  is **noetherian** if every submodule is finitely generated.

$A$  is **noetherian** if every finitely generated  $A$ -module is noetherian. □

**Example 2.6.**  $\text{Sym}(U[1])(S) = U^{\otimes S}$  and multiplication is given by concatenation:

$$\begin{array}{ccc} (\text{Sym}(U[1]) \otimes \text{Sym}(U[1]))(S) & \longrightarrow & \text{Sym}(U[1])(S) \\ \parallel & & \parallel \\ \bigoplus_{S=S_1 \sqcup S_2} U^{\otimes S_1} \otimes U^{\otimes S_2} & \longrightarrow & U^{\otimes S} \end{array}$$

Note: it is not exactly meaningful to speak of the product of two elements, but it is meaningful to ask about the set of elements generated by an element.  $\square$

**Theorem 2.7.** *Let  $\mathbf{k}$  be noetherian. A tca  $A$  finitely generated in degree  $\leq 1$  is noetherian.*

(We will not prove this now)

**Question 2.8.** *Let  $\mathbf{k}$  be noetherian and  $A$  be a finitely generated tca. Is  $A$  noetherian?*

**2.2. Alternative descriptions.** Let  $A_U = \text{Sym}(U[1])$  with  $U \cong \mathbf{k}^d$ . Pick a basis  $x_1, \dots, x_d$  for  $U$ . Define a category  $\mathbf{FI}_d$  as follows: the objects are finite sets and a morphism  $S \rightarrow T$  is an injective map  $f: S \rightarrow T$  together with a function  $g: T \setminus f(S) \rightarrow \{1, \dots, d\}$ .

An  **$\mathbf{FI}_d$ -module** is a functor  $\mathbf{FI}_d \rightarrow \text{Mod}_{\mathbf{k}}$ .

**Proposition 2.9.** *The category of  $\mathbf{FI}_d$ -modules is equivalent to the category of  $A_U$ -modules.*

*Proof sketch.* Let  $M$  be an  $A_U$ -module. Define  $N: \mathbf{FI}_d \rightarrow \text{Mod}_{\mathbf{k}}$  by  $N(S) = M(S)$ .

Given  $f: S \rightarrow T$  and  $g: T \setminus f(S) \rightarrow \{1, \dots, d\}$ , consider the multiplication map

$$\begin{array}{ccc} (A_U \otimes M)(T) & \longrightarrow & M(T) \\ \parallel & & \parallel \\ \bigoplus_{T=T_1 \sqcup T_2} U^{\otimes T_1} \otimes M(T_2) & \longrightarrow & M(T) \end{array}$$

Set  $T_2 = f(S)$ ; then  $g$  picks out a basis element  $v \in U^{\otimes T_1}$  so define  $N_{(f,g)}$  to be the restriction to  $v \otimes M(T_2) \rightarrow M(T)$ .  $\square$

Special case:  $\text{Sym}(\mathbf{k}[1])$ -modules are equivalent to functors on the category  $\mathbf{FI}$  of finite sets and injections.

**2.3. Polynomial functors.** Now assume that  $\mathbf{k}$  is an infinite field. Write  $\text{Vec}_{\mathbf{k}}$  for the category of finite dimensional vector spaces over  $\mathbf{k}$ .

**Definition 2.10.** A functor  $F: \text{Vec}_{\mathbf{k}} \rightarrow \text{Vec}_{\mathbf{k}}$  is **polynomial** if for all  $V, V'$ , the map

$$\text{Hom}_{\text{Vec}_{\mathbf{k}}}(V, V') \rightarrow \text{Hom}_{\text{Vec}_{\mathbf{k}}}(F(V), F(V'))$$

is defined by polynomial functions. Let  $\text{Pol}_{\mathbf{k}}$  be the category of polynomial functors.  $\square$

$\text{Pol}_{\mathbf{k}}$  has a monoidal structure:

$$(F \otimes F')(V) = F(V) \otimes_{\mathbf{k}} F'(V)$$

and a symmetry  $\tau$  which interchanges the factors.

Note that a polynomial functor is naturally a direct sum of its homogeneous parts, so we have  $\text{Pol}_{\mathbf{k}} = \bigoplus_{d \geq 0} \text{Pol}_{\mathbf{k}, d}$ .

Define a functor  $\Phi: \text{Fun}(\mathbf{FB}, \text{Vec}_{\mathbf{k}}) \rightarrow \text{Pol}_{\mathbf{k}}$  by  $V \mapsto \Phi_V$  where  $\Phi_V$  is defined by:

$$\Phi_V(W) = \bigoplus_{n \geq 0} (V(\{1, \dots, n\}) \otimes_{\mathbf{k}} W^{\otimes n})_{\Sigma_n}$$

Here  $\Sigma_n$  is the symmetric group, which acts on  $W^{\otimes n}$  by permuting factors, and the subscript denotes coinvariants, i.e., tensoring with the trivial representation.

Both categories  $\text{Vec}_{\mathbf{k}}$  and  $\text{Pol}_{\mathbf{k}}$  have “locally finite” subcategories: in  $\text{Vec}_{\mathbf{k}}$ , consider those functors such that  $\dim V(S) < \infty$  for all  $S$ , and for  $\text{Pol}_{\mathbf{k}}$  consider those functors so that the degree  $d$  piece is a finite length functor. We will restrict attention to these from now on.

**Lemma 2.11.**  $\Phi$  is a tensor functor.

*Proof.* Pick  $V, V' \in \text{Fun}(\mathbf{FB}, \text{Vec}_{\mathbf{k}})$ . Write  $V_i = V(\{1, \dots, i\})$ .

$$\begin{aligned} (\Phi_V \otimes \Phi_{V'})(W) &= \Phi_V(W) \otimes \Phi_{V'}(W) \\ &= \left( \bigoplus_{n \geq 0} (V_n \otimes W^{\otimes n})_{\Sigma_n} \right) \otimes \left( \bigoplus_{n \geq 0} (V'_n \otimes W^{\otimes n})_{\Sigma_n} \right) \\ &= \bigoplus_{n \geq 0} \bigoplus_{i+j=n} (V_i \otimes W^{\otimes i})_{\Sigma_i} \otimes (V'_j \otimes W^{\otimes j})_{\Sigma_j} \\ &= \bigoplus_{n \geq 0} \bigoplus_{i+j=n} (\text{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} (V_i \otimes V'_j) \otimes W^{\otimes n})_{\Sigma_n} \\ &= \Phi_{V \otimes V'}(W). \end{aligned}$$

The fourth equality comes from associativity of tensor products (interpreting induction as a tensor product), and the last comes from the fact that the tensor product on  $\mathcal{V}_{\mathbf{k}}$  is defined by induction if we choose representatives for each isomorphism class of sets.  $\square$

#### 2.4. Schur functors.

**Definition 2.12.** A **partition**  $\lambda$  is a decreasing sequence of positive integers  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . We represent this as a Young diagram by drawing  $\lambda_i$  boxes left-justified in the  $i$ th row, starting from top to bottom. The **dual partition**  $\lambda^\dagger$  is obtained by letting  $\lambda_i^\dagger$  be the number of boxes in the  $i$ th column of  $\lambda$ .  $\square$

**Example 2.13.** Let  $\lambda = (4, 3, 1)$ . Then  $\lambda^\dagger = (3, 2, 2, 1)$ .  $\square$

**Definition 2.14.** Let  $E$  be a vector space. Let  $\lambda$  be a partition with  $n$  parts and write  $m = \lambda_1$ . We use  $S^n E$  to denote the  $n$ th symmetric power of  $E$ . The **Schur functor**  $\mathbf{S}_\lambda(E)$  is the image of the map

$$\bigwedge^{\lambda_1^\dagger} E \otimes \dots \otimes \bigwedge^{\lambda_m^\dagger} E \xrightarrow{\Delta} E^{\otimes \lambda_1^\dagger} \otimes \dots \otimes E^{\otimes \lambda_m^\dagger} = E^{\otimes \lambda_1} \otimes \dots \otimes E^{\otimes \lambda_n} \xrightarrow{\mu} S^{\lambda_1} E \otimes \dots \otimes S^{\lambda_n} E,$$

where the maps are defined as follows. First,  $\Delta$  is the product of the comultiplication maps  $\bigwedge^i E \rightarrow E^{\otimes i}$  given by  $e_1 \wedge \dots \wedge e_i \mapsto \sum_{w \in \Sigma_i} \text{sgn}(w) e_{w(1)} \otimes \dots \otimes e_{w(i)}$ . The equals sign is interpreted as follows: pure tensors in  $E^{\otimes \lambda_1^\dagger} \otimes \dots \otimes E^{\otimes \lambda_m^\dagger}$  can be interpreted as filling the Young diagram of  $\lambda$  with vectors along the columns, which can be thought of as pure tensors in  $E^{\otimes \lambda_1} \otimes \dots \otimes E^{\otimes \lambda_n}$  by reading via rows. Finally,  $\mu$  is the multiplication map  $E^{\otimes i} \rightarrow S^i E$  given by  $e_1 \otimes \dots \otimes e_i \mapsto e_1 \dots e_i$ .

In particular, note that  $\mathbf{S}_\lambda E = 0$  if the number of parts of  $\lambda$  exceeds  $\text{rank } E$ .  $\square$

**Example 2.15.** Take  $\lambda = (3, 2)$ . Then the map is given by

$$\begin{aligned} (e_1 \wedge e_2) \otimes (e_3 \wedge e_4) \otimes e_5 &\mapsto \begin{array}{|c|c|c|} \hline e_1 & e_3 & e_5 \\ \hline e_2 & e_4 & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline e_2 & e_3 & e_5 \\ \hline e_1 & e_4 & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline e_1 & e_4 & e_5 \\ \hline e_2 & e_3 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline e_2 & e_4 & e_5 \\ \hline e_1 & e_3 & \\ \hline \end{array} \\ &\mapsto (e_1 e_3 e_5 \otimes e_2 e_4) - (e_2 e_3 e_5 \otimes e_1 e_4) - (e_1 e_4 e_5 \otimes e_2 e_3) + (e_2 e_4 e_5 \otimes e_1 e_3) \end{aligned}$$

□

**2.5. Schur–Weyl duality.** If  $\text{char}(\mathbf{k}) = 0$ , then both  $\mathcal{V}_{\mathbf{k}}$  and  $\text{Pol}_{\mathbf{k}}$  are semisimple categories and the simple objects are naturally indexed by partitions.

The irreducibles of  $\Sigma_n$  are the Specht modules  $\mathbf{M}_{\lambda}$  for  $|\lambda| = n$  and so the irreducibles for  $\mathcal{V}_{\mathbf{k}}$  are given by the  $\mathbf{M}_{\lambda}$ .

The irreducible polynomial functors are the Schur functors  $\mathbf{S}_{\lambda}$ .

Given a vector space  $V$ , there are commuting actions of  $\mathbf{GL}(V)$  and  $\Sigma_n$  on  $V^{\otimes n}$ .

**Theorem 2.16** (Schur–Weyl duality). *If  $\text{char}(\mathbf{k}) = 0$ , then*

$$V^{\otimes n} = \bigoplus_{\substack{\lambda \\ \ell(\lambda) \leq \dim(V) \\ |\lambda| = n}} \mathbf{S}_{\lambda}(V) \boxtimes \mathbf{M}_{\lambda}.$$

**Proposition 2.17.** *If  $\text{char}(\mathbf{k}) = 0$ , then  $\Phi: \mathcal{V}_{\mathbf{k}} \rightarrow \text{Pol}_{\mathbf{k}}$  is an equivalence.*

*Proof.* Schur–Weyl duality says that  $\Phi(\mathbf{M}_{\lambda}) = \mathbf{S}_{\lambda}$ . Now use that both categories are semisimple and every locally finite object is a direct sum of these objects. □

So to describe tca’s in characteristic 0, we can use polynomial functors.

One more reduction: let  $\mathbf{k}^{\infty} = \bigcup_{d \geq 0} \mathbf{k}^d$  where  $\mathbf{k}^d \subset \mathbf{k}^{d+1}$  is included as the span of the first  $d$  basis vectors. Schur functors satisfy

$$\mathbf{S}_{\lambda}(\mathbf{k}^{\infty}) = \bigcup_{d \geq 0} \mathbf{S}_{\lambda}(\mathbf{k}^d), \quad \mathbf{S}_{\lambda}(\mathbf{k}^d) = \mathbf{S}_{\lambda}(\mathbf{k}^{\infty})^{\mathbf{GL}(\mathbf{k}^{[d+1, \infty)})}.$$

In particular, the functor

$$\begin{aligned} \text{Pol}_{\mathbf{k}} &\rightarrow \mathbf{GL}_{\infty}(\mathbf{k})\text{-modules} \\ F &\mapsto F(\mathbf{k}^{\infty}) \end{aligned}$$

is an equivalence.

So we come to another equivalent definition of tca’s in characteristic 0:

**Definition 2.18.** If  $\text{char}(\mathbf{k}) = 0$ , then a tca is an associative, commutative, unital  $\mathbf{k}$ -algebra  $A$  equipped with a (locally finite) polynomial action of  $\mathbf{GL}_{\infty}(\mathbf{k})$  such that the multiplication map is equivariant. An  $A$ -module is a module over  $A$  with a compatible action of  $\mathbf{GL}_{\infty}(\mathbf{k})$ . □

The algebra  $\text{Sym}(U[1])$  becomes  $\text{Sym}(U \otimes \mathbf{k}^{\infty})$  with the action of  $\mathbf{GL}_{\infty}(\mathbf{k})$  on the  $\mathbf{k}^{\infty}$  factor.

### 3. symcl results

Reference: [arXiv:1206.2233](https://arxiv.org/abs/1206.2233)

In this lecture, we focus on the tca  $A = \text{Sym}(\mathbf{C}[1])$ . This is the algebra  $\mathbf{C}[x_1, x_2, \dots]$  with the action of  $\mathbf{GL}(\infty) = \mathbf{GL}_{\infty}(\mathbf{C})$  by linear changes of coordinates. Let  $\mathfrak{m} = \bigoplus_{d > 0} A_d$  be the maximal ideal.

Recall that  $A$ -modules are equivalent to functors from  $\mathbf{FI}$ , the category of finite sets and injective maps, to the category of  $\mathbf{C}$ -vector spaces.

Recall also Schur functors  $\mathbf{S}_\lambda = \mathbf{S}_\lambda(\mathbb{C}^\infty)$  from last lecture; they are the simple  $\mathbf{GL}(\infty)$ -representations and that an  $A$ -module is finitely generated if it is a quotient of  $A \otimes V$  where  $V$  is a finite direct sum of  $\mathbf{S}_\lambda$ .

**3.1. Torsion  $A$ -modules.** The simple  $A$ -modules are Schur functors  $\mathbf{S}_\lambda$  where  $\mathfrak{m} = \bigoplus_{d>0} A_d$  acts trivially.

**Proposition 3.1.** *If  $\text{Ext}_A^n(\mathbf{S}_\lambda, \mathbf{S}_\mu) \neq 0$  for  $n > 0$ , then  $|\mu| > |\lambda|$  and  $\text{Ext}_A^n(\mathbf{S}_\lambda, \mathbf{S}_\mu)$  is finite-dimensional.*

*Proof.* The Koszul complex gives a projective resolution of  $\mathbf{S}_\lambda$ :

$$\mathbf{S}_\lambda \otimes (\cdots \rightarrow A \otimes \bigwedge^2 \rightarrow A \otimes \bigwedge^1 \rightarrow A) \rightarrow \mathbf{S}_\lambda \rightarrow 0.$$

Now apply  $\text{Hom}_A(-, \mathbf{S}_\mu)$  and take homology. If  $\text{Ext}^n \neq 0$  for  $n > 0$ , then  $\mathbf{S}_\mu$  appears in  $\mathbf{S}_\lambda \otimes A \otimes \bigwedge^n$  and in particular  $|\mu| > |\lambda|$ .  $\square$

Let  $\text{Mod}_A^{\text{tors}}$  be the category of finitely generated  $A$ -modules which are annihilated by some power of  $\mathfrak{m}$ .

**Corollary 3.2.**  *$\text{Mod}_A^{\text{tors}}$  has enough injectives, and every object has a finite injective resolution.*

*Proof.* This follows from this directed property.  $\square$

A priori, injectives in  $\text{Mod}_A^{\text{tors}}$  need not be injective as  $A$ -modules but we will see later that they are.

**3.2. Structure of projective  $A$ -modules.**

**Proposition 3.3.** *The indecomposable projective  $A$ -modules are  $A \otimes \mathbf{S}_\lambda$  for each partition  $\lambda$ .*

**Definition 3.4.** If  $\lambda_i \leq \mu_i$  for all  $i$ , write  $\lambda \subseteq \mu$ . If also  $\lambda_i \geq \mu_{i+1}$  for all  $i$ , and  $|\mu| - |\lambda| = d$ , write  $\mu/\lambda \in \text{HS}_d$  (horizontal strip).  $\square$

$\mu/\lambda \in \text{HS}_d$  can be represented graphically:

**Example 3.5.**  $(4, 2)/(2, 1) \in \text{HS}_3$  but  $(4, 3)/(2, 1) \notin \text{HS}_4$ :

$$\begin{array}{|c|c|c|c|} \hline & & x & x \\ \hline & x & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & & x & x \\ \hline & x & x & \\ \hline \end{array}.$$

$\square$

Two important facts about the projectives  $A \otimes \mathbf{S}_\lambda$ :

**Theorem 3.6** (Pieri's rule).

$$\text{Sym}^d \otimes \mathbf{S}_\lambda = \bigoplus_{\mu, \mu/\lambda \in \text{HS}_d} \mathbf{S}_\mu.$$

**Theorem 3.7** (Olver). *The submodule of  $A \otimes \mathbf{S}_\lambda$  generated by  $\mathbf{S}_\mu$  is the sum of all  $\mathbf{S}_\nu$  that appear in the decomposition of both  $A \otimes \mathbf{S}_\lambda$  and  $A \otimes \mathbf{S}_\mu$ .*

For  $D \geq \lambda_1$ , set

$$L_\lambda^{\geq D} = \bigoplus_{d \geq D} \mathbf{S}_{(d, \lambda_1, \lambda_2, \dots)}.$$

Then  $L_\lambda^{\geq D}$  can be realized a quotient module of  $A \otimes \mathbf{S}_{(D, \lambda)}$ .

This is the unique  $A$ -module structure so that  $L_\lambda^{\geq D}$  is generated by  $\mathbf{S}_{(D, \lambda)}$ .

**Proposition 3.8.** *There is a filtration of  $A$ -submodules*

$$0 = F_{-1} \subset F_0 \subset \cdots \subset F_{\lambda_1} = A \otimes \mathbf{S}_\lambda$$

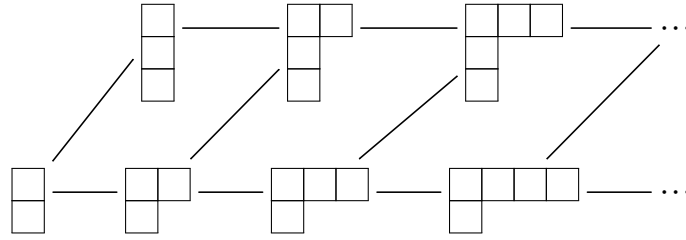
*such that*

$$F_i/F_{i-1} \cong \bigoplus_{\substack{\nu \\ \lambda/\nu \in \text{HS}_i}} L_\nu^{\geq \lambda_1}.$$

**Example 3.9.**  $\lambda = (1, 1)$ , i.e.,  $\mathbf{S}_\lambda = \wedge^2$ . Then have

$$0 \rightarrow L_{1,1}^{\geq 1} \rightarrow A \otimes \mathbf{S}_{1,1} \rightarrow L_1^{\geq 1} \rightarrow 0.$$

Submodule structure:



Top part is  $L_{1,1}^{\geq 1}$ , bottom part is  $L_1^{\geq 1}$ . □

*Proof.*  $A \otimes \mathbf{S}_\lambda$  is multiplicity-free. Let  $F_i$  be sum of  $\mathbf{S}_\mu$  in  $A \otimes \mathbf{S}_\lambda$  such that

$$i \geq \mu_1 - (|\mu| - |\lambda|) = \sum_{j \geq 1} \lambda_j - \sum_{j \geq 2} \mu_j.$$

So if  $\mathbf{S}_\mu \in F_i$  and  $\mu \subseteq \nu$ , then  $\mathbf{S}_\nu \in F_i$ , so  $F_i$  is an  $A$ -submodule. The subquotient identification is a combinatorial exercise together with the uniqueness property of  $L_\nu^{\geq \lambda_1}$  being generated by  $\mathbf{S}_{(\lambda_1, \nu)}$ . □

**3.3. Serre quotient.** Let  $\text{Mod}_A$  be the category of finitely generated  $A$ -modules. We study it by stratifying it. Let  $\text{Mod}_A^{\text{tors}}$  be the subcategory of finite length  $A$ -modules and define

$$\text{Mod}_K = \text{Mod}_A / \text{Mod}_A^{\text{tors}}.$$

Intuition:  $K = \text{Frac}(A)$  and  $\text{Mod}_K$  is the category of “coherent sheaves” on “ $\text{Proj}(A)$ ”

To be precise, the objects of  $\text{Mod}_K$  are the objects of  $\text{Mod}_A$ , and

$$\text{Hom}_{\text{Mod}_K}(M, N) = \text{colim} \text{Hom}_{\text{Mod}_A}(M', N/N')$$

where the colimit is over all  $M' \subseteq M$  and  $N' \subseteq N$  such that  $M/M'$  and  $N'$  are finite length.

An alternative description: the fraction field  $K$  of  $A$  has a  $\mathbf{GL}(\infty)$ -action and  $\text{Mod}_K$  is the category of finitely generated semi-linear representations:  $K$ -vector spaces  $V$  with a  $\mathbf{GL}(\infty)$ -action such that

$$g.(\alpha v) = (g.\alpha)(g.v)$$

for  $g \in \mathbf{GL}(\infty)$ ,  $\alpha \in K$ , and  $v \in V$ .

**Remark 3.10.** One can show that  $\text{Mod}_A^{\text{tors}} \simeq \text{Mod}_K$ . The intuition is as follows: let  $P \subset \text{GL}(\infty)$  be the stabilizer of a nonzero vector in  $\mathbf{C}^\infty$ . For instance, pick the vector  $e_1$  so that  $P = \begin{pmatrix} 1 & \mathbf{C}^\infty \\ 0 & \text{GL}(\infty) \end{pmatrix}$ . Then  $P$  is the semidirect product  $\text{GL}(\infty) \ltimes \mathbf{C}^\infty$  and so  $P$ -modules are the same as  $A$ -modules!

But  $\text{Mod}_K$  can be thought of as equivariant sheaves on  $\mathbf{C}^\infty \setminus \{0\}$ . Since this is a homogeneous space, these are determined by their fibers over a given point, for example over  $e_1$ , so that  $\text{Mod}_K$  is equivalent to the category of finite length  $P$ -modules.  $\square$

**Lemma 3.11.** (a) For  $D' \geq D$ , the inclusion  $L_\lambda^{\geq D'} \subseteq L_\lambda^{\geq D}$  becomes an isomorphism in  $\text{Mod}_K$ .  
 (b) The resulting object, called  $L_\lambda$ , is simple in  $\text{Mod}_K$ .

*Proof.* (a) the quotient  $L_\lambda^{\geq D}/L_\lambda^{\geq D'}$  is finite length.

(b) all submodules of  $L_\lambda^{\geq D}$  are of the form  $L_\lambda^{\geq D'}$ .  $\square$

**Corollary 3.12.** Every object of  $\text{Mod}_K$  has finite length (i.e., the Krull dimension of  $\text{Mod}_A$  is 1). Every simple object is of the form  $L_\lambda$ .

*Proof.* Every finitely generated  $A$ -module is a quotient of a finite direct sum of  $A \otimes \mathbf{S}_\lambda$ . Now use the explicit filtration.  $\square$

**Theorem 3.13.** The image of  $A \otimes \mathbf{S}_\lambda$  in  $\text{Mod}_K$  is an indecomposable injective object; call it  $Q_\lambda$ . Every indecomposable injective is of this form.

Every object in  $\text{Mod}_K$  has finite injective dimension.

Proof omitted for first point; I could not find a reasonably simple explanation.

For the second point, we can use the equivalence  $\text{Mod}_A^{\text{tors}} \simeq \text{Mod}_K$ .

### 3.4. Section functor.

**Theorem 3.14.** The localization functor  $T: \text{Mod}_A \rightarrow \text{Mod}_K$  has a right adjoint  $S: \text{Mod}_K \rightarrow \text{Mod}_A$ .

We can define it as follows: every element in the image of  $T(M) \rightarrow N$  where  $M \in \text{Mod}_A$  is *polynomial*, i.e., generates a polynomial subrepresentation of  $N$ . So  $S(N)$  is the submodule of all polynomial elements.

$S$  is the **section functor**. We omit the proof that  $S(N)$  is a finitely generated  $A$ -module.

**Definition 3.15.** A (finitely generated)  $A$ -module  $M$  is **saturated** if  $\text{Ext}_A^i(N, M) = 0$  for  $i = 0, 1$  and all torsion  $A$ -modules.  $\square$

**Lemma 3.16.** The modules  $L_\lambda^{\geq \lambda_1}$  and  $A \otimes \mathbf{S}_\lambda$  are saturated.

*Proof.* Explicit calculations; proof omitted.  $\square$

A few basic properties of  $S$  (follows from Gabriel's results);

**Proposition 3.17.** (a)  $S$  is left exact

(b)  $S$  takes injective objects to injective objects

(c) For  $M' \in \text{Mod}_K$ ,  $S(M')$  is saturated and the adjunction  $T(S(M')) \rightarrow M'$  is an isomorphism

(d) For  $M \in \text{Mod}_A$ , the adjunction  $M \rightarrow S(T(M))$  is an isomorphism if and only if  $M$  is saturated.



### 3.5. Structure of injective $A$ -modules.

**Corollary 3.18.**  $A \otimes \mathbf{S}_\lambda$  is injective.

*Proof.*  $A \otimes \mathbf{S}_\lambda$  is saturated, so is isomorphic to  $T(S(A \otimes \mathbf{S}_\lambda)) = T(Q_\lambda)$  and  $Q_\lambda$  is injective. Now use that  $S$  preserves injectives.  $\square$

Let  $\text{Mod}_A^{\text{tors}}$  be the category of finitely generated torsion  $A$ -modules.

**Lemma 3.19.** An injective object in  $\text{Mod}_A^{\text{tors}}$  is also injective in  $\text{Mod}_A$ .

*Proof.* Let  $I$  be an injective object in  $\text{Mod}_A^{\text{tors}}$  and let  $M$  be an arbitrary  $A$ -module. If  $M$  and  $I$  have no Schur functors  $\mathbf{S}_\lambda$  in common then any extension

$$0 \rightarrow I \rightarrow E \rightarrow M \rightarrow 0$$

splits uniquely in  $\text{Pol}_\mathbb{C}$  and hence also as  $A$ -modules.

For  $n \gg 0$ ,  $\mathfrak{m}^n M$  and  $I$  have no common Schur functors. The short exact sequence

$$0 \rightarrow \mathfrak{m}^n M \rightarrow M \rightarrow M/\mathfrak{m}^n M \rightarrow 0$$

leads to

$$\text{Ext}_A^1(M/\mathfrak{m}^n M, I) \rightarrow \text{Ext}_A^1(M, I) \rightarrow \text{Ext}_A^1(\mathfrak{m}^n M, I)$$

The rightmost group is 0 by choice of  $n$  and the leftmost group is 0 since  $M/\mathfrak{m}^n M$  is torsion.  $\square$

Let  $I_\lambda$  be the injective envelope of  $\mathbf{S}_\lambda \in \text{Mod}_A^{\text{tors}}$ . This can be described as the restriction of  $\mathbf{S}_\lambda$  from  $\mathbf{GL}(\infty)$  to  $P$  under the interpretation that finite length  $A$ -modules are the same as finite length  $P$ -modules.

**Proposition 3.20.** Every injective module is a direct sum of  $A \otimes \mathbf{S}_\lambda$  and  $I_\mu$ .

*Proof.* Let  $I$  be an injective module. Let  $I_{\text{tors}}$  be its maximal torsion submodule – this is injective in  $\text{Mod}_A^{\text{tors}}$  and hence is injective in  $\text{Mod}_A$ , so it splits off:  $I = I_{\text{tors}} \oplus I'$ . Then  $I'$  is saturated, so  $I' = S(T(I'))$ . By adjunction, for any module  $M$ , we have  $\text{Hom}_A(M, I') = \text{Hom}_K(T(M), T(I'))$ , so  $T(I')$  is injective in  $\text{Mod}_K$ . So  $T(I') = \bigoplus_\lambda Q_\lambda$  and so  $I' = \bigoplus_\lambda (A \otimes \mathbf{S}_\lambda)$ .  $\square$

**Theorem 3.21.** Every finitely generated  $A$ -module  $M$  has finite injective dimension.

*Proof.*  $T(M)$  has a finite injective resolution  $\mathbf{I}^\bullet$  in  $\text{Mod}_K$ . So we get  $S(T(M)) \rightarrow S(\mathbf{I}^\bullet)$ . The kernel and cokernel of  $M \rightarrow S(T(M))$  are torsion, and torsion modules have finite injective dimension since  $\text{Mod}_A^{\text{tors}}$  is directed. Now combine the injective resolutions to get one for  $M$ .  $\square$

**Corollary 3.22.** Let  $\mathcal{P}$  be a property of finitely generated  $A$ -modules such that

- (1) (2 out of 3): If  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  and  $\mathcal{P}$  holds for two of the  $M_i$ , then it also holds for the third
- (2)  $\mathcal{P}$  holds for all  $\mathbf{S}_\lambda$
- (3)  $\mathcal{P}$  holds for all  $A \otimes \mathbf{S}_\lambda$

Then  $\mathcal{P}$  holds for all modules.

*Proof.* (1) and (2) imply  $\mathcal{P}$  holds for all torsion injectives  $I_\lambda$ . Now use that every f.g. module has a finite injective resolution.  $\square$

3.6. **Hilbert series.** Hilbert series of  $A$ -modules make no sense since dimensions are infinite. But can use **FI** description. Recall that a polynomial functor  $M$  is the same as a functor  $\tilde{M}: \mathbf{FB} \rightarrow \text{Vec}_{\mathbb{C}}$ . The **Hilbert series** of  $M$  is

$$H_M(t) = \sum_{n \geq 0} \dim_{\mathbb{C}} \tilde{M}_n t^n.$$

**Proposition 3.23.**  $H_M(t)$  is a rational function of the form  $f(t)/(1-t)^d$  for some  $d$ . Equivalently, the function  $n \mapsto \dim_{\mathbb{C}} \tilde{M}_n$  is polynomial for  $n \gg 0$ .

*Proof.* Hilbert series is additive with respect to exact sequences so this rationality result satisfies 2 out of 3. It is obvious for  $\mathbf{S}_{\lambda}$ . Also

$$(A \otimes \mathbf{S}_{\lambda})_n = \text{Ind}_{\Sigma_{n-|\lambda|} \times \Sigma_{|\lambda|}}^{\Sigma_n} \mathbf{C} \boxtimes \mathbf{M}_{\lambda}$$

so has dimension  $(\dim \mathbf{M}_{\lambda}) \binom{n}{|\lambda|}$  and hence also has the desired form.  $\square$

This captures little information. For example, does not distinguish representations with the same dimension. Every object  $F \in \mathcal{V}_{\mathbb{C}}$  has a character  $\text{ch}(F)$  defined as the trace of the diagonal matrix with entries  $x_1, x_2, \dots$  acting on  $F(\mathbb{C}^{\infty})$ . Define **enhanced Hilbert series** of  $M$ :

$$\tilde{H}_M(t) = \sum_{n \geq 0} \text{ch}(M_n) t^n$$

Define  $s_{\lambda} = \text{ch}(\mathbf{S}_{\lambda})$  (Schur function).

(Schur–Weyl duality says that we could also use characters of symmetric groups instead of these trace functions)

**Proposition 3.24.**  $\tilde{H}_M(t) = p_M(t) \prod_{i \geq 1} (1 - x_i t)^{-1} + q_M(t)$  where  $p_M, q_M$  are polynomials in  $s_{\lambda}$  and  $t$ .

*Proof.* This is obvious for simples  $\mathbf{S}_{\lambda}$ . Note that

$$\tilde{H}_{\mathbf{S}_{\lambda} \otimes A}(t) = \text{ch}(\mathbf{S}_{\lambda}) \prod_{i \geq 1} (1 - x_i t)^{-1}$$

and that this property satisfies 2 out of 3.  $\square$

3.7. **Local cohomology.** We stated earlier that the category of finitely generated  $A$ -modules has enough injectives (in fact, every module has a finite injective resolution).

Given an  $A$ -module, define  $H_{\mathfrak{m}}^0(M)$  to be the largest submodule annihilated by some power of  $\mathfrak{m} = \bigoplus_{d > 0} A_d$ . This is clearly left-exact, so we can define its right-derived functors  $H_{\mathfrak{m}}^i$ . This is **local cohomology**.

We will just state some basic facts about local cohomology and skip the technical proofs. We wish to highlight analogies with local cohomology over polynomial rings.

**Definition 3.25.** Let  $M$  be an  $A$ -module. Define  $d_M(n)$  to be the depth of  $A(\mathbb{C}^n)$ -module  $M(\mathbb{C}^n)$  (recall that this is the length of the longest regular sequence in  $\mathfrak{m}(\mathbb{C}^n)$ ).

Fact: if  $M$  is not projective, then the function  $d_M(n)$  is independent of  $n$  for  $n \gg 0$ .

This stable value is the **depth** of  $M$ . If  $M$  is projective, it has infinite depth.  $\square$

We have a version of Grothendieck’s vanishing theorem:

(Recall the classical version says that  $H_{\mathfrak{m}}^i(M) = 0$  if  $i < \text{depth}(M)$  or if  $i > \dim(M)$ )

**Proposition 3.26.** *Let  $M \neq 0$  be an  $A$ -module. Then  $\inf\{d \mid H_{\mathfrak{m}}^d(M) \neq 0\}$  is the depth of  $M$  (convention:  $\inf(\emptyset) = \infty$ ) and  $\sup\{d - 1 \mid H_{\mathfrak{m}}^d(M) \neq 0\}$  is the injective dimension of  $T(M)$  (convention:  $\sup(\emptyset) = 0$  and injective dimension of 0 is  $-1$ ).*

There is another left-exact functor on  $A$ -modules defined by  $M \mapsto ST(M)$ . Following the analogy of  $\text{Mod}_K$  being coherent sheaves on  $\text{Proj}(A)$ , its right derived functors are “(coherent) sheaf cohomology”. We have the following analogue of the relation between local and sheaf cohomology:

**Proposition 3.27.** *Let  $M$  be an  $A$ -module. Then there is an exact sequence*

$$0 \rightarrow H_{\mathfrak{m}}^0(M) \rightarrow M \rightarrow S(T(M)) \rightarrow H_{\mathfrak{m}}^1(M) \rightarrow 0$$

and for each  $i > 1$ ,

$$H_{\mathfrak{m}}^{i+1}(M) = R^i S(T(M)).$$

$q_M(t)$  from Proposition 3.24 can be thought of as the measure of failure for  $n \mapsto \dim_{\mathbb{C}} \widetilde{M}_n$  to be a polynomial for all  $n \geq 0$ . In fact, it comes from local cohomology:

**Proposition 3.28.**  $q_M(t) = \sum_{i \geq 0} (-1)^i \widetilde{H}_{H_{\mathfrak{m}}^i(M)}(t)$ .

*Proof.* Both sides are additive with respect to exact sequences, so just need to check simples and projectives by 2 out of 3 property. For simples, have  $M = H_{\mathfrak{m}}^0(M)$  and  $q_M(t) = \widetilde{H}_M(t)$ . For projectives,  $q_M(t) = 0$  and  $H_{\mathfrak{m}}^i(M) = 0$  for all  $i$ .  $\square$

**3.8. Regularity & Koszul duality.** Let  $\mathbb{C}$  be the residue field  $A/\mathfrak{m}$ . If  $M$  is a finitely generated  $A$ -module, then the Tor groups  $\text{Tor}_i^A(M, \mathbb{C})$  are objects of  $\mathcal{V}_{\mathbb{C}}$ , so are naturally  $\mathbb{Z}_{\geq 0}$ -graded and record information about the minimal projective resolutions of  $M$ : there exists a minimal projective resolution

$$\cdots \rightarrow \mathbf{F}_2 \rightarrow \mathbf{F}_1 \rightarrow \mathbf{F}_0 \rightarrow M \rightarrow 0$$

where  $\mathbf{F}_i = \text{Tor}_i^A(M, \mathbb{C}) \otimes A$ .

The **(Castelnuovo–Mumford) regularity** of  $M$  is

$$\text{reg}(M) = \sup_i \{j \mid \text{Tor}_i^A(M, \mathbb{C})_{i+j} \neq 0\}.$$

**Theorem 3.29.** *If  $M$  is finitely generated, then  $\text{reg}(M) < \infty$ .*

*Proof.* This is obvious for  $M = A \otimes \mathbf{S}_{\lambda}$ . For  $M = \mathbf{S}_{\lambda}$ , we have the Koszul complex

$$\mathbf{S}_{\lambda} \otimes (\cdots \rightarrow A \otimes \bigwedge^2 \rightarrow A \otimes \bigwedge^1 \rightarrow A) \rightarrow \mathbf{S}_{\lambda} \rightarrow 0$$

which shows that  $\text{reg}(M) = |\lambda|$ . The long exact sequence of Tor shows that finite regularity satisfies 2 out of 3.  $\square$

There is a duality on  $\mathcal{V}_{\mathbb{C}}$ :  $F^{\vee}(V) := F(V^*)^*$ .

There is more structure:  $\text{Tor}_{\bullet}^A(M, \mathbb{C})$  is a comodule over the exterior (co)algebra  $B := \bigoplus_{d \geq 0} \bigwedge^d$  and it can be shown to be finitely cogenerated, and so  $\text{Tor}_{\bullet}^A(M, \mathbb{C})^{\vee}$  is a finitely generated module. The action preserves linear strands, so that this module is a direct sum of  $\text{Tor}_{\bullet}^A(M, \mathbb{C})_{\bullet+i}^{\vee}$  for various  $i$ , and only nonzero for finitely many values of  $i$ .

**Theorem 3.30.**  $M \mapsto \mathrm{Tor}_{\bullet}^A(M, \mathbf{C})^{\vee}$  can be extended to an equivalence

$$D^b(\mathrm{Mod}_A) \simeq D^b(\mathrm{Mod}_B)^{\mathrm{op}}.$$

We omit the technical details but point out one interesting feature.

There is another operation on  $\mathcal{V}_{\mathbf{C}}: \mathbf{S}_{\lambda}^{\dagger} := \mathbf{S}_{\lambda^+}$ . This turns out to be a monoidal equivalence. Note that  $A^{\dagger} = B$ . So we get an interesting auto-equivalence

$$\mathcal{F}: D^b(\mathrm{Mod}_A) \simeq D^b(\mathrm{Mod}_A)^{\mathrm{op}}.$$

This can be used to deduce the following symmetry. For a complex  $\mathbf{F}_{\bullet}$  of  $A$ -modules, define  $\widetilde{H}_{\mathbf{F}_{\bullet}}(t) = \sum_i (-1)^i \widetilde{H}_{\mathbf{F}_i}(t)$ .

Recall the auto-equivalence

$$\mathcal{F}: D^b(\mathrm{Mod}_A) \simeq D^b(\mathrm{Mod}_A)^{\mathrm{op}}.$$

**Proposition 3.31.** *Let  $M \in D^b(\mathrm{Mod}_A)$  and write  $\widetilde{H}_M(t) = p_M(t) \prod_{i \geq 1} (1 - x_i t)^{-1} + q_M(t)$ . Then  $\widetilde{H}_{\mathcal{F}(M)}(t) = q_M(-t) \prod_{i \geq 1} (1 - x_i t)^{-1} + p_M(-t)$ . In other words,*

$$p_{\mathcal{F}(M)}(t) = q_M(-t), \quad q_{\mathcal{F}(M)}(t) = p_M(-t).$$

This has an interesting consequence: define the **Poincaré series** of  $M$  as

$$P_M(q, t) = \sum_{i \geq 0} q^i H_{\mathrm{Tor}_i^A(M, \mathbf{C})}(t).$$

**Theorem 3.32.** *If  $M$  is finitely generated then  $P_M(q, t)$  is of the form  $f(q, t)/(1 - qt)^d$ .*

*Proof.* Set  $\mathcal{F}_i(M) = \mathrm{Tor}_{\bullet+i}^A(M, \mathbf{C})^{\vee, \dagger}$ . Then

$$P_M(q, t) = \sum_{i \geq 0} q^{-i} H_{\mathcal{F}_i(M)}(qt).$$

There are finitely many  $i$  such that  $\mathcal{F}_i(M) \neq 0$ , and each one is a finitely generated  $A$ -module. Now use previous results.  $\square$

Also get analogous statement for enhanced Poincaré series

$$\widetilde{P}_M(q, t) = \sum_{i \geq 0} q^i \widetilde{H}_{\mathrm{Tor}_i^A(M, \mathbf{C})}(t).$$

Interesting note: the Poincaré series is not additive with respect to short exact sequences, so a priori the rationality property above might not satisfy the 2 out of 3 property.

**3.9. Summary of other results.** A number of results about  $\mathbf{FI}_1$  in char. 0 extend to  $\mathbf{FI}_d$ . We just state the results:

**Theorem 3.33.** *Let  $M$  be a finitely generated  $\mathbf{FI}_d$ -module (equivalently,  $A = \mathrm{Sym}(U \otimes \mathbf{C}^{\infty})$ -module). Then:*

- (a)  $H_M(t)$  and  $\widetilde{H}_M(t)$  are rational functions.
- (b)  $\mathrm{Tor}_{\bullet+i}^A(M, \mathbf{C})^{\vee, \dagger}$  is a finitely generated  $A$ -module for each  $i$  and is nonzero for only finitely many  $i$ . In particular  $\mathrm{reg}(M) < \infty$ .
- (c)  $P_M(q, t)$  and  $\widetilde{P}_M(q, t)$  are rational functions.

## 4. General theory of tca's

### 4.1. Noetherian properties.

**Conjecture 4.1.** *A finitely generated tca  $A$  is noetherian, i.e., every finitely generated  $A$ -module is noetherian.*

Some partial progress:

**Theorem 4.2.** *Arbitrary noetherian coefficient ring. If  $A$  is finitely generated in degree  $\leq 1$ , then  $A$  is noetherian.*

If  $\lambda$  is a partition, define  $\ell(\lambda)$  to be the number of nonzero parts. Define  $\ell(\mathbf{S}_\lambda) = \ell(\lambda)$  and for a general polynomial functor  $F = \bigoplus \mathbf{S}_\lambda$ , define  $\ell(F) = \sup\{\ell(\lambda)\}$ . Say that  $F$  is **bounded** if  $\ell(F) < \infty$ .

A simple consequence of Littlewood–Richardson rule:  $\ell(F \otimes F') = \ell(F) + \ell(F')$ .

**Theorem 4.3.** *Characteristic 0. If  $A$  is bounded, then  $A$  is noetherian.*

**Example 4.4.** If  $A$  is finitely generated in degree  $\leq 1$ , it is bounded.

For any tca  $A$  and  $r > 0$ , the sum of all  $\mathbf{S}_\lambda$  with  $\ell(\lambda) > r$  is an ideal and so there is a bounded quotient  $A^{\leq r}$ .  $\square$

The advantage of char. 0 is the polynomial functor interpretation. In particular, we can specialize. By definition, a sequence

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$$

of polynomial functors is exact if the same holds upon evaluation on all vector spaces. Also,  $\mathbf{S}_\lambda(\mathbf{C}^n) \neq 0$  if and only if  $n \geq \ell(\lambda)$ . So we get

**Lemma 4.5.** *If  $M$  is an  $A$ -module and  $\ell(M) \leq n$ , then the specialization map to  $\mathbf{C}^n$  gives an isomorphism of submodule lattices*

$$\{A\text{-submodules of } M\} \rightarrow \{A(\mathbf{C}^n)\text{-submodules of } M(\mathbf{C}^n)\}$$

*Proof of Theorem 4.3.* If  $A$  is bounded, then every finitely generated projective is too, so every finitely generated  $A$ -module is bounded.

If  $n \geq \max(\ell(A), \ell(M))$ , then we can specialize to  $\mathbf{C}^n$  to show  $M$  is noetherian.

This follows since  $M(\mathbf{C}^n)$  is a finitely generated module over a finitely generated algebra  $A(\mathbf{C}^n)$ .  $\square$

### 4.2. Gröbner methods. Reference: [arXiv:1409.1670](https://arxiv.org/abs/1409.1670)

To show that tca's finitely generated in degree  $\leq 1$  are noetherian, it suffices to handle free ones, i.e.,  $\text{Sym}(U[1])$ . Recall that its module category is equivalent to  $\mathbf{FI}_d$ -modules.

Reminder: the objects of  $\mathbf{FI}_d$  are finite sets and a morphism  $S \rightarrow T$  is an injective map  $f: S \rightarrow T$  together with a function  $g: T \setminus f(S) \rightarrow [d]$  where  $[d] := \{1, \dots, d\}$ .

Define a category  $\mathbf{OI}_d$  same as  $\mathbf{FI}_d$  except the objects are ordered sets and the injection  $f: S \rightarrow T$  is required to be order-preserving (no condition on  $g$ ).

For  $\mathbb{C} \in \{\mathbf{FI}_d, \mathbf{OI}_d\}$ , and each  $n \geq 0$ , define a projective module  $P_{\mathbb{C},n}$  by  $P_{\mathbb{C},n}(S) = \mathbf{k}[\text{Hom}_{\mathbb{C}}([n], S)]$ . These are projective generators of the category of  $\mathbb{C}$ -modules.

**Theorem 4.6.** *The category of  $\mathbf{OI}_d$ -modules is noetherian.*

Assume  $\mathbf{k}$  is a field for simplicity.

It suffices to prove that  $P_{\mathbf{OI}_d, n}$  is noetherian for each  $n \geq 0$ . Set  $P = P_{\mathbf{OI}_d, n}$ .

All of the information of  $P$  is contained in the graded vector space

$$\bigoplus_{m \geq n} P([m])$$

together with the operations

$$\mathbf{k}[\mathrm{Hom}_{\mathbf{OI}_d}([m], [m'])] \otimes P([m]) \rightarrow P([m']).$$

A **monomial** of  $P$  is a basis vector  $e_{(f,g)} \in \mathbf{k}[\mathrm{Hom}_{\mathbf{OI}_d}([n], [m])]$  corresponding to a morphism  $(f, g): [n] \rightarrow [m]$ .

A monomial is encoded by a word  $w$  in  $\{0, \dots, d\}$  of length  $m$  where 0 is used  $n$  times: given  $(f, g)$ , replace the image of  $f([n])$  by 0 and mark the other elements using  $g$ .

Write  $e_w$  instead of  $e_{(f,g)}$ .

A word  $w$  contains  $w'$  as a **subword** if  $w'$  is obtained from  $w$  by deleting some letters. Then write  $w \geq w'$ .

For the following lemma, we consider all words in  $\{0, \dots, d\}$  (ignoring the restriction on 0):

**Proposition 4.7** (Higman's lemma). *Given  $w_1, w_2, \dots$ , there exist  $i < j$  such that  $w_i \leq w_j$ .*

*Proof.* Call a sequence violating this condition **bad**. Assume they exist.

Pick a minimal bad sequence in the sense that  $\ell(w_1)$  is minimized amongst all of them, and  $\ell(w_i)$  is minimal amongst all bad sequences starting with  $w_1, \dots, w_{i-1}$ .

Let  $v_i$  be the first letter in  $w_i$  and let  $w'_i$  be the result of deleting the first letter from  $w_i$ . Then for some infinite subsequence  $i_1, i_2, \dots$ ,  $v$  is constant. By minimality, the sequence  $w_1, \dots, w_{i_1-1}, w'_{i_1}, w'_{i_2}, \dots$  is not bad.

If  $w_i \leq w'_{i_j}$ , then  $w_i \leq w_{i_j}$ .

If  $w'_{i_j} \leq w'_{i_k}$ , then  $w_{i_j} \leq w_{i_k}$ , so in either case we get a contradiction.  $\square$

A submodule of  $P$  is **monomial** if it is spanned by the monomials it contains.

**Lemma 4.8.** *The submodule generated by  $e_{w'} \in P$  is the span of all  $e_w$  such that  $w \geq w'$ .*

**Lemma 4.9.** *Every monomial submodule of  $P$  is finitely generated.*

*Proof.* Suppose not; pick  $M \subset P$  monomial and not finitely generated.

So we can find monomials  $w_1, w_2, w_3, \dots$  such that  $w_i \not\leq w_j$  for all  $i < j$ .

These don't exist by Higman's lemma.  $\square$

Order  $<$  words first by length and then lexicographically. Given an element  $x = \sum \alpha_w e_w \in M$ , define  $\mathrm{init}(x) = \max\{e_w \mid \alpha_w \neq 0\}$ . Define

$$\mathrm{init}(M)([n]) = \mathbf{k}\{\mathrm{init}(x) \mid x \in M([n])\}.$$

**Lemma 4.10.** (a)  *$\mathrm{init}(M)$  is an  $\mathbf{OI}_d$ -submodule of  $P$ .*

(b) *If  $M' \subseteq M$  and  $\mathrm{init}(M) = \mathrm{init}(M')$ , then  $M = M'$ .*

*Proof.* (a) Observe that if  $w \geq w'$ , then  $w > w'$ , so that for all morphisms  $(f, g)$  and  $x \in M$ , we have  $M_{(f,g)} \mathrm{init}(x) = \mathrm{init}(M_{(f,g)} x)$ .

(b) Suppose  $M' \subsetneq M$ . Pick  $x \in M \setminus M'$  with  $\mathrm{init}(x)$  minimal. There exists  $y \in M'$  such that  $\mathrm{init}(x) = \mathrm{init}(y)$ . So  $x - y \in M \setminus M'$  and  $\mathrm{init}(x - y) < \mathrm{init}(x)$ . Contradiction.  $\square$

**Corollary 4.11.**  *$P$  is noetherian. In particular, finitely generated  $\mathbf{OI}_d$ -modules are noetherian.*

*Proof.* Pick  $M \subseteq P$ . Then  $\text{init}(M)$  is finitely generated by say  $v_1, \dots, v_r$ .

Pick  $x_1, \dots, x_r$  with  $\text{init}(x_i) = v_i$  and let  $M'$  be the submodule generated by  $x_i$ .

Then  $M' \subseteq M$  and  $\text{init}(M') = \text{init}(M)$  so  $M = M'$ .

Finally, noetherianity is preserved under finite direct sums and quotients.  $\square$

**Theorem 4.12.** *Finitely generated  $\mathbf{FI}_d$ -modules are noetherian.*

*Proof.* It suffices to show that  $P = P_{\mathbf{FI}_d, n}$  is noetherian. There is a forgetful functor

$$\Phi: \mathbf{OI}_d \rightarrow \mathbf{FI}_d$$

so we can pullback along  $\Phi$ . This pullback preserves strict inclusions and  $\Phi^*(P) \cong P_{\mathbf{OI}_d, n}^{\oplus n!}$ , so we are done.  $\square$

**Remark 4.13.** The monomial replacement (Gröbner degeneration) can also be used to get general rationality results for Hilbert series but we do not discuss this here.  $\square$

**4.3. Shortcomings.** The next example of a tca is  $\text{Sym}(\mathbf{k}[2])$ . Its module category also has a functor description (this is true in general).

$\mathbf{FI}^{(2)}$  is category whose objects are finite sets and a morphism  $S \rightarrow T$  is an injection  $f: S \rightarrow T$  and perfect matching on  $T \setminus f(S)$  (i.e., decomposition into 2-element subsets).

The category of  $\text{Sym}(\mathbf{k}[2])$ -modules is equivalent to the category of  $\mathbf{FI}^{(2)}$ -modules.

We could try the strategy above. Define  $\mathbf{OI}^{(2)}$  in the same way but with ordered sets and injections.

We run into the following obstacle:

**Example 4.14.** The category of  $\mathbf{OI}^{(2)}$ -modules is not noetherian:  $P = P_{\mathbf{OI}^{(2)}, 0}$  contains infinitely generated monomial submodules.

An example is given by letting  $m_n$  be the perfect matching on  $[2n]$  with edges  $(i, i+3)$  with  $i$  odd. Then  $m_3, m_4, \dots$  generates such a monomial submodule.  $\square$

**Remark 4.15.** The non-noetherian property of  $\mathbf{OI}^{(2)}$  says that monomial submodules need not be finitely generated. However, the proof above only needs information about initial submodules, which have some additional properties. It is plausible that initial submodules (of say submodules of  $P_{\mathbf{OI}^{(2)}, 0}$ ) are finitely generated even if the general monomial submodule is not.  $\square$

In the final section, we will begin an investigation of degree 2 tca's.

## 5. Degree 2 tca's

Reference: [arXiv:1302.5859](https://arxiv.org/abs/1302.5859)

We now focus on tca's generated in degree 2. Work over char. 0. There are 2 irreducible representations of  $\Sigma_2$ : trivial and sign, and the corresponding tca's they generate are

$$\text{Sym}(\text{Sym}^2 \mathbf{C}^\infty), \quad \text{Sym}\left(\bigwedge^2 \mathbf{C}^\infty\right)$$

(described as polynomial functors).

We begin with the following observation:

$$\text{Sym}(\text{Sym}^2) = \bigoplus_{\lambda} \mathbf{S}_{2\lambda}, \quad \text{Sym}\left(\bigwedge^2\right) = \bigoplus_{\lambda} \mathbf{S}_{(2\lambda)^+}$$

so they are multiplicity-free. Note that  $\text{Sym}(\text{Sym}^2)$  is the same as  $\text{Sym}(\mathbf{k}[2])$  defined earlier.

**Proposition 5.1** (various authors). *The ideal generated by  $\mathbf{S}_{2\lambda}$  contains all  $\mathbf{S}_{2\mu}$  with  $\mu \supseteq \lambda$ . In particular, every ideal of  $\text{Sym}(\text{Sym}^2)$  is finitely generated. Similar statement for  $\text{Sym}(\wedge^2)$ .*

It is not known if each projective module  $\text{Sym}(\text{Sym}^2) \otimes \mathbf{S}_\lambda$  is noetherian; similarly for  $\text{Sym}(\wedge^2) \otimes \mathbf{S}_\lambda$ . Using the transpose operation, the two problems are interconnected.

We focus on  $\text{Sym}(\text{Sym}^2)$  for simplicity of exposition.

**5.1. Infinite orthogonal group.** Set  $\mathbf{V} = \mathbf{C}^\infty = \bigcup_{n \geq 0} \mathbf{C}^n$  with basis  $e_i$  and equip it with orthogonal form  $\omega$ :

$$\omega\left(\sum_{i \geq 1} a_i e_i, \sum_{j \geq 1} b_j e_j\right) = \sum_{k \geq 1} (a_{2k-1} b_{2k} + a_{2k} b_{2k-1}).$$

Let  $\mathbf{O}(\infty) \subset \mathbf{GL}(\infty)$  be subgroup preserving  $\omega$ . Then  $\mathbf{O}(\infty) = \bigcup_{n \geq 1} \mathbf{O}(2n)$ .

A representation of  $\mathbf{O}(\infty)$  is **algebraic** if it is a subquotient of a finite direct sum of  $\mathbf{V}^{\otimes n}$ . Denote the category  $\text{Rep}(\mathbf{O})$ .

**Remark 5.2.** The form  $\omega$  provides a map  $\text{Sym}^2(\mathbf{V}) \rightarrow \mathbf{C}$  which does not split. So  $\text{Rep}(\mathbf{O})$  is not semisimple.  $\square$

A morphism  $S \rightarrow T$  in  $\mathbf{FI}^{(2)}$  is an injection  $f: S \rightarrow T$  and a perfect matching on  $T \setminus f(S)$ ; this gives a map  $\mathbf{V}^{\otimes T} \rightarrow \mathbf{V}^{\otimes S}$ : apply  $\omega$  to the 2-element subsets of  $T \setminus f(S)$  and identify  $\mathbf{V}^{\otimes f(S)} = \mathbf{V}^{\otimes S}$ . This defines a functor

$$\mathcal{K}: (\mathbf{FI}^{(2)})^{\text{op}} \rightarrow \text{Rep}(\mathbf{O}).$$

Let  $\text{Rep}_{\mathcal{C}}$  denote the category of functors  $\mathcal{C} \rightarrow \text{Vec}_{\mathbf{C}}$ . Use this to define

$$\begin{aligned} \Phi_{\mathcal{K}}: (\text{Rep}_{(\mathbf{FI}^{(2)})^{\text{op}}})^{\text{op}} &\rightarrow \text{Rep}(\mathbf{O}) \\ M &\mapsto \text{Hom}(M, \mathcal{K}) \end{aligned}$$

where the  $\text{Hom}$  is the set of natural transformations. Also define

$$\begin{aligned} \Psi_{\mathcal{K}}: \text{Rep}(\mathbf{O}) &\rightarrow (\text{Rep}_{(\mathbf{FI}^{(2)})^{\text{op}}})^{\text{op}} \\ N &\mapsto \text{Hom}_{\mathbf{O}}(N, \mathcal{K}). \end{aligned}$$

Note we have an identification

$$\begin{aligned} (\text{Rep}_{(\mathbf{FI}^{(2)})^{\text{op}}})^{\text{op}} &= \text{Rep}_{\mathbf{FI}^{(2)}} \\ M &\mapsto (S \mapsto M(S)^*). \end{aligned}$$

**Theorem 5.3.**  $\Phi_{\mathcal{K}}$  defines an equivalence between the finite length objects of  $\text{Rep}_{\mathbf{FI}^{(2)}}$  and  $\text{Rep}(\mathbf{O})$  with inverse  $\Psi_{\mathcal{K}}$ . In fact, identifying  $\text{Rep}_{\mathbf{FI}^{(2)}} \simeq \text{Mod}_{\text{Sym}(\text{Sym}^2 \mathbf{C}^\infty)}$ , this defines a monoidal equivalence.

*Proof.* Sketch:

- (1)  $\Phi_{\mathcal{K}}$  preserves simple objects (discussed next)
- (2) Both  $\Phi_{\mathcal{K}}$  and  $\Psi_{\mathcal{K}}$  are left-exact; if  $M$  is finite length, then  $\text{len}(M) \geq \text{len}(\Phi_{\mathcal{K}}(M))$  and similarly for  $\Psi_{\mathcal{K}}$ .
- (3) There are injective maps  $\text{id}_{\text{Rep}(\mathbf{O})} \rightarrow \Phi_{\mathcal{K}} \Psi_{\mathcal{K}}$  and  $\text{id}_{\text{Rep}_{\mathbf{FI}^{(2)}}} \rightarrow \Psi_{\mathcal{K}} \Phi_{\mathcal{K}}$ . Since length can't go up, the map  $M \rightarrow \Psi_{\mathcal{K}} \Phi_{\mathcal{K}} M$  is an isomorphism if  $M$  is finite length.

We omit discussion of monoidal structure.  $\square$



The simple objects of  $\text{Rep}(\mathbf{O})$  are the irreducible representations  $\mathbf{M}_\lambda$  of  $\Sigma_n$  where  $n = |\lambda|$ . The image of  $\mathbf{M}_\lambda$  in  $\mathcal{K}$  must land in the intersection of the kernels of  $\mathbf{V}^{\otimes n} \rightarrow \mathbf{V}^{\otimes(n-2)}$  (call this  $\mathbf{V}_0^{\otimes n}$ ).

**Lemma 5.4.**  $\mathbf{S}_{[\lambda]}\mathbf{V} := \text{Hom}_{\Sigma_n}(\mathbf{M}_\lambda, \mathbf{V}_0^{\otimes n})$  is a simple object of  $\text{Rep}(\mathbf{O})$ . Furthermore,  $\mathbf{S}_{[\lambda]}\mathbf{V}$  and  $\mathbf{S}_{[\mu]}\mathbf{V}$  are isomorphic if and only if  $\lambda = \mu$ .

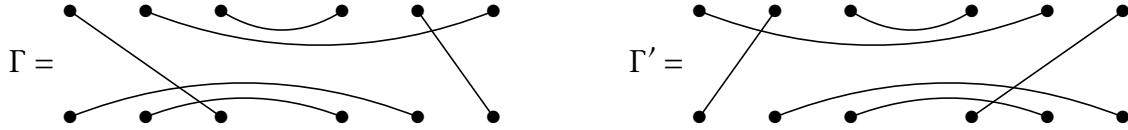
*Proof.* When  $\dim \mathbf{V} < \infty$ , a classical construction of Weyl says that  $\text{Hom}_{\Sigma_n}(\mathbf{M}_\lambda, \mathbf{V}_0^{\otimes n})$  is an irreducible representation of  $\mathbf{O}(\mathbf{V})$  provided it is nonzero. We omit the technical details.  $\square$

An analogy: finite length algebraic representations of  $\mathbf{O}(\infty)$  play the role of polynomial functors and  $\mathbf{FI}^{(2)}$  plays the role of  $\mathbf{FB}$ : so  $\mathcal{K}$  provides a generalized Schur–Weyl duality.

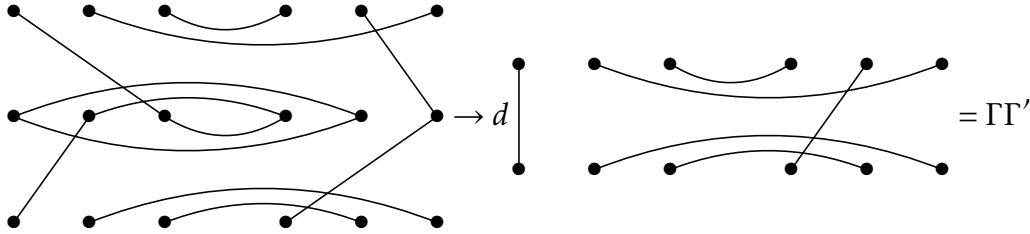
There are two versions of Schur–Weyl duality for  $\mathbf{O}(\mathbf{V})$  when  $\dim \mathbf{V} < \infty$ . The first we have seen:

$$\mathbf{V}_0^{\otimes n} = \bigoplus_{\lambda} \mathbf{M}_\lambda \boxtimes \mathbf{S}_{[\lambda]}\mathbf{V} \quad (\Sigma_n \times \mathbf{O}(\mathbf{V})\text{-equivariant})$$

The second involves the Brauer algebra  $\mathcal{B}_{d,n}$  ( $d = \dim \mathbf{V}$ ). Objects are perfect matchings on  $[n] \sqcup [n]$  represented pictorially as (for  $n = 6$ ):



The product is obtained via top-down concatenation. For each loop, multiply by  $d$ :



The second version of Schur–Weyl duality is

$$\mathbf{V}^{\otimes n} = \bigoplus_{\lambda} \mathbf{N}_\lambda \boxtimes \mathbf{S}_{[\lambda]}\mathbf{V} \quad (\mathcal{B}_{d,n} \times \mathbf{O}(\mathbf{V})\text{-equivariant})$$

where  $\mathbf{N}_\lambda$  are irreducibles of  $\mathcal{B}_{d,n}$ .

Note that  $\mathbf{FI}^{(2)}$  is built out of Brauer-like diagrams.

There is a similar story with the symplectic group. Also with  $\mathbf{FI}$ : the orthogonal group is replaced by the general affine group (this is the stabilizer of a nonzero linear map  $\mathbf{V} \rightarrow \mathbf{C}$  sending  $e_i \mapsto \delta_{1,i}$ ).

**5.2. Next steps.** The equivalence  $\Phi_{\mathcal{K}}$  gives some handle on the finite length modules of  $A = \text{Sym}(\text{Sym}^2 \mathbf{C}^\infty)$ .

Thinking geometrically, the space of infinite symmetric matrices is stratified by rank. Let  $\text{Mod}_A^{\leq r}$  be the category of finitely generated modules set-theoretically supported on the rank  $r$  locus. These modules are bounded and the quotients  $\text{Mod}_A^{\leq r} / \text{Mod}_A^{\leq r-1}$  seem to behave well (we have just described the case  $r = 0$ ).