Infinite-dimensional combinatorial commutative algebra

Steven Sam

University of California, Berkeley

September 20, 2014

Basic theme: there are many "axes" in commutative algebra which are governed by algebraic structures

- (obvious) If M is a graded module, consider M_i as i varies
- $\operatorname{Tor}_{i}^{R}(M, \mathbf{k})^{*}$ as *i* varies; module over Ext algebra
- (vague) M(n) module over algebra A(n) for special classes, have compatibilities between A(n) and A(n') and between M(n) and M(n').
 (sources: Hilbert scheme of n points in A², moduli space of n points on P¹, equivariant pure free resolutions over C[x₁,...,x_n], etc.)

Today's talk: some structures arising from (secants of) Segre embeddings and finiteness theorems arising from them

Work of Draisma-Kuttler

Theorem (Draisma-Kuttler)

Fix r. There is a constant d(r) so that the rth secant variety of a Segre embedding is set-theoretically defined by equations of degree $\leq d(r)$.

Alternatively: "Bounded rank tensors are cut out by bounded degree equations"

The main point: it doesn't matter how many projective spaces we take or what their dimensions are – we can find d(r) that works for all of them!

Examples: d(1) = 2, d(2) = 3 (Landsberg–Manivel, Raicu), d(3) = 4 (Yang Qi), $d(4) \ge 9$ (Strassen)

Infinite-dimensional tensors

First step: given r, it suffices to work with projective spaces of some dim. n-1 (depending only on r). Fix vector space V of dim. n and functional $\varphi: V \to \mathbf{k}$. Define

$$V^{\otimes p} \to V^{\otimes (p-1)}$$

 $v_1 \otimes \cdots \otimes v_p \mapsto \varphi(v_p) v_1 \otimes \cdots \otimes v_{p-1}.$

Set $V^{\otimes \infty} = \lim_{p} V^{\otimes p}$.

Two subvarieties of interest in $V^{\otimes p}$: $X_p^{\leq r}$: rth secant variety of Segre $Y_p^{\leq r}$: flattening variety — tensors such that all flattenings (into matrices) have rank at most r

Set
$$X_{\infty}^{\leq r} = \lim_{p} X_{p}^{\leq r}$$
 and $Y_{\infty}^{\leq r} = \lim_{p} Y_{p}^{\leq r}$.
Then $X_{\infty}^{\leq r} \subseteq Y_{\infty}^{\leq r}$.

Some symmetries

There is more structure on $X_{\infty}^{\leq r}$ and $Y_{\infty}^{\leq r}$: the group $G_p := S_p \ltimes \mathbf{GL}(V)^p$ acts on $V^{\otimes p}$ and preserves $X_p^{\leq r}$ and $Y_p^{\leq r}$.

There are natural embeddings $G_{\rho} \subset G_{\rho+1}$ compatible with our inverse system, so $G_{\infty} = \bigcup_{\rho} G_{\rho}$ acts on $V^{\otimes \infty}$ and preserves $X_{\infty}^{\leq r}$ and $Y_{\infty}^{\leq r}$.

Theorem (Draisma–Kuttler)

 $Y_{\infty}^{\leq r} \subset V^{\otimes \infty}$ is cut out by finitely many G_{∞} -orbits of polynomials (certain determinants of size r + 1).

Let Π be a monoid acting on a topological space X. Then X is Π -**Noetherian** if every descending chain of Π -stable closed subsets $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ eventually stabilizes.

If X is a Π -Noetherian variety, then every Π -stable subvariety is cut out set-theoretically by finitely many Π -orbits of equations.

Theorem (Draisma-Kuttler)

 $Y_{\overline{\infty}}^{\leq r}$ is G_{∞} -Noetherian.

Combining this with the previous result, we get

Theorem

 $X_{\infty}^{\leq r} \subset V^{\otimes \infty}$ is cut out set-theoretically by finitely many G_{∞} -orbits of equations. In particular, there is a finite bound on their degrees.

How about ideals? Look to Segre embeddings for motivation.

The simplest Segre embedding:

$${\sf P}^1 imes {\sf P}^1 \subset {\sf P}^3$$

Interpretation: 2×2 matrices of rank ≤ 1 , so cut out by single determinant equation $x_{11}x_{12} - x_{12}x_{21}$.

More convenient to write this as

$$\mathsf{P}(V_1) imes \mathsf{P}(V_2) \subset \mathsf{P}(V_1 \otimes V_2)$$

where $\dim(V_1) = \dim(V_2) = 2$.

Two ways to get equations from smaller Segres:

• Flattening (reduce number of projective spaces):

$$\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \times \mathbf{P}(V_{n+1}) \subset \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n \otimes V_{n+1}) \\ \subset \mathbf{P}(V_1 \otimes \cdots \otimes (V_n \otimes V_{n+1}))$$

(can be done in many ways)

• Functoriality (reduce dimensions of projective spaces). Given $V_i \rightarrow V'_i$, have

In both cases, can pullback equations. Basic observation: all equations can be generated from the 2×2 determinant using these operations.

Δ -modules

 Δ -modules axiomatize the two operations (flattening and functoriality) that we have just seen. We define them shortly.

The previous discussion can be summarized as: the equations of the Segre embedding is a principal Δ -module (generated by 1 element)

We understand the equations quite well, but Δ -modules apply to other situations:

- Any family of varieties closed under flattening and functoriality fit in the framework. The set of such families is closed under join and taking tangents (so includes higher secants and tangential varieties of Segres — poorly understood!)
- They extend beyond equations to arbitrary order syzygies (more precisely, higher Tor groups of Segre embeddings poorly understood!)

An abstract approach is cleanest. Fix a field ${\bf k}.$ Let ${\rm Vec}^\Delta$ be the following category:

- Objects are pairs (*I*, {*V_i*}_{*i*∈*I*}), *I* a finite set and *V_i* a finite-dimensional vector space.
- A morphism (I, {V_i}) → (J, {W_j}) is a surjection f: J → I together with linear maps V_i → ⊗_{j∈f⁻¹(i)} W_j.

Intuition: the surjection f encodes **flattenings** and the linear maps encode **functoriality**.

Definition

A Δ -module is a *polynomial* functor $\operatorname{Vec}^{\Delta} \to \operatorname{Vec}$.

(Intuitively, polynomial just means that morphisms transform like polynomial functions.)

Examples of Δ -modules

- Tautological example: $(I, \{V_i\}) \mapsto \bigotimes_{i \in I} V_i$
- Ambient space: $R: (I, \{V_i\}) \mapsto \operatorname{Sym}^{\bullet}(\bigotimes_{i \in I} V_i)$
- Segre: $S: (I, \{V_i\}) \mapsto \bigoplus_{d \ge 0} (\bigotimes_{i \in I} \operatorname{Sym}^d(V_i))$
- Tor modules: $T_i: (I, \{V_i\}) \mapsto \operatorname{Tor}_i^{R(I, \{V_i\})}(\mathbf{k}, S(I, \{V_i\})).$
- coordinate rings of secants, tangents, ...
- Tor modules of secants, tangents, ...

Finite generation

A Δ -module is a *polynomial* functor $F : \operatorname{Vec}^{\Delta} \to \operatorname{Vec}$.

An **element** of *F* is an element $x \in F(I, \{V_i\})$ for some object $(I, \{V_i\}) \in \text{Vec}^{\Delta}$.

The submodule of F generated by a collection of elements is the smallest submodule containing all of them.

F is **finitely generated** if it can be generated by finitely many elements.

Theorem (Sam–Snowden)

Tor modules of Segre embeddings are finitely generated Δ -modules.

Originally proven by Snowden in characteristic 0.

Noetherian property

F is **Noetherian** if every submodule is finitely generated.

Theorem (Sam–Snowden)

Finitely generated Δ -modules are Noetherian. In particular, they have resolutions by finitely generated projective Δ -modules.

The first part was originally proven by Snowden in characteristic 0 under further assumptions.

So the following result seems to be new, even in characteristic 0:

Theorem (Sam–Snowden)

Tor modules of Segre embeddings are finitely presented Δ -modules.

Lingering questions

• Fix *r*. Are the equations of the *r*th secant variety of the Segre embedding a finitely generated Δ-module?

It can be shown that, for fixed d, the degree d equations form a finitely generated Δ -module, so we are asking: is the ideal defined by equations of bounded degree (stronger property than provided by Draisma-Kuttler).

• How about the Tor modules? If we know that the equations of the *r*th secant variety of the Segre are defined in bounded degree, does this imply the same for all higher Tor modules?

For the Segre, this follows from the existence of a quadratic Gröbner basis.

 Analogy: Δ-modules are like vector spaces and we really want to study Δ-algebras like (I, {V_i}) → Sym(⊗_{i∈I} V_i).

- Hilbert series: Use characters of the general linear group to encode a Δ-module into a Hilbert series. We can show that they are rational functions.
- **Regularity:** Minimal projective resolutions are tricky to define due to the action of general linear groups. 2 ways to fix this:
 - work in char. 0 where representation theory is semisimple,
 - $\bullet\,$ modify the category so that group actions don't appear

After fixing this, we can define Castelnuovo–Mumford regularity. *Is it finite?*

 Krull dimension: Gabriel defined Krull dimension objects in any Abelian category A: the zero object has Kdim −1. Let A^{≤d} be the subcategory of objects of Kdim ≤ d. An object has Kdim ≤ d + 1 if its image in A/A^{≤d} has finite length. Can we compute this for Δ-modules? Is it connected to combinatorial properties of Hilbert series?