"Gröbner methods for generic representation theory and the artinian conjecture"
GDR conference: Algebraic topology and applications
Clermont-Ferrand, October 22, 2014
Steven Sam (joint with Andrew Putman, Andrew Snowden)

1. Definitions

Lannes–Schwartz artinian conjecture:

Theorem 1.1 (Putman, Sam, Snowden). *Let $\mathbf{F}_q$ be a finite field and let $\mathrm{Vec}_{\mathbf{F}_q}$ be the category of finite-dimensional vector spaces. Every finitely generated functor $\mathrm{Vec}_{\mathbf{F}_q} \to \mathrm{Vec}_{\mathbf{F}_q}$ is noetherian.*

Some definitions:

Let $\mathcal{C}$ be a category with set of isoclasses $|\mathcal{C}|$ and $\mathbf{k}$ a ring. A functor $F\colon \mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ can be encoded as a graded $\mathbf{k}$-module $\bigoplus_{x\in|\mathcal{C}|} F(x)$ with operations coming from morphisms in $\mathcal{C}$:

$$\mathbf{k}[\mathrm{Hom}_{\mathcal{C}}(x,y)] \otimes F(x) \to F(y).$$

A **subfunctor** of F is a graded submodule closed under these operations.

Given subset $S \subset \bigoplus F(x)$, the subfunctor generated by S is smallest subfunctor containing S. F is **finitely generated** if it has a finite generating set.

F is **noetherian** if every subfunctor of F is finitely generated.

2. Motivation

Let $\mathcal{F}(q)$ be category of functors $\mathrm{Vec}_{\mathbf{F}_q} \to \mathrm{Vec}_{\mathbf{F}_q}$.

For vector space V, get projective $P_V \in \mathcal{F}(q)$ given by $W \mapsto \mathbf{F}_q[\mathrm{Hom}(V,W)]$

Finite generation is equivalent to being a quotient of $P_{V_1} \oplus \cdots \oplus P_{V_r}$.

Noetherian is equivalent to every finitely generated object having projective resolution by finitely generated P_V.

(This is useful since various subcategories have come up:)

2.1. Steenrod algebra.

For simplicity, $q = p$ is prime.

Let $\mathcal{F}_\omega(q)$ be the subcategory of locally finite functors $\mathrm{Vec}_{\mathbf{F}_q} \to \mathrm{Vec}_{\mathbf{F}_q}$ (i.e., union of its finite length subfunctors).

Let A be mod. p Steenrod algebra and let $\mathcal{U}$ be the category of unstable A-modules. There is a functor

$$\begin{aligned} f\colon \mathcal{U} &\to \mathcal{F}_\omega(p) \\ M &\mapsto (V \mapsto \mathrm{Hom}_{\mathcal{U}}(M, \mathrm{H}^*\mathrm{B}V)') \end{aligned}$$

($'$ is continuous dual) which induces an equivalence $\mathcal{U}/\mathcal{N} \simeq \mathcal{F}_\omega(p)$. Here $\mathcal{N} = \{x \in \mathcal{U} \mid f(x) = 0\}$ is the category of nilpotent modules. (Result of Henn, Lannes, Schwartz)

2.2. Cohomology.

Pirashvili identified $\mathrm{Tor}_*^{\mathcal{F}(q)}(\mathrm{id},\mathrm{id})$ with the topological Hochschild homology of $\mathbf{F}_q$, so general Tor (or Ext) can be interpreted as THH with twisted coefficients.

Theorem 2.1 (Betley, Suslin). *For finite length F and F',*

$$\mathrm{Ext}^*_{\mathcal{F}(q)}(F,F') \cong \mathrm{Ext}^*_{\mathbf{GL}_k(\mathbf{F}_q)}(F(\mathbf{F}_q^k), F'(\mathbf{F}_q^k))$$

for $k \gg 0$.

3. Proof idea: Gröbner degenerations

Goal: given $\mathcal{C}$ and $\mathbf{k}$, how to prove that $\mathcal{C}$ is noetherian? i.e., show f.g. functors $\mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ are noetherian?

General strategy:

- To prove that f.g. functors $\mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ are noetherian, enough to do so for projectives $P_x : y \mapsto \mathbf{k}[\mathrm{Hom}_{\mathcal{C}}(x, y)]$ since they generate the category.
- A set S of morphisms $\{x \to z_i\}$ generates a subfunctor of P_x called a **monomial subfunctor**.
- In many instances, it is easy to prove that monomial subfunctors are f.g. (this becomes a combinatorial problem)
- Want to reduce study of subfunctors to monomial subfunctors. General technique: **Gröbner degenerations** (borrowed from commutative algebra / algebraic geometry)

  The idea is that morphisms $f : x \to z$ are monomials in P_x, and so given a **term ordering** $<$ on monomials, can define initial terms
  $$\mathrm{init}(\sum a_f \cdot f) = \max\{f \mid a_f \neq 0\}$$
  and initial subfunctors
  $$\mathrm{init}(M) = \mathbf{k}\{\mathrm{init}(f) \mid f \in M\}.$$
  But $<$ should satisfy strong property: $f < g$ implies $hf < hg$.

Problem: for $\mathcal{C} = \mathrm{Vec}_{\mathbf{F}_q}$, most morphisms differ by an automorphism, so there aren't many monomial submodules.

There is a further reduction technique: Given
$$\Phi : \mathcal{C}' \to \mathcal{C},$$
can pullback functors.

If we assume that pullback Φ^* preserves finite generation and Φ is essentially surjective, then we can reduce problem to showing $\mathcal{C}'$ is noetherian.

For $\mathcal{C} = \mathrm{Vec}_{\mathbf{F}_q}$, define $\mathcal{C}'$ to be the category whose objects are ordered finite sets and a morphism $S \to T$ is a surjection $f : T \to S$ satisfying $\min f^{-1}(i) < \min f^{-1}(j)$ for $i < j \in S$.

Define $\Phi(S) = \mathrm{Hom}_{\mathbf{F}_q}(\mathbf{F}_q[S], \mathbf{F}_q)$; exercise to show Φ preserves finite generation.

Now $\mathcal{C}'$ has many monomial modules (note its automorphism groups are trivial) and can carry out Gröbner degeneration program.

Remark 3.1. Previous work on finiteness of functor categories relied on ***polynomial*** property, i.e., focused on functors annihilated by certain shift operator. This works for category of finite sets and injections (Church, Ellenberg, Farb, Nagpal) but cannot address larger functors. □

4. Applications: Homological stability

The same ideas apply to similar contexts.

Let R be a commutative ring.

$\mathbf{VIC}_R$ is category of finite rank free R-modules; a morphism $V \to W$ is a pair of maps $V \to W \to V$ whose composition is identity.

$\mathbf{SI}_R$ is category of finite rank free symplectic R-modules; morphism $V \to W$ is an injection compatible with forms.

Theorem 4.1 (Putman, Sam). *Let* $\mathbf{k}$ *be commutative noetherian ring and assume* R *is finite.*

Finitely generated $\mathbf{VIC}_R$ *and* $\mathbf{SI}_R$*-modules are noetherian.*

Corollary 4.2 (Putman, Sam). *Let* $\mathbf{k}$ *be commutative noetherian ring and assume* R *is finite.*

Let $F\colon \mathbf{VIC}_R \to \mathrm{Mod}_{\mathbf{k}}$ *be finitely generated. Then for each* $i \geq 0$, $\mathrm{H}_i(\mathbf{GL}_n(R); F(R^n))$ *is independent of* n *for* $n \gg 0$.

Let $F\colon \mathbf{SI}_R \to \mathrm{Mod}_{\mathbf{k}}$ *be finitely generated. Then for each* $i \geq 0$, $\mathrm{H}_i(\mathbf{Sp}_{2n}(R); F(R^{2n}))$ *is independent of* n *for* $n \gg 0$.

Some more applications:

Let K be a number field, $\mathcal{O}_K$ its ring of integers, pick nonzero ideal $\alpha \subset \mathcal{O}_K$.

It is known that homology of $\mathbf{Sp}_{2n}(\mathcal{O}_K)$ stabilizes for large n, but not for *congruence subgroups.*

Define group-valued functor on $\mathbf{SI}_{\mathcal{O}_K}$ by $V \mapsto \mathbf{Sp}(\alpha, V)$ where $\mathbf{Sp}(\alpha, V)$ is kernel of $\mathbf{Sp}(V) \to \mathbf{Sp}(V \otimes_{\mathcal{O}_K} \mathcal{O}_K/\alpha)$.

Use this to define functor

$$\begin{aligned} \mathcal{H}_i(\alpha, \mathbf{k})\colon \mathbf{SI}_{\mathcal{O}_K/\alpha} &\to \mathrm{Mod}_{\mathbf{k}} \\ V &\mapsto \mathrm{H}_i(\mathbf{Sp}(\alpha, V); \mathbf{k}) \end{aligned}$$

Theorem 4.3 (Putman, Sam). $\mathcal{H}_i(\alpha, \mathbf{k})$ *is finitely presented. In particular, the* i*th homology of* $\mathbf{Sp}(\alpha, \mathcal{O}_K^{2n})$ *admits a finite description as we vary over all* n.

There are similar results for congruence subgroups of $\mathbf{GL}_n(\mathcal{O}_K)$ but more notation needed.

Also can make statements for congruence subgroups for mapping class groups and automorphism groups of free groups.