Representation stability and finite linear groups
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1. Homological stability

A sequence (of groups, spaces, etc.) $X_1 \to X_2 \to X_3 \to \cdots$ satisfies **homological stability** if for each $i \geq 0$, the maps on homology $\mathrm{H}_i(X_n) \to \mathrm{H}_i(X_{n+1})$ are isomorphisms for $n \gg 0$.

To make this concrete, consider the following example:
$Y_n = \{(x_1, \dots, x_n) \in \mathbf{C}^n \mid x_i \neq x_j \text{ if } i \neq j\}$.
$X_n = Y_n/\Sigma_n$

Remark 1.1. Let B_n be the braid group on n strands.
The homology of B_n is the same as the homology of the following topological space: X_n is the space of unordered n-tuples of distinct complex numbers (X_n is a $K(B_n, 1)$) □

There is an inclusion $B_n \to B_{n+1}$ as first n strands.
(This corresponds to a map $X_n \to X_{n+1}$ which "adds a point near infinity".)

Theorem 1.2 (Arnol'd). *$\{B_n\}$ satisfies homological stability.*

The stable value coincides with the homology of $\Omega^2_0 \mathrm{S}^2$.
There is a surjection $B_n \to \Sigma_n$, let PB_n be the kernel (pure braid group). It turns out that Y_n is a $K(PB_n, 1)$, so the homology of both coincide.
However, $\mathrm{H}_1(PB_n) \cong \mathbf{Z}^{\binom{n}{2}}$, so $\{PB_n\}$ does not satisfy homological stability.

Is there a way to fix this? Two remarks:

(1) (By general homological algebra,) Σ_n acts on $\mathrm{H}_*(PB_n)$, so should use this action. (In fact, $\mathrm{H}_1(PB_n)$ is the permutation representation of Σ_n on 2-element subsets of $\{1, \dots, n\}$.)
(2) (Another interpretation:) think of $\mathrm{H}_i(X_n) \to \mathrm{H}_i(X_{n+1})$ as "multiplication by t" so that $\bigoplus_n \mathrm{H}_i(X_n)$ is a module over $\mathbf{Z}[t]$. Replacing $\mathbf{Z}$ by a field $\mathbf{k}$, homological stability says that this module is a direct sum of torsion and free $\mathbf{k}[t]$-modules, which is automatic for *finitely generated* $\mathbf{k}[t]$-modules. (Same idea is important in *persistent homology*)

Suggestion: replace $\mathbf{Z}[t]$ by a larger "ring" so that $\bigoplus_n \mathrm{H}_i(PB_n)$ is finitely generated.

2. Representation stability

(What do we have?) Set $[n] = \{1, \dots, n\}$. For every injection $f\colon [n] \to [n+k]$ there is an inclusion $PB_n \to PB_{n+k}$ which induces $f_*\colon \mathrm{H}_i(PB_n) \to \mathrm{H}_i(PB_{n+k})$.
So modules M over our ring should have grading $M = \bigoplus_n M_n$ and for every injective map $f\colon [n] \to [n+k]$, have a map $f_*\colon M_n \to M_{n+k}$ that satisfy $(fg)_* = f_* g_*$.
These are **FI-modules**: functors from FI to abelian groups (or $\mathbf{k}$-modules for a ring $\mathbf{k}$).
Now we intuitively see what it means to be finitely generated, and it turns out:

For $i \geq 0$, $\bigoplus_n \mathrm{H}_i(PB_n)$ is a finitely generated FI-module.

(We can use the functorial language to define submodules, quotients, etc., or use the more intuitive language of graded groups with operators.)

(When calculating, one needs to take kernels and quotients.)

Finite generation always passes to quotients. But it need not pass to submodules (kernels): if it does, then the module is **noetherian**.

(The following is the fundamental algebraic property that allows one to prove finite generation statements:)

Theorem 2.1 (Church–Ellenberg–Farb–Nagpal). *If $\mathbf{k}$ is a noetherian ring, then finitely generated FI-modules over $\mathbf{k}$ are noetherian (i.e., finite generation is inherited by submodules).*

Can replace FI by other categories $\mathcal{C}$ (to handle other settings where $\mathrm{H}_i(X_n)$ has an action of some group Γ_n.)

(Going back to $\mathbf{k}[t]$, we have a question:)

Question 2.2. *What is the structure of representation stability? i.e., Classify finitely generated $\mathcal{C}$-modules, or at least classify them modulo torsion (= finite length) $\mathcal{C}$-modules.*

Sam–Snowden: gave classification for finitely generated FI-modules modulo torsion ones when $\mathbf{k}$ is a field of char. 0. (Positive char. seems to be difficult.)

3. Finite linear groups

(Linear analogues of above story:)

Pick integer $\ell \geq 2$. Define congruence subgroups

$\mathbf{SL}_n(\mathbf{Z},\ell) = \ker(\mathbf{SL}_n(\mathbf{Z}) \to \mathbf{SL}_n(\mathbf{Z}/\ell))$

$\mathbf{Sp}_{2n}(\mathbf{Z},\ell) = \ker(\mathbf{Sp}_{2n}(\mathbf{Z}) \to \mathbf{Sp}_{2n}(\mathbf{Z}/\ell))$

$\mathrm{Aut}(F_n,\ell) = \ker(\mathrm{Aut}(F_n) \to \mathbf{GL}_n(\mathbf{Z}/\ell))$

$\mathrm{Mod}_{g,1}(\ell) = \ker(\mathrm{Mod}_{g,1} \to \mathbf{Sp}_{2g}(\mathbf{Z}/\ell))$

$\{\mathbf{SL}_n(\mathbf{Z})\}$ and $\{\mathbf{Sp}_{2n}(\mathbf{Z})\}$ satisfy homological stability but congruence subgroups do not.

Remark 3.1. Moduli space of principally polarized n-dimensional complex abelian varieties is quotient $\mathfrak{H}_n/\mathbf{Sp}_{2n}(\mathbf{Z})$ (where $\mathfrak{H}_n$ is space of $n \times n$ symmetric complex matrices with positive-definite imaginary part) and $\mathfrak{H}_n/\mathbf{Sp}_{2n}(\mathbf{Z},\ell)$ is moduli space of ppav with level ℓ structure.

Other examples in our framework:

- $\mathbf{SL}_n(\mathbf{Z}) \leftrightarrow$ volume 1 flat metrics on n-torus
- Aut(free group) $\leftrightarrow$ "Auter space" = moduli of marked metric graphs with basepoint
- mapping class group of a surface $\leftrightarrow$ moduli of curves □

(We can use a similar setup, but category $\mathcal{C}$ is now more complicated.)

Let R be a commutative ring. $\mathrm{VIC}(R)$ is the category of f.g. free R-modules and a morphism $R^n \to R^{n+k}$ is an R-linear injection $f\colon R^n \to R^{n+k}$ plus a direct sum decomposition $R^{n+k} = f(R^n) \oplus C$ (with $C \cong R^k$). (This corresponds to a choice of embedding $\mathbf{GL}_n(R)$ into $\mathbf{GL}_{n+k}(R)$.)

Given subgroup $U \subset R^\times$, let $\mathrm{VIC}(R,U)$ be the same except that $\det(f) \in U$ whenever $k = 0$ above.

There is also a symplectic version $\mathrm{SI}(R)$.

Theorem 3.2 (Putman–Sam). *If R is finite and $\mathbf{k}$ is noetherian, then finitely generated $\mathcal{C}$-modules over $\mathbf{k}$ are noetherian for $\mathcal{C} \in \{\mathrm{VIC}(R,U), \mathrm{SI}(R)\}$.*

Remark 3.3. We prove a stronger statement that f.g. modules have finite "Gröbner bases" building off work with Snowden (and older work of Hilbert, Higman, Hillar–Sullivant, ...), □

Corollary 3.4 (Putman–Sam). (a) $\bigoplus_n \mathrm{H}_i(\mathbf{SL}_n(\mathbf{Z},\ell))$ *is a finitely generated* $\mathrm{VIC}(\mathbf{Z}/\ell, 1)$*-module*
(b) $\bigoplus_n \mathrm{H}_i(\mathbf{Sp}_{2n}(\mathbf{Z},\ell))$ *is a finitely generated* $\mathrm{SI}(\mathbf{Z}/\ell)$*-module*

(Earlier work of Putman and CEF show a stronger version of (a): it is a finitely generated FI-module)

(Proof idea: generalize a spectral sequence argument of Quillen)

(This theorem extends to congruence subgroups of Aut(free group) and mapping class groups of surfaces and more general arithmetic groups)

(As a consequence, we prove a conjecture coming from the study of the Steenrod algebra and algebraic K-theory)

Lannes–Schwartz artinian conjecture:

Corollary 3.5 (Putman, Sam, Snowden). *Let* $\mathcal{C}$ *be the category of finite dimensional* $\mathbf{F}_q$*-vector spaces. Then finitely generated* $\mathcal{C}$*-modules over* $\mathbf{F}_q$ *are noetherian.*

Corollary 3.6 (Putman–Sam). *If* $\mathbf{k}$ *is noetherian,* R *is finite, and* M *is a f.g.* $\mathrm{VIC}(R, U)$*-module, then* $\mathrm{H}_i(\mathbf{GL}_n(R, U); M_n) \to \mathrm{H}_i(\mathbf{GL}_{n+1}(R, U); M_{n+1})$ *is an isomorphism for* $n \gg 0$ *(twisted homological stability). Similar statement for* $\mathrm{SI}(R)$*-modules.*

4. Other topics

- The structure of representation stability for these linear categories $\mathcal{C}$ is much more challenging than for FI: the category of FI-modules (over a field) has "Krull dimension 1" whereas the linear versions have infinite Krull dimension.
- FI-modules sit in the world of twisted commutative algebras, which connects to geometry of tensors
- Understanding $\mathrm{VIC}(\mathbf{Z})$ and $\mathrm{SI}(\mathbf{Z})$ is important for study of Torelli groups
- Artinian conjecture is first step towards more general finiteness properties of unstable modules over the Steenrod algebra.