Twisted commutative algebras
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Steven Sam (joint with Andrew Snowden)

1. Motivating example: determinantal varieties

- E, F are $\mathbf{k}$-vector spaces of dimensions e, f, assume $e \geq f$.
- $X = \mathrm{Hom}(E, F)$ affine space of linear maps $E \to F$
- $A = \mathrm{Sym}(E \otimes F^*)$ = coordinate ring of X.
- $X(r)$ = subvariety of rank $\leq r$ matrices
- $A(r)$ = coordinate ring of $X(r)$

Problem: Calculate minimal free resolution $\mathbf{F}(r)_\bullet$ of $A(r)$ over A
Alternatively, calculate $\mathrm{Tor}^A_\bullet(A(r), \mathbf{k}) = \mathbf{F}(r) \otimes_A \mathbf{k}$.

Some history:

- Auslander–Buchsbaum (1957): A/I Cohen–Macaulay iff $\mathrm{Tor}^A_j(A/I, \mathbf{k}) = 0$ for all $j > \operatorname{codim} V(I)$
- Eagon–Northcott (1962): constructed $\mathbf{F}(f-1)_\bullet$. In particular, $A(f-1)$ is Cohen–Macaulay.
- Eagon–Hochster (1971): showed $A(r)$ is Cohen–Macaulay for all r.
- Kempf (1973): gave geometric construction of $\mathbf{F}(f-1)_\bullet$.

Work on $X \times \mathbf{P}(F) =: \varepsilon$. Have short exact sequence on $\mathbf{P}(F)$:

$$0 \to \mathscr{R} \to F \otimes \mathscr{O}_{\mathbf{P}(F)} \to \mathscr{O}(1) \to 0, \qquad (\mathscr{R} = \Omega^1_{\mathbf{P}(F)}(1))$$

ε has subvariety Z = total space of $\mathscr{H}om(E, \mathscr{R})$. Its image under $Z \to \varepsilon \xrightarrow{\pi} X$ is $X(f-1)$.
Z is cut out by a section of $\mathscr{H}om(E, \mathscr{O}(1))$ so get Koszul complex:

$$\cdots \to \mathscr{O}_\varepsilon \otimes \bigwedge^2 E \otimes \mathscr{O}(-2) \to \mathscr{O}_\varepsilon \otimes E \otimes \mathscr{O}(-1) \to \mathscr{O}_\varepsilon \to \mathscr{O}_Z \to 0.$$

Check: $\mathrm{R}^i \pi_* \mathscr{O}_Z = 0$ for $i > 0$.
So derived projection formula gives

$$\begin{aligned}\mathrm{Tor}^A_i(A(f-1), \mathbf{k}) &= \bigoplus_{j \geq 0} \mathrm{H}^j(\mathbf{P}(F); \bigwedge^{i+j} E \otimes \mathscr{O}(-i-j)) \\ &= \begin{cases} \mathbf{k} & \text{if } i = 0 \\ \bigwedge^{i+f-1} E \otimes \bigwedge^f F^* \otimes \mathrm{Sym}^{i-1}(F)^* & \text{if } 1 \leq i \leq e-f+1 \end{cases}.\end{aligned}$$

Lascoux (1978) extended Kempf's construction to calculate $\mathrm{Tor}^A_i(A(r), \mathbf{Q})$ for all r. We replace $\mathbf{P}(F)$ with Grassmannian and need Borel–Weil–Bott theorem (hence restriction to char. 0)

Main idea: $X(r)$, $A(r)$ are functorial in E, F. In particular, have action of $\mathbf{GL}(E) \times \mathbf{GL}(F)$.

Remarks:

(1) In general, Betti numbers depend on char., but not known in general. They are independent of char. iff $r \geq f-3$ or $r=0$ (Eagon–Northcott, Akin–Buchsbaum–Weyman, Hashimoto)
(2) By functoriality, can replace E,F by vector bundles. This has geometric applications to equations/syzygies of curves (e.g., Gruson–Lazarsfeld–Peskine, Schreyer)

2. Twisted commutative algebras

Guiding question: How does equivariance force simple behavior? Or how can we exploit it in a useful way?

Work over field of char. 0.
Let Vec be the category of vector spaces
Intuitively, a twisted commutative algebra is a nice functor from Vec to commutative rings

An endofunctor of Vec is **polynomial** if it is a subquotient of a direct sum of functors $V \mapsto V^{\otimes d}$ (category of endofunctors of Vec is Abelian). Let Pol be the category of polynomial functors.

This includes symmetric and exterior powers and Schur functors $\mathbf{S}_\lambda$ (labeled by integer partitions λ):

There is a natural action of symmetric group Σ_d on $V^{\otimes d}$ and the multiplicity spaces are functorial in V and called Schur functors (evaluated on V).

Pol has a tensor structure: $(\mathscr{F}\otimes\mathscr{G})(V) := \mathscr{F}(V)\otimes\mathscr{G}(V)$.

A **tca** $\mathscr{A}$ is a commutative algebra in $(Pol,\otimes)$, i.e., $\mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ such that ...
An $\mathscr{A}$-module is $\mathscr{M}$ with $\mathscr{A}\otimes\mathscr{M}\to\mathscr{M}$ such that ...
$\mathscr{M}$ is f.g. if it is a quotient of $\mathscr{A}\otimes V$ for some finite length $V\in Pol$.

Examples:

(1) Fix F. Set $E\mapsto \mathrm{Sym}(E\otimes F)$. Call this tca $\mathrm{Sym}(F\langle 1\rangle)$.
(2) $E\mapsto A(f-1)$, which is quotient of $\mathrm{Sym}(F\langle 1\rangle)$.
(3) Fix d. Then $E\mapsto$ coordinate ring of Grassmannian $\mathbf{Gr}(d,E)$. etc.

Set $\ell(\lambda)=\max\{r \mid \lambda_r\neq 0\}$ and $\ell(\bigoplus_{\lambda\in I}\mathbf{S}_\lambda)=\max\ell(\lambda)$, so ℓ defined on Pol.
Fact: $\ell(\mathscr{F}\otimes\mathscr{G})=\ell(\mathscr{F})+\ell(\mathscr{G})$.
An object $\mathscr{M}$ of Pol is **bounded** if $\ell(\mathscr{M})<\infty$.

If $\mathscr{A}$ is bounded tca, and $\mathscr{M}$ is f.g. $\mathscr{A}$-module, then $\mathscr{M}$ is bounded.
If $\dim V\geq \ell(\mathscr{M})$, have bijection

$$\{\mathscr{A}\text{-submodules of }\mathscr{M}\}\cong\{\mathbf{GL}(V)\text{-invariant }\mathscr{A}(V)\text{-submodules of }\mathscr{M}(V)\}$$

Conclusion: if $\mathscr{A}$ is a bounded tca and $\mathscr{A}(W)$ Noetherian for $\dim W\geq\ell(\mathscr{A})$, then f.g. $\mathscr{A}$-modules are Noetherian.

Example: $\mathrm{Sym}(F\langle 1\rangle)\cong(\bigoplus_{d\geq 0}\mathrm{Sym}^d)^{\otimes f}$ is bounded. ($\mathrm{Sym}^d=\mathbf{S}_d$ so $\ell(\mathrm{Sym}^d)=1$)

3. Some problems

All tca's generated in degree 1 are bounded.
tca's generated in degree 2 can be unbounded, e.g.,

$$E \mapsto \mathrm{Sym}(\mathrm{Sym}^2(E)) \text{ and } E \mapsto \mathrm{Sym}(\bigwedge^2(E)).$$

Problems:

(1) Is every f.g. $\mathrm{Sym}(\bigwedge^2)$-module Noetherian?
(2) Does $\mathrm{Sym}(\bigwedge^3)$ have ascending chain condition for ideals? How about just for prime ideals?

4. FI-modules

Given sequence of Σ_n-representations $(M_n)_n$, get element of Pol defined by

$$V \mapsto \bigoplus_n (M_n \otimes V^{\otimes n})_{\Sigma_n}.$$

Schur–Weyl duality: this is an equivalence between Pol and sequences of Σ_n-representations (this is char. 0 phenomena)

$\otimes$ in Pol becomes induction product:

$$(M \otimes N)_n = \bigoplus_{i+j=n} \mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n}(M_i \boxtimes N_j)$$

Let FI be category of finite sets with injections as morphisms. **FI-module** (introduced by Church–Ellenberg–Farb) is a functor $FI \to Vec$. Under Schur–Weyl duality, an FI-module becomes a module over $\mathrm{Sym}(\mathbf{C}\langle 1\rangle)$.

Example:

- Let X be a smooth manifold. Let $X^{(n)}$ be configuration space of n ordered points on X. For fixed i, $(\mathrm{H}^i(X^{(n)}))$ is an FI-module (induced by forgetful maps). (It is f.g. if X is connected, oriented, dim at least 2 by Church)
- Fix $g \geq 2$. $\mathscr{M}_{g,n}$ = Deligne–Mumford moduli of genus g curves with n marked points. Then $\mathrm{H}^i(\mathscr{M}_{g,n})$ is an FI-module. (This is f.g. by Jimenez Rolland)

5. Another motivation: Segre embeddings

$\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_r) \subset \mathbf{P}(V_1 \otimes \cdots \otimes V_r)$
Want to understand Tor as $\dim(V_i)$ vary and as r varies
tca's not suitable to allow r to vary

Snowden introduced Δ-modules. Roughly this is a sequence of Σ_n-equivariant functors $\mathscr{F}_n : Vec^{\times n} \to Vec$ with maps

$$\mathscr{F}_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}) \to \mathscr{F}_{n+1}(V_1, \ldots, V_n, V_{n+1}).$$

For fixed i, $\{V_1, \ldots, V_n \mapsto \mathrm{Tor}_i^A(Segre, \mathbf{k})$ is a **finitely generated** Δ-module.

For fixed $\mathscr{F}$ and $d \gg 0$, $(\mathbf{C}^d \mapsto \mathscr{F}_n(\mathbf{C}^d, \ldots, \mathbf{C}^d))_n$ is a sequence of Σ_n-reps. Under Schur–Weyl duality, get object of Pol. It is a f.g. module over a bounded tca, and was used to prove "rationality" of Hilbert series.