Infinite rank classical groups and specialization

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(joint work with Andrew Snowden, Jerzy Weyman)

Throughout we work over the field of complex numbers $\mathbf{C}$.

The polynomial representation theory of the general linear group $\mathbf{GL}_n$ has a well-behaved limit for $n \to \infty$. On the level of character rings, this is the passage from the ring of symmetric polynomials in n variables to the ring of symmetric functions in infinitely many variables. One possible model for the $n \to \infty$ limit is the category $\mathbf{Pol}$ of polynomial functors from the category of vector spaces to itself. The category $\mathbf{Pol}$ is a semisimple Abelian category, it has a tensor product, and its simple objects are the Schur functors $\mathbf{S}_\lambda$, where λ ranges over all integer partitions. See [5, Chapter 1] for an exposition. $\mathbf{Pol}$ is characterized by a universal property: for any Abelian symmetric monoidal $\mathbf{C}$-linear category $\mathcal{A}$, a symmetric monoidal tensor functor $\mathbf{Pol} \to \mathcal{A}$ is equivalent to the choice of an object $A \in \mathcal{A}$. This universal property gives a specialization functor $\mathbf{Pol} \to \mathrm{Rep}(\mathbf{GL}_n)$ which simply evaluates a polynomial functor on the vector representation $\mathbf{C}^n$ of $\mathbf{GL}_n$. This specialization functor is exact, and is fundamental for many stabilization properties of the polynomial representation theory of $\mathbf{GL}_n$.

We remark that the category $\mathbf{Pol}$ has a natural grading via $\deg(\mathbf{S}_\lambda) = |\lambda|$ and many examples of algebra objects (twisted commutative algebras) in the category of graded-finite polynomial functors arise in classical algebraic geometry. This perspective is pursued in [9, 10].

In some cases, considering only polynomial representations is too restrictive (for example, when studying the adjoint representation of $\mathbf{GL}_n$), and also it is natural to ask about analogues of the category $\mathbf{Pol}$ for the orthogonal and symplectic groups. Such a category was introduced and studied by several authors [1, 6, 7, 8] and can be described as follows. Let $\mathbf{V} = \bigcup_n V_n$ be the union of the defining representations of either the series $\mathbf{GL}_n$, $\mathbf{O}_n$, or $\mathbf{Sp}_{2n}$, and also let $\mathbf{V}_* = \bigcup_n V_n^*$ be the union of the dual of the defining representations. Then $\mathrm{Rep}(\mathbf{O})$ and $\mathrm{Rep}(\mathbf{Sp})$ are defined as the category of representations of either $\mathbf{O}_\infty = \bigcup_n \mathbf{O}_n$ or $\mathbf{Sp}_\infty = \bigcup_n \mathbf{Sp}_{2n}$ which are subquotients of finite direct sums of the tensor spaces $\mathbf{V}^{\otimes N}$. To define $\mathrm{Rep}(\mathbf{GL})$ we use the mixed tensor spaces $\mathbf{V}^{\otimes N} \otimes \mathbf{V}_*^{\otimes M}$.

These categories are no longer semisimple. For example, the map $\mathbf{V} \otimes \mathbf{V}_* \to \mathbf{C}$ in $\mathrm{Rep}(\mathbf{GL})$ does not split. However, they still have a tensor structure. The simple objects V_λ are indexed by partitions in the orthogonal and symplectic case, and by pairs of partitions in the general linear case. The injective envelope of V_λ in the first two cases is $\mathbf{S}_\lambda(\mathbf{V})$ and the injective envelope of $V_{\lambda,\mu}$ is $\mathbf{S}_\lambda(\mathbf{V}) \otimes \mathbf{S}_\mu(\mathbf{V}_*)$ in the third case.

In [11], we showed that $\mathrm{Rep}(G)$ is equivalent to the category of finite length modules over certain twisted commutative algebras. For $G = \mathbf{O}$ and $G = \mathbf{Sp}$, the algebras are $\mathrm{Sym}(\mathrm{Sym}^2)$ and $\mathrm{Sym}(\bigwedge^2)$, which can be thought of as the coordinate rings of the space of infinite size symmetric and skew-symmetric matrices, respectively. For $G = \mathbf{GL}$, there is a similar description in terms of a bivariate twisted commutative algebra which can be thought of as the coordinate ring of the space of infinite size generic matrices. The utility of this perspective is that one can now use techniques from commutative algebra to study these categories.

Also in [11], we characterized these categories in terms of universal properties. Let G be one of $\{\mathbf{O}, \mathbf{Sp}, \mathbf{GL}\}$ and let $\mathcal{A}$ be an Abelian symmetric monoidal $\mathbf{C}$-linear category. Then a left-exact symmetric monoidal tensor functor $\mathrm{Rep}(G) \to \mathcal{A}$ is equivalent to the choice of:

- ($G = \mathbf{O}$): a pair (A, ω) with $A \in \mathcal{A}$ and $\omega \colon \mathrm{Sym}^2 A \to \mathbf{C}$.
- ($G = \mathbf{Sp}$): a pair (A, ω) with $A \in \mathcal{A}$ and $\omega \colon \bigwedge^2 A \to \mathbf{C}$.
- ($G = \mathbf{GL}$): a triple (A, A', ω) with $A, A' \in \mathcal{A}$ and $\omega \colon A \otimes A' \to \mathbf{C}$.

Using these universal properties, we can define specialization functors

$$\begin{aligned} \Gamma_n \colon \mathrm{Rep}(\mathbf{O}) &\to \mathrm{Rep}(\mathbf{O}_n), \\ \Gamma_{2n} \colon \mathrm{Rep}(\mathbf{Sp}) &\to \mathrm{Rep}(\mathbf{Sp}_{2n}), \\ \Gamma_n \colon \mathrm{Rep}(\mathbf{GL}) &\to \mathrm{Rep}(\mathbf{GL}_n). \end{aligned}$$

They are guaranteed to be left-exact, but will not be right-exact. For example, in $\mathrm{Rep}(\mathbf{Sp})$, we have the injective resolution

$$0 \to V_{1,1,1,1,1,1} \to \bigwedge^6 \mathbf{V} \to \bigwedge^4 \mathbf{V} \to 0,$$

and applying Γ_4, we get a complex concentrated in cohomological degree 1, so $\mathrm{R}^1\Gamma_4(V_{1,1,1,1,1,1}) = \mathbf{C}$.

In fact, in [12], we showed that the derived specialization functors of a simple object are nonzero in at most 1 cohomological degree. The rule to calculate the derived functors can be obtained from a Weyl group action and is analogous to the Borel–Weil–Bott theorem. We explain this now for $\mathrm{Rep}(\mathbf{Sp})$. Let W be the Weyl group of type BC_∞ which acts on integer sequences $(a_1, a_2, \dots)$ with generators $s_i(\dots, a_i, a_{i+1}, \dots) = (\dots, a_{i+1}, a_i, \dots)$ $(i \geq 1)$ and $s_0(a_1, a_2, \dots) = (-a_1, a_2, \dots)$. For $w \in W$, set $\ell(w)$ to be the minimal number of s_i needed to generate w. Given a partition λ, let $\lambda^\dagger$ be its transposed partition. Finally, set $\rho^{2n} = (-(n+1), -(n+2), \dots)$. Then one of two possibilities occurs:

- There exists a non-identity $w \in W$ such that $w(\lambda^\dagger + \rho^{2n}) - \rho^{2n} = \lambda^\dagger$. In this case, $\mathrm{R}^i\Gamma_{2n}(V_\lambda) = 0$ for all $i \geq 0$.
- There exists a unique $w \in W$ such that $w(\lambda^\dagger + \rho^{2n}) - \rho^{2n} = \mu^\dagger$ where μ is a partition with at most n parts. In this case, $\mathrm{R}^{\ell(w)}\Gamma_{2n}(V_\lambda) = V_\mu$ (where μ is interpreted as a dominant weight for $\mathbf{Sp}_{2n}$) and $\mathrm{R}^i\Gamma_{2n}(V_\lambda) = 0$ for $i \neq \ell(w)$.

There are similar rules for the orthogonal case and general linear case, in which case one uses the Weyl groups of type D_∞ and A_∞, respectively. We refer the reader to [12] for more details.

We close by mentioning that the character rings of the categories $\mathrm{Rep}(G)$ and the ring homomorphisms induced from the specialization functors were studied in [2, 3, 4].

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