Torus actions, multi-partitions, and crystals

Steven Sam   Peter Tingley

Massachusetts Institute of Technology

January 6, 2012

- We will focus on $\hat{\mathfrak{sl}}_n(\mathbf{C})$ and combinatorial data associated to its highest-weight representations.
- Roughly speaking, $\hat{\mathfrak{sl}}_n(\mathbf{C})$ is a certain extension of the loop algebra $\mathfrak{sl}_n(\mathbf{C}[t, t^{-1}])$
- It is generated by lowering and raising operators e_i and f_i $(i = 0, \ldots, n-1)$ and a degree operator d
- $\hat{\mathfrak{sl}}_n$ (and its quantum version) has a combinatorially defined action on the space of all partitions (Fock space realization):

$$e_i\lambda = \sum_{\mu} \mu, \quad f_i\lambda = \sum_{\nu} \nu, \quad d\lambda = N_0(\lambda)\lambda$$

The sum is defined so that λ/μ and ν/λ are i-nodes (boxes whose content is $i \pmod{n}$) and $N_0(\lambda)$ is the number of 0-nodes.

- To define quantum $\hat{\mathfrak{sl}}_n$, we work over the field $\mathbf{C}(q)$. The action of e_i, f_i, and d is similar to before:

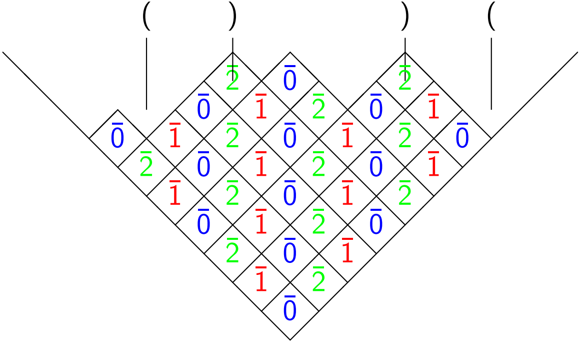
$$e_i\lambda = \sum_\mu q^{-N_i^l(\lambda,\mu)}\mu, \quad f_i\lambda = \sum_\nu q^{N_i^r(\nu,\lambda)}\nu, \quad d\lambda = q^{N_0(\lambda)}\lambda$$

  where $N_i^r(\nu,\lambda)$ is the number of addable i-nodes minus the number of removable i-nodes of λ to the right (English notation) of ν/λ, and N_i^l is defined similarly, except "left" replaces "right".
- After passing to a suitable "$q \to 0$" limit, and augmenting e_i and f_i in a certain way, we are left with operators $\widetilde{e}_i$ and $\widetilde{f}_i$ which act as partial permutations of the set of partitions
- The remaining object is a crystal, and we can define them axiomatically. They are the subject of this talk.

To define $\widetilde{e}_i\lambda$ and $\widetilde{f}_i\lambda$ in crystal:

- Draw partition λ in Russian notation.
- For each addable i-node, write "(" and for each removable i-node, write ")" and order them from left-to-right. Now cancel all paired parentheses
- $\widetilde{e}_i\lambda$ is obtained by removing the rightmost remaining removable i-node, or 0 if not possible; $\widetilde{f}_i\lambda$ is obtained by adding the leftmost remaining addable i-node, or 0 if not possible
- Under this action, the set of partitions breaks up into infinitely many connected components, all isomorphic to one another. The component containing the empty partition is the set of n-regular partitions (every part appears with multiplicity $< n$)

Defining the actions of $\widetilde{f}_2$ and $\widetilde{e}_2$ on $(7, 6, 6, 6, 5, 3, 3)$:



So $\widetilde{f}_2(7, 6, 6, 6, 5, 3, 3) = (7, 6, 6, 6, 5, 3, 3, 1)$ and
$\widetilde{e}_2(7, 6, 6, 6, 5, 3, 3) = (7, 6, 6, 6, 5, 3, 2)$.

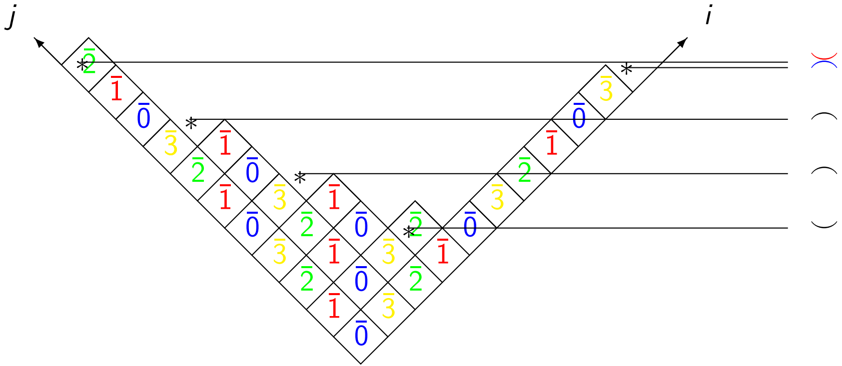
Fayers' family of realizations for $B(\Lambda_0)$

Fayers showed that the left-to-right ordering of the parentheses from the previous slide can be augmented as follows:

- Let ξ be an irrational number (rational numbers are okay, but would need to take "upper" or "lower" limits)
- Define the height of a box (i, j) as $i + j\xi$ (i is the row index and j is the column index)

This gives a "different" crystal. In fact, only the connected component containing the empty partition is in general a crystal (but it is isomorphic to the Misra–Miwa–Hayashi realization).

This component consists of ξ-regular partitions: there is no box s whose hook length is divisible by n and whose arm length is $\lfloor \mathrm{hook}(s)/(\xi + 1) \rfloor$.



This example is for slope $\xi = 1 + \varepsilon$ for sufficiently small $\varepsilon > 0$.
$\widetilde{f}_2$ adds a box at $\frown$ and $\widetilde{e}_2$ removes the box at $\smile$.

Fayers' proof is complicated and difficult to generalize to other representations of $\hat{\mathfrak{sl}}_n$. Another approach:

- Associated to any symmetric Kac–Moody algebra (so including $\hat{\mathfrak{sl}}_n$) and two weights $\mathbf{v}$ and $\mathbf{w}$ ($\mathbf{Z}_{\geq 0}$-labelings of Dynkin diagram), Nakajima defined symplectic manifolds (quiver varieties) $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ and distinguished Lagrangian subvarieties $\mathfrak{L}(\mathbf{v}, \mathbf{w}) \subset \mathfrak{M}(\mathbf{v}, \mathbf{w})$.
- Let $\mathrm{Irr}(\mathfrak{L}(\mathbf{v}, \mathbf{w}))$ denote the set of irreducible components. Saito constructed crystal operators on $\coprod_{\mathbf{v}} \mathrm{Irr}(\mathfrak{L}(\mathbf{v}, \mathbf{w}))$.
- To get combinatorial crystals out of this action, one can find a combinatorial enumeration of these irreducible components and translate Saito's crystal operators.

We take a "Morse theory" approach to get combinatorics.

- There is a natural torus T acting on $\mathfrak{M}(\mathbf{v}, \mathbf{w})$. Choose a 1-parameter subgroup $\iota\colon \mathbf{C}^* \to T$.
- $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ is compact, so we can let points flow to infinity under this torus action $x \mapsto \lim_{t\to\infty} \iota(t) \cdot x$.
- This gives a map from irreducible components of $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ to irreducible components of the fixed point variety of ι
- Technical assumption: if ι is "positive" then this map is injective.
- So we can reduce to understanding fixed points of a torus action (and tangent space / eigenvalue-type calculations)
- If ι is generic, then the fixed points of ι and T will coincide

In the case of $\hat{\mathfrak{sl}}_n$, Nakajima's quiver varieties have alternate descriptions.
The Dynkin diagram of $\hat{\mathfrak{sl}}_n$ is an n-cycle on $[0, n-1]$, and we are interested in the weight $\mathbf{w}$ which is 1 at 0 and 0 everywhere else.

- $\mathbf{Hilb}^N$ is the Hilbert scheme of N points in the plane, i.e., the set of ideals $I \subset \mathbf{C}[x, y]$ such that $\dim_{\mathbf{C}} \mathbf{C}[x, y]/I = N$.
- Let ζ be a primitive nth root of unity. The group $\mathbf{Z}/n$ of nth roots acts on $\mathbf{C}[x, y]$ by $\zeta^k p(x, y) := p(\zeta^k x, \zeta^{-k} y)$.
- If I is fixed under this action, we get an eigenvalue decomposition of $\mathbf{C}[x, y]/I$. The $\mathbf{Z}/n$ fixed points of $\mathbf{Hilb}^N$ is a disjoint union of $\mathbf{Hilb}^N(\mathbf{v})$ over all such eigenvalue decompositions, and we have $\mathbf{Hilb}^N(\mathbf{v}) \cong \mathfrak{M}(\mathbf{v}, \mathbf{w})$.

- The torus T is 2-dimensional and acts by scaling variables
- The Lagrangian $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ is the set of ideals in $\mathbf{Hilb}^N(\mathbf{v})$ that are supported at the origin. T-fixed points are monomial ideals, which can be identified with partitions of N
- Given irrational ξ from before, we have an "embedding" $\iota_\xi \colon \mathbf{C}^* \to T$ by $z \mapsto (z, z^\xi)$. This doesn't actually make sense–to interpret its meaning, we find a rational number a/b which closely approximates ξ and define $z \mapsto (z^b, z^a)$. The closeness of this approximation changes with $\mathbf{v}$.
- The flow to infinity is something like "reverse initial ideal" because it picks out lowest degree terms
- By working out which components go to which monomial ideals and identifying Saito's crystal operators, we recover Fayers' construction.

This framework easily generalizes to other $\mathbf{w}$ (higher-level representations).

- Let W be a $\mathbf{Z}/n$-representation whose eigenvalue multiplicities is given by $\mathbf{w}$.
- The Hilbert scheme is replaced by a "moduli space of framed torsion-free sheaves on $\mathbf{P}^2$"
- The Lagrangian is now the set of $\mathbf{Z}/n$-stable submodules of $W \otimes \mathbf{C}[x, y]$ whose quotient is $N = \sum_i \mathbf{v}_i$ dimensional, has eigenvalue multiplicities according to $\mathbf{v}$, and is supported at the origin.
- The torus T is now larger, but the fixed points will be monomial submodules, which are naturally indexed by multi-partitions. The irrational number ξ now becomes a "slope datum" and Fayers' construction generalizes.
- We could only identify the multi-partitions in the component of the empty partition under certain hypotheses on the slope datum.

- The reason $\hat{\mathfrak{sl}}_n$ was so nice is because the T-fixed points were isolated and easy to enumerate.
- When the Dynkin diagram has vertices of valency ≥ 3, this fails. For D_4 we worked out some examples and the fixed point varieties always seemed to be rational varieties (usually iterated Grassmannian bundles, or blowups/blowdowns). So we might ask if this is always the case.
- In affine type D, one has combinatorial models for crystals in terms of "Young walls" and it would be interesting if the above framework could recover them.