# Free resolutions, combinatorics, and geometry

Steven Sam

Massachusetts Institute of Technology

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- I am interested in systems of polynomial equations (ideals). To understand ideals, one is usually forced to study generalized systems of equations (modules).
- To such a system, one can calculate its minimal free resolution, which is a measure of how complicated the interactions between the equations are.
- The numerical shadow of the resolution is the graded Betti table, which is a finite table of positive integers.
- There is a natural addition on graded Betti tables (related to direct sums of modules) and this semigroup structure is complicated.
- However, the $\mathbf{R}_{\geq 0}$-cone spanned by this semigroup is described by recent results of Eisenbud–Schreyer.
- There is a lot of rich (combinatorial, geometric, representation-theoretic) structure behind this cone, and this thesis is about trying to understand it better.

- $K$ is a field, $A = K[x_1, \ldots, x_n]$ with standard grading
- $A(-i)$ denotes $A$ with a grading shift, i.e., $A(-i)_j = A_{j-i}$
- $M$ denotes a graded finitely generated module

A **minimal free resolution** of $M$ is a sequence of homomorphisms

$$\cdots \to \mathbf{F}_i \xrightarrow{d_i} \mathbf{F}_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} \mathbf{F}_0 \to M \to 0$$

such that

- each $\mathbf{F}_i$ is a graded free $A$-module (write $\mathbf{F}_i = \bigoplus_j A(-j)^{\oplus \beta_{i,j}}$),
- it is exact, i.e., $\mathrm{image}(d_i) = \ker(d_{i-1})$ for all $i$,
- it is graded, i.e., $d_i((\mathbf{F}_i)_j) \subset (\mathbf{F}_{i-1})_j$ for all $i, j$,
- it is minimal, i.e., none of the nonzero entries of the matrices $d_i$ are scalars from $K$

Basic facts:

- $\mathbf{F}_\bullet$ is unique up to isomorphism,
- $\mathbf{F}_i = 0$ for $i > n$ (Hilbert)

We define the **graded Betti table** $\beta(M) = (\beta_{i,j})$.

Set $A = K[x, y, z]$ and let $M = A/(x, y, z) \cong K$. Then the minimal free resolution is

$$0 \to A(-3) \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} A(-2)^3 \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} A(-1)^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} A$$

The general pattern for $n$ variables and $M = A/(x_1, \dots, x_n)$ is that $\mathbf{F}_i = A(-i)^{\oplus \binom{n}{i}}$. This is the **Koszul complex**.

If we replace $x_1, \dots, x_n$ with arbitrary (homogeneous) polynomials $f_1, \dots, f_n \in K[x_1, \dots, x_m]$, then substituting $f_i$ into $x_i$ in the Koszul complex gives a minimal free resolution if and only if the solution set of $f_1 = \cdots = f_n = 0$ is $m - n$ dimensional (complete intersection).

# Example: Hilbert–Burch complex

Let $A = K[x_{1,1}, \dots, x_{n,n+1}]$ and define $n \times (n+1)$ matrix $\varphi = (x_{i,j})$. Let $(-1)^i \Delta_i$ be the determinant of the submatrix of $\varphi$ gotten by deleting the $i$th column. Set $M = A/(\Delta_1, \dots, \Delta_{n+1})$. Then its minimal free resolution is

$$0 \to A(-n-1)^n \xrightarrow{\varphi} A(-n)^{n+1} \xrightarrow{\begin{pmatrix} \Delta_1 & \cdots & \Delta_{n+1} \end{pmatrix}} A$$

(the fact that $d_1 d_2 = 0$ is Laplace expansion)

Given $n+1$ (homogeneous) polynomials $f_1, \dots, f_{n+1} \in K[y_1, \dots, y_m]$ such that the solution set $f_1 = \cdots = f_{n+1} = 0$ has dimension $m-2$ and is "Cohen–Macaulay", there exists a $\varphi$ as above and a substitution $x_{i,j} \mapsto g_{i,j}(y)$ so that $\Delta_i \mapsto f_i$ for all $i$.

- We will restrict attention to finite length modules, i.e., modules that are finite-dimensional over $K$
- A module has a **pure resolution** if each $\mathbf{F}_i$ is generated in a single degree, i.e., for each $i$, $\beta_{i,j} \neq 0$ for at most one value of $j$. In this case, the $j$ for which $\beta_{i,j} \neq 0$ is denoted $d_i$, and $d = (d_0, d_1, \ldots . d_n)$ is the degree sequence.
- If $M$ has a pure resolution, then the Herzog–Kühl equations say that there is a rational number $c$ such that

$$\beta_{i,d_i}(M) = c \prod_{j \neq i} |d_i - d_j|^{-1}.$$

**Theorem (Eisenbud–Fløystad–Weyman, Eisenbud–Schreyer)**

*For $d_0 < d_1 < \cdots < d_n$, there exists a finite length module $M$ whose resolution is pure with degree sequence $d$.*

By Herzog–Kühl, each degree sequence defines a ray in the space of all Betti tables. The cone spanned by these rays is the **Boij–Söderberg cone**.

**Theorem (Eisenbud–Schreyer)**

*Every Betti table of a finite length module $M$ is contained in the Boij–Söderberg cone.*

So for some positive integer $N$, $N\beta(M)$ is a positive integer linear combination of Betti tables of modules with pure resolution.

**Theorem (Erman)**

*The semigroup spanned by actual Betti tables is finitely generated (if we bound the degrees that may appear in the Betti table).*

```
i1 : R = ZZ/101[x_1..x_3];
i2 : random(R^2,R^{3:-2}++R^{3:-3});
               2       6
o2 : Matrix R  <--- R
i3 : betti res coker o2
              0 1 2 3
o3 = total: 2 6 7 3
         0: 2 . . .
         1: . 3 . .
         2: . 3 . .
         3: . . 6 .
         4: . . 1 3
i4 : loadPackage "BoijSoederberg";
i5 : decompose o3

      3 /        0 1 2 3\     1 /        0  1  2  3\     1 /        0 1  2 3\
o5 = (-)|total: 3 7 7 3| + (--)|total: 8 35 42 15| + (--)|total: 2 7 14 9|
      7 |     0: 3 . . .|    14 |     0: 8  .  .  .|    14 |     0: 2 .  . .|
        |     1: . 7 . .|       |     1: .  .  .  .|       |     1: . .  . .|
        |     2: . . . .|       |     2: . 35  .  .|       |     2: . 7  . .|
        |     3: . . 7 .|       |     3: .  . 42  .|       |     3: . .  . .|
        \     4: . . . 3/       \     4: .  .  . 15/       \     4: . . 14 9/
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Define $d \leq d'$ if $d_i \leq d'_i$ for $i = 0, \ldots, n$. The Boij–Söderberg cone is a geometric realization of this poset, and hence one gets a simplicial decomposition.
With respect to this triangulation, every Betti table can **uniquely** be expressed as a positive linear combination of Betti tables of pure resolutions (assuming we have chosen a normalized value for each ray).
We can also interpret this poset structure module-theoretically:

**Theorem (Berkesch–Erman–Kummini–S.)**

*$d \leq d'$ if and only if there exist modules $M$ and $M'$ with pure resolutions of type $d$ and $d'$, respectively, such that* $\mathrm{Hom}(M', M)_{\leq 0} \neq 0$.

$\mathrm{Hom}(M', M)_{\leq 0}$ is the set of homomorphisms $M' \to M$ that do not increase degree of elements

It is natural to ask what the decompositions of Betti tables means in terms of modules.
Naive guess: for any module $M$, some high multiple $M^{\oplus N}$ has a filtration such that the quotient modules have pure resolutions.

**Example**

Let $M = K[x, y]/(x, y^2)$. Then

$$\beta(M) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

but any submodule or quotient module of $M^{\oplus 3N}$ is annihilated by $y^2$, so must have a degree 2 relation.

Some cases where pure filtrations do exist are given by Eisenbud–Erman–Schreyer.

How to fix the previous example: there is a flat deformation of $M^{\oplus 3}$ to $M' = A/(x, y^2) \oplus A/(x^2, y) \oplus A/(x+y, (x-y)^2)$, and this does possess a pure filtration. [Take the submodule spanned by $(1, 1, 1)$.]

**Conjecture:** The Boij–Söderberg decomposition of $\beta(M)$ corresponds to a pure filtration of a flat deformation of some high multiple of $M$.

Note: When $K$ is an infinite field, Pardue showed that two quotients of a fixed free $A$-module are in the same deformation class if and only if they have the same Hilbert function.

- Eisenbud–Schreyer introduced a bilinear pairing between Betti tables and cohomology tables of vector bundles on projective space $\mathbf{P}^{n-1}$ to produce the facet inequalities of the Boij–Söderberg cone.
- Given a coherent sheaf $\mathcal{F}$ on $\mathbf{P}^{n-1}$ its **cohomology table** is $\gamma(\mathcal{F}) = (\dim_K \mathrm{H}^i(\mathbf{P}^{n-1}; \mathcal{F}(j)))_{i,j}$. These also form a cone.
- The analogue of pure resolution is a **supernatural bundle**: $\mathcal{E}$ is supernatural if for all $j$, $\gamma_{i,j}$ is nonzero for at most one $i$, and $\gamma_{\bullet,j} = 0$ for exactly $n-1$ distinct values of $j$ (the roots of $\mathcal{E}$). The cohomology table of a supernatural bundle is determined up to scalar if we fix its roots.

### Theorem (Eisenbud–Schreyer)

*The extremal rays of the cone of cohomology tables of vector bundles are given by the supernatural bundles. The "completion" of this cone is the cone of cohomology tables of coherent sheaves.*

The cone is naturally triangulated by a poset structure on roots, and a module-theoretic interpretation of this poset also exists.

- Given an embedded projective variety $X \subset \mathbf{P}^n$, we can find a finite map $\pi\colon X \to \mathbf{P}^{\dim X}$ (Noether normalization).
- $\pi_*$ gives an inclusion of the cone of cohomology tables of vector bundles / coherent sheaves of $X$ into the same cone for $\mathbf{P}^{\dim X}$.
- This is an equality if and only if $X$ has an Ulrich bundle / sheaf. A sheaf $\mathcal{U}$ is **Ulrich** if $\mathrm{H}^i(X; \mathcal{U}(j)) = 0$ whenever (1) $i > 0$ and $j > 0$, (2) $j = -1, \ldots, -\dim X$, or (3) $i < \dim X$ and $j < -\dim X$
- This definition is equivalent to $\pi_* \mathcal{U} \cong \mathcal{O}^t_{\mathbf{P}^{\dim X}}$ for some $t$.
- It is also equivalent to $\mathrm{H}^0_*(\mathcal{U}) = \bigoplus_{d \in \mathbf{Z}} \mathrm{H}^0(X; \mathcal{U}(d))$ having a pure free resolution of type $(0, 1, 2, \ldots, n - \dim X)$ over the homogeneous coordinate ring of $\mathbf{P}^n$.
- The existence of Ulrich sheaves in general is conjectural. It is known for: (1) complete intersections, (2) Grassmannians, (3) $\dim X = 1$, (4) Veronese re-embeddings of an existing example
- I constructed them for matrix Schubert varieties

Let $G = \mathbf{GL}_n(\mathbf{Q})$. The irreducible polynomial representations of $G$ are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0)$ with at most $n$ parts.

One way to construct these is as the image of a certain map

$$\bigwedge^{\lambda'_1} \otimes \cdots \otimes \bigwedge^{\lambda'_r} \to \mathrm{Sym}^{\lambda_1} \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n}.$$

where $\lambda'$ is the dual partition of $\lambda$, i.e., $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$.

This construction is functorial and makes sense over any base ring (not just $\mathbf{Q}$), i.e., we can plug in any $R$-module $V$ into the above map, the image is denoted $\mathbf{S}_\lambda V$ and is a representation of $\mathbf{GL}(V)$ when $V$ is (locally) free.

To a partition $\lambda$, we can associate the **Schur polynomial** $s_\lambda(x) = s_\lambda(x_1, \ldots, x_n)$ as follows. A **semistandard Young tableau** is a way of filling in the diagram of $\lambda$ with the numbers $1, \ldots, n$ such that the numbers are weakly increasing along rows and along columns, with no repeats in any column. Such a filling gives a monomial in the $x_i$, and $s_\lambda$ is the sum over all such SSYT.

$$\lambda = (5,4,2), \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 4 \\ \hline 2 & 2 & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-4} 4 & 5 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array} \quad \mapsto x_1^2 x_2^4 x_3 x_4^3 x_5$$

Connection to $\mathbf{S}_\lambda V$: pick a basis $e_1, \ldots, e_n$ for $V$. Then $\mathbf{S}_\lambda V$ has a basis $\{v\}$ such that $\mathrm{diag}(x_1, \ldots, x_n)$ acts on $v$ as $x_1^{m_1} \cdots x_n^{m_n}$ for some nonnegative integers $m_1, \ldots, m_n$. Summing these expressions over all such $v$ gives $s_\lambda(x_1, \ldots, x_n)$.

In fact, the previous definition can be "superfied". More precisely, we have $\mathbf{Z}/2$-graded versions of our favorite multilinear functors such as $\bigwedge$ and Sym, so we can construct a $\mathbf{Z}/2$-graded module $\mathbf{S}_\lambda V$ when $V = V_0 \oplus V_1$ has a $\mathbf{Z}/2$-grading.
The case of interest is a 2-term complex $\varphi\colon V_1 \to V_0$. In this case, $\mathbf{S}_\lambda V$ inherits a differential from $\varphi$, and becomes a chain complex.

Let's specialize to the case rank $V_1 = m$ and rank $V_0 = n$, $\lambda = (m-r, \dots, m-r)$ ($n-r$ times) and $\varphi$ is "generic". [This will mean depth $I_{r+1}(\varphi) \geq (m-r)(n-r)$.] In this case, $\mathbf{S}_\lambda \varphi$ is acyclic and its cokernel $M$ is a rank 1 module supported on the ideal defined by $I_{r+1}(\varphi)$.

This generalizes to the situation of vector bundles $V_1 = E$ and $V_0 = F$, and implies a formula for degeneracy loci in terms of the Chern roots of $E$ and $F$ (Thom–Porteous formula).

We also have **double Schur polynomials** $s_\lambda(x/y)$: instead of just filling in with the numbers $1, \dots, n$, also allow $-1, \dots, -m$. The weakly increasing rule applies, along with no repeats of positive numbers in a column, and no repeats of negative numbers in any row. Such a filling gives a monomial in $x$ and $y$, and the sum over all such fillings gives $s_\lambda(x/y)$.

This polynomial plays the same role for the Schur complex $\mathbf{S}_\lambda(\varphi)$ as $s_\lambda(x)$ does for the Schur functor. [We should really talk about the general linear super(group/algebra) though.]

Consider $\varphi\colon E \to F$ over $X$. Let $Z_r = \{x \in X \mid \mathrm{rank}(\varphi|_x) < r\}$. The Thom–Porteous formula says that the dual class of $[Z_r]$ in cohomology is obtained by substituting the Chern roots of $E$ and $F$ for $y$ and $x$, respectively, into $s_\lambda(x/y)$ where $\lambda = ((m-r)^{n-r})$.

Go back to a map of bundles $\varphi\colon E \to F$. Now assume that $E$ has a filtration $E_1 \subset \cdots \subset E_n = E$ and $F$ has a quotient filtration $F = F_m \twoheadrightarrow F_{m-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1$ where $E_i, F_j$ are vector bundles of ranks $i, j$, and each $E_{i+1}/E_i$ is a line bundle.

Given a permutation $w \in \Sigma_N$, define

$$\begin{aligned} r_w(p,q) &= \#\{i \le p \mid w(i) \le q\} \\ D_w(\varphi) &= \{x \in X \mid \mathrm{rank}(\varphi|_x\colon E_p|_x \to F_q|_x) \le r_w(p,q)\}. \end{aligned}$$

Fulton gave a formula for $[D_w(\varphi)]$ in terms of the Chern roots of $E$ and $F$. Instead of double Schur polynomials, one gets double Schubert polynomials.

Given $w \in \Sigma_N$, its Rothe diagram is

$$D(w) = \{(i, w(j)) \mid i < j, w(i) > w(j)\}$$

(using matrix indexing notation). A **balanced labeling** is a filling of $D(w)$ with nonzero integers such that:

- No entries in the $i$th row are bigger than $i$, and no entries in the $j$th column are smaller than $-j$.
- No repeats of positive integers in any column, and no repeats of negative numbers in any row.
- Balanced condition: given $(i, j)$, its hook is

  $$H(i,j) = \{(i', j) \in D(w) \mid i' \geq i\} \cup \{(i, j') \in D(w) \mid j' \geq j\}.$$

  For any $(i, j)$, if the entries in $H(i, j)$ are rearranged in increasing order starting from top right going to bottom left, then the $(i, j)$ entry must stay the same.

$$w = 25413, \qquad \begin{array}{|c|c|c|c|} \hline 1 & . & . & . \\ \hline 2 & . & -1 & -2 \\ \hline 3 & . & 3 & . \\ \hline \end{array} \mapsto x_1 x_2 x_3^2 y_1 y_2$$

Each balanced labeling gives a monomial in $x$ and $y$, and the sum over all such is the double Schubert polynomial $\mathfrak{S}_w(x, y)$. Schubert polynomials obtained by setting $y_i = 0$.

Is there an analogue of Schubert functors and Schubert complexes? Schubert functors introduced by Kraśkiewicz and Pragacz, definition is functorial and similar to Schur functors. So can be made into a $\mathbf{Z}/2$-graded version.

The Schubert functors have an action of the group of invertible upper triangular matrices, and the Schubert polynomials give the character of this representation. Similar story for Schubert complexes.

- Back to the situation of $\varphi\colon E \to F$ where both $E$ and $F$ have filtrations, but now assume they are free $\mathbf{Z}$-modules, and $\varphi$ consists of indeterminates. This induces ordered bases on $E$ and $F$.
- $w \in \Sigma_N$ is a permutation. Recall $r_w(p, q) = \#\{i \leq p \mid w(i) \leq q\}$.
- Let $I_w(\varphi)$ be the ideal generated by the minors of size $r_w(p, q) + 1$ inside of the upper left $p \times q$ submatrix of $\varphi$ for all $(p, q)$. Let $X_w$ be the variety defined by $I_w(\varphi)$.

**Theorem (S.)**

*The Schubert complex is acyclic and resolves a rank 1 module supported on $X_w$.*

Since the Schubert complex has linear differentials by construction, this rank 1 module is an Ulrich module.

If we specialize the variables of $\varphi$, then acyclicity of the Schubert complex is controlled by the depth of $I_w(\varphi)$ in general.

Now let $R$ be a regular local ring of dimension $n$.
($R$ could be of mixed characteristic, like $\mathbf{Z}_{(p)}[\![x_1, \ldots, x_{n-1}]\!]$.)

By a result of Auslander–Buchsbaum, this condition on local rings is equivalent to all modules having finite free resolutions.

We can no longer speak about grading, but we can ask about Betti sequences (ranks of the terms in the free resolution). Let $\varepsilon_i$ be the $i$th standard basis vector of $\mathbf{R}^n$.

**Theorem (Berkesch–Erman–Kummini–S.)**

*The extremal rays of the closure of the cone of Betti sequences of $R$-modules of finite length are the vectors $\varepsilon_i + \varepsilon_{i+1}$ for $i = 0, \ldots, n-1$.*

Surprising consequence: The linear functionals defining the cone above are given by partial Euler characteristics (that they must be nonnegative is easy), so this result essentially says there are no other obstructions to being a Betti sequence of a module (at least up to scalar multiple).

So for example, some multiple of a small perturbation of $(1, 1, 10^{10}, 10^{10}, \dots, 10^{10}, 1, 1, 10^{100}, 10^{100}, 1, 1)$ is a Betti sequence, and you get even wilder behavior.

Let $Q = R/(f)$ be a hypersurface. Set $n = \dim(R)$ and $d = \mathrm{ord}(f)$. Free resolutions over $Q$ may be infinite in length, but become periodic of period 2 after $n$ steps, i.e., $\mathbf{F}_{i+2} \cong \mathbf{F}_i$ and $d_{i+2} = d_i$ for $i \geq n$.

**Conjecture.** The extremal rays of the closure of the cone of Betti sequences of finite length $Q$-modules are $\varepsilon_i + \varepsilon_{i+1}$ for $i = 0, \ldots, n-2$, together with

$$\frac{d-1}{d}\varepsilon_{n-2} + \sum_{i=n-1}^{\infty} \varepsilon_i, \qquad \frac{1}{d}\varepsilon_{n-2} + \sum_{i=n-1}^{\infty} \varepsilon_i.$$

- The cone spanned by these rays contains the cone of Betti sequences.
- Equality is known for $n = 2$.
- Asymptotic equality is also true: if we fix $n$ and take the union over all $d$, then we get equality.