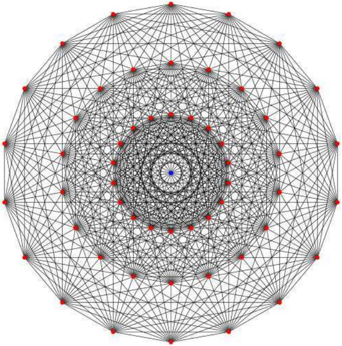
Combinatorics and geometry of E_7

Steven Sam

University of California, Berkeley

October 4, 2012



Classically, abstract groups did not exist. They arose as the symmetries of a geometric object.

Aut(Icosahedron) $=$
Alternating group A_5

We will examine the group $W(\mathrm{E}_7)$ of order $2903040 = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$
The relevant geometry:

- ordered collections of 7 points in projective plane $\mathbf{P}^2$
- quartic curves in $\mathbf{P}^2$ (genus 3)
- Abelian 3-folds (and their Kummer quotients)

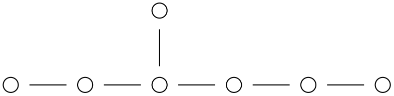
A classification problem

Start with a finite connected graph Γ. For every node $i \in \Gamma$, introduce a generator s_i and construct a group with relations

$$\begin{aligned} s_i^2 &= 1 && \\ (s_i s_j)^2 &= 1 && \text{if } i \text{ and } j \text{ are not adjacent} \\ (s_i s_j)^3 &= 1 && \text{if } i \text{ and } j \text{ are adjacent} \end{aligned}$$

Geometrically, the s_i are reflections in a Euclidean space.

When is this group $W(\Gamma)$ finite?



Dynkin diagrams

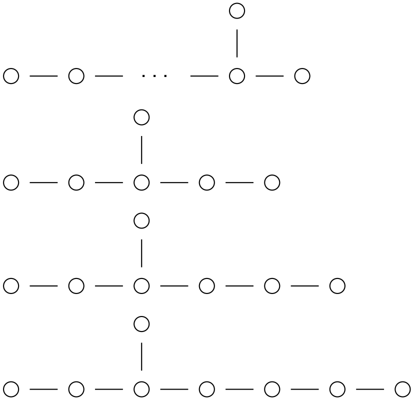
A_n

D_n

E_6

E_7

E_8

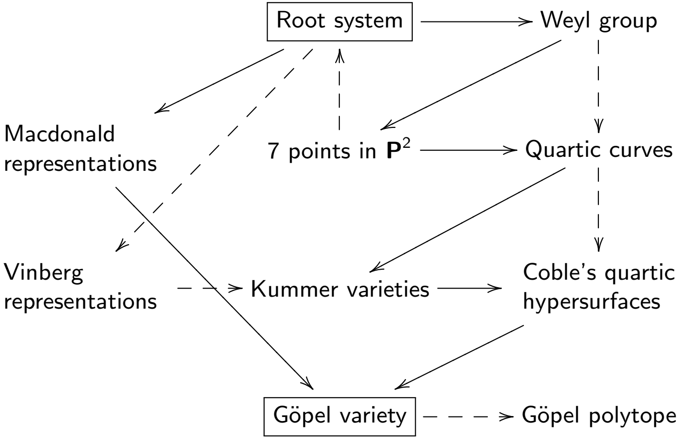


Theorem (Coxeter)

$W(\Gamma)$ is finite if and only if Γ appears above.

E_7 in classical algebraic geometry (overview)

Input: the finite group $W(\mathrm{E}_7)$



Output: Moduli space for the geometric objects under consideration

Each vector v in Euclidean space gives a reflection s_v:
s_v negates v and fixes the hyperplane orthogonal to it.

Take the following $63 = \binom{8}{2} + \binom{7}{3}$ vectors in $\mathbf{R}^8$:

$$\begin{aligned} e_i - e_j \qquad & (1 \le i < j \le 8), \\ \frac{1}{2}\Big(e_8 + \sum_{i \in \sigma} e_i - \sum_{j \notin \sigma} e_j\Big) \qquad & \sigma \subset \{1, 2, \ldots, 7\},\ |\sigma| = 3 \end{aligned}$$

The group generated by all s_v is isomorphic to $W(\mathrm{E}_7)$.

Arthur Cayley found a remarkable bijection between these 63 vectors and the nonzero vectors in the finite vector space $\mathbf{F}_2^6$.
(i.e., length 6 0-1 bitstrings under XOR)

A. Cayley, *J. Reine Angew. Mathe.* **87** (1879), 165–169.

	000	100	010	110	001	101	011	111
000		236	345	137	467	156	124	257
100	237	67	136	12	157	48	256	35
010	245	127	23	68	134	357	15	47
110	126	13	78	145	356	25	46	234
001	567	146	125	247	45	17	38	26
101	147	58	246	34	16	123	27	367
011	135	347	14	57	28	36	167	456
111	346	24	56	235	37	267	457	18

$$\langle x, y\rangle = x_1y_4 + x_2y_5 + x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3, \qquad (x, y \in \mathbf{F}_2^6)$$

$$\mathbf{Sp}_6(\mathbf{F}_2) = \{g \in \mathbf{GL}_6(\mathbf{F}_2) \mid \langle gx, gy\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbf{F}_2^6\}.$$

The table gives bijection between 63 vectors and $\mathbf{F}_2^6 \setminus 0$ (example: $247 \leftrightarrow 001110$) that preserves orthogonality.

Theorem (Cayley?)

$W(\mathrm{E}_7) \cong \mathbf{Z}/2 \times \mathbf{Sp}_6(\mathbf{F}_2)$

Given 7 ordered points $p_1, \ldots, p_7$ in general position in $\mathbf{P}^2$, do a change of coordinates so that

$$p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1], p_4 = [1:1:1].$$

In these coordinates, define $\sigma([x:y:z]) = [x^{-1} : y^{-1} : z^{-1}]$, so this gives us a birational involution

$$(p_1, \ldots, p_7) \mapsto (p_1, \ldots, p_4, \sigma(p_5), \sigma(p_6), \sigma(p_7))$$

on the (GIT) moduli space $(\mathbf{P}^2)^7$ of 7 ordered points in $\mathbf{P}^2$.

For $i = 1, \ldots, 6$, let s_i be the involution that swaps p_i and p_{i+1}. Then $s_1, \ldots, s_6, \sigma$ generates the group $W(\mathrm{E}_7)$ and this gives a birational action of $W(\mathrm{E}_7)$ on $(\mathbf{P}^2)^7$.

Given 7 points $p_1, \ldots, p_7 \in \mathbf{P}^2$ in general position, the space of cubic polynomials vanishing on them is 3-dimensional; fix a basis q_1, q_2, q_3. This gives a map

$$\begin{aligned} q\colon \mathbf{P}^2 \setminus \{p_1, \ldots, p_7\} &\to \mathbf{P}^2 \\ [x:y:z] &\mapsto [q_1(x,y,z) : q_2(x,y,z) : q_3(x,y,z)]. \end{aligned}$$

Consider the graph of q in $\mathbf{P}^2 \times \mathbf{P}^2$. The closure is a **del Pezzo surface** X and the projection $\pi_2\colon X \to \mathbf{P}^2$ is of degree 2, i.e., for almost all $x \in \mathbf{P}^2$, $\pi_2^{-1}(x)$ is 2 points.
The locus where this fails is defined by a quartic polynomial.

The resulting **quartic curve** (up to change of coordinates) is independent of the $W(\mathrm{E}_7)$-orbit of $p_1, \ldots, p_7$ under the action from the last slide. It is a smooth projective curve of genus 3.

For any smooth projective curve C of genus g, the set of all isomorphism classes of line bundles on C forms a group under tensor product.

Those of degree 0 also have the structure of a g-dimensional smooth projective variety $\mathcal{J}(C)$, the **Jacobian** of C. Alternatively, using differential forms, we could define $\mathcal{J}(C)$ as a special quotient of the form $\mathbf{C}^g/(\mathbf{Z}^g + \tau\mathbf{Z}^g)$

The **Kummer variety** $\mathcal{K}(C)$ of C is the quotient of $\mathcal{J}(C)$ by the involution $L \mapsto L^{-1}$ (the inverse map on line bundles). It naturally admits an embedding in projective space $\mathbf{P}^{2^g-1}$.

For our plane quartic, we have $g = 3$, and we will consider the Kummer variety as a subvariety of $\mathbf{P}^7$.

There is a natural subgroup of automorphisms in $\mathbf{GL}_8$ acting on $\mathcal{K}(C) \subset \mathbf{P}^7$. Let $(x_{ijk})_{i,j,k \in \mathbf{Z}/2}$ be the coordinates on $\mathbf{P}^7$. This comes from translation by 2-torsion points in $\mathcal{J}(C)$.

The **(finite) Heisenberg group** H is generated by the 6 operators

$$\begin{array}{ll} x_{ijk} \mapsto (-1)^i x_{ijk} & x_{ijk} \mapsto x_{i+1,j,k} \\ x_{ijk} \mapsto (-1)^j x_{ijk} & x_{ijk} \mapsto x_{i,j+1,k} \\ x_{ijk} \mapsto (-1)^k x_{ijk} & x_{ijk} \mapsto x_{i,j,k+1} \end{array}$$

The Heisenberg group $\widetilde{H}$ is obtained by adding all scalar matrices. The action of $\widetilde{H}$ preserves $\mathcal{K}(C)$.

Connecting to $W(\mathrm{E}_7)$: Let $N(\widetilde{H}) = \{g \in \mathbf{GL}_8 \mid g\widetilde{H}g^{-1} = \widetilde{H}\}$ be the normalizer. Then $N(\widetilde{H})/\widetilde{H} \cong \mathbf{Sp}_6(\mathbf{F}_2)$.

Arthur Coble (1878–1966) showed that $\mathcal{K}(C)$ is the singular locus of a quartic hypersurface $\mathcal{Q}(C)$ in $\mathbf{P}^7$, and that this is the unique such quartic hypersurface with this property.

Explicitly, given C, this means that there is a unique homogeneous quartic polynomial F_C so that $\mathcal{K}(C)$ is the solution set of the partial derivatives of F_C.

By uniqueness, the equation of $\mathcal{Q}(C)$ will be an invariant of the finite Heisenberg group H. The space of invariant quartic polynomials is 15-dimensional. So this equation has the following form:

$$
\begin{aligned}
F_C \quad = \quad & r \cdot (x_{000}^4 + x_{001}^4 + x_{010}^4 + x_{011}^4 + x_{100}^4 + x_{101}^4 + x_{110}^4 + x_{111}^4) \\
& + s_{001} \cdot (x_{000}^2 x_{001}^2 + x_{010}^2 x_{011}^2 + x_{100}^2 x_{101}^2 + x_{110}^2 x_{111}^2) \\
& + s_{010} \cdot (x_{000}^2 x_{010}^2 + x_{001}^2 x_{011}^2 + x_{100}^2 x_{110}^2 + x_{101}^2 x_{111}^2) \\
& + s_{011} \cdot (x_{000}^2 x_{011}^2 + x_{001}^2 x_{010}^2 + x_{100}^2 x_{111}^2 + x_{101}^2 x_{110}^2) \\
& + s_{100} \cdot (x_{000}^2 x_{100}^2 + x_{001}^2 x_{101}^2 + x_{010}^2 x_{110}^2 + x_{011}^2 x_{111}^2) \\
& + s_{101} \cdot (x_{000}^2 x_{101}^2 + x_{001}^2 x_{100}^2 + x_{010}^2 x_{111}^2 + x_{011}^2 x_{110}^2) \\
& + s_{110} \cdot (x_{000}^2 x_{110}^2 + x_{001}^2 x_{111}^2 + x_{010}^2 x_{100}^2 + x_{011}^2 x_{101}^2) \\
& + s_{111} \cdot (x_{000}^2 x_{111}^2 + x_{001}^2 x_{110}^2 + x_{010}^2 x_{101}^2 + x_{011}^2 x_{100}^2) \\
& \quad + t_{001} \cdot (x_{000} x_{010} x_{100} x_{110} + x_{001} x_{011} x_{101} x_{111}) \\
& \quad + t_{010} \cdot (x_{000} x_{001} x_{100} x_{101} + x_{010} x_{011} x_{110} x_{111}) \\
& \quad + t_{011} \cdot (x_{000} x_{011} x_{100} x_{111} + x_{001} x_{010} x_{101} x_{110}) \\
& \quad + t_{100} \cdot (x_{000} x_{001} x_{010} x_{011} + x_{100} x_{101} x_{110} x_{111}) \\
& \quad + t_{101} \cdot (x_{000} x_{010} x_{101} x_{111} + x_{001} x_{011} x_{100} x_{110}) \\
& \quad + t_{110} \cdot (x_{000} x_{001} x_{110} x_{111} + x_{010} x_{011} x_{100} x_{101}) \\
& \quad + t_{111} \cdot (x_{000} x_{011} x_{101} x_{110} + x_{001} x_{010} x_{100} x_{111})
\end{aligned}
$$

Question: what conditions are imposed on the coefficients r, s, t?
Let $\mathcal{G}$ be (the closure of) the set of all $[r : s_{100} : \cdots : t_{111}]$ (**Göpel variety**) such that the solution set of the partial derivatives of the above polynomial is the Kummer variety $\mathcal{K}(C)$ of some plane quartic curve C.

Theorem (Ren–S.–Schrader–Sturmfels)

The 6*-dimensional Göpel variety* $\mathcal{G}$ *has degree* 175 *in* $\mathbf{P}^{14}$*. The homogeneous coordinate ring of* $\mathcal{G}$ *is Gorenstein, it has the Hilbert series*

$$\frac{1+8z+36z^2+85z^3+36z^4+8z^5+z^6}{(1-z)^7},$$

and its defining prime ideal is minimally generated by 35 *cubics and* 35 *quartics. The graded Betti table of this ideal in the polynomial ring* $\mathbf{Q}[r, s_{001}, \ldots, t_{111}]$ *in* 15 *variables equals*

	0	1	2	3	4	5	6	7	8
total:	1	70	609	1715	2350	1715	609	70	1
0:	1	.	.	.	.	.	.	.	.
1:	.	.	.	.	.	.	.	.	.
2:	.	35	21	.	.	.	.	.	.
3:	.	35	588	1715	2350	1715	588	35	.
4:	.	.	.	.	.	.	21	35	.
5:	.	.	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	.	.	1

Recall that $\mathbf{Sp}_6(\mathbf{F}_2) = N(\widetilde{H})/\widetilde{H}$. So the space of coefficients r, s, t in the equation of the Coble quartic is a linear representation of $\mathbf{Sp}_6(\mathbf{F}_2)$.

Back to the root system:

- There are 135 collections of 7 roots which are pairwise orthogonal, and $W(\mathrm{E}_7)$ acts transitively on them.
- Each collection gives a degree 7 polynomial (take the product of the corresponding linear functionals), and the linear span of these 135 polynomials is 15-dimensional.
- This gives a linear representation of $W(\mathrm{E}_7)$, which is a special instance of a **Macdonald representation**.

Ignoring the $\mathbf{Z}/2$ factor, this is the same representation as above.

Let $c_1, \dots, c_7$ be a fixed set of pairwise orthogonal roots and do a change of coordinates to them. Then the matching of the two representations is as follows:

$$\begin{aligned}
r &= 4c_1c_2c_3c_4c_5c_6c_7 \\
s_{001} &= c_1c_2c_7(c_3^4 - 2c_3^2c_4^2 + c_4^4 - 2c_3^2c_5^2 - 2c_4^2c_5^2 + c_5^4 - 2c_3^2c_6^2 - 2c_4^2c_6^2 - 2c_5^2c_6^2 + c_6^4) \\
&\vdots \\
t_{111} &= c_4(-c_1^4c_2^2 + c_1^2c_2^4 + c_1^4c_3^2 - c_2^4c_3^2 - c_1^2c_3^4 + c_2^2c_3^4 - c_1^4c_5^2 - 2c_1^2c_2^2c_5^2 + 2c_2^2c_3^2c_5^2 + c_3^4c_5^2 \\
&\quad + c_1^2c_5^4 - c_3^2c_5^4 + 2c_1^2c_2^2c_6^2 + c_2^4c_6^2 - 2c_1^2c_3^2c_6^2 - c_3^4c_6^2 + 2c_1^2c_5^2c_6^2 - 2c_2^2c_5^2c_6^2 + c_5^4c_6^2 \\
&\quad - c_2^2c_6^4 + c_3^2c_6^4 - c_5^2c_6^4 + c_1^4c_7^2 - c_2^4c_7^2 + 2c_1^2c_3^2c_7^2 - 2c_2^2c_3^2c_7^2 + 2c_2^2c_5^2c_7^2 - 2c_3^2c_5^2c_7^2 \\
&\quad - c_5^4c_7^2 - 2c_1^2c_6^2c_7^2 + 2c_3^2c_6^2c_7^2 + c_6^4c_7^2 - c_1^2c_7^4 + c_2^2c_7^4 + c_5^2c_7^4 - c_6^2c_7^4)
\end{aligned}$$

So we can think of the r, s, t as functions on $\mathbf{R}^7$. Complexify this to $\mathbf{C}^7$, and we get a map on $\mathbf{P}^6$ (only defined on an open dense set)

$$\begin{aligned}
\mathbf{P}^6 &\dashrightarrow \mathbf{P}^{14} \\
[c_1 : \cdots : c_7] &\mapsto [r(c) : s_{100}(c) : \cdots : t_{111}(c)]
\end{aligned}$$

The closure of the image is the Göpel variety $\mathcal{G}$.

What is the "core" or "essence" of an algebraic variety? Here are two possible answers:

- Tropicalization: The Macdonald parametrization of the Göpel variety can be modified to construct a 35-dimensional toric variety. This leads to a "tropical" degeneration and a possible definition of a tropical moduli space of genus 3 curves with *level structure*.
- Arithmetic invariant theory: The Coble and Kummer can be constructed using *degeneracy loci* and are related to *Vinberg's θ-representations*. This is interesting to study over number fields or fields of positive characteristic.
  The θ-representation is $\bigwedge^4 V$ of $\mathbf{GL}(V)$ where $\dim V = 8$

In both cases, it really helps to have explicit equations!