Combinatorics and geometry of E_7

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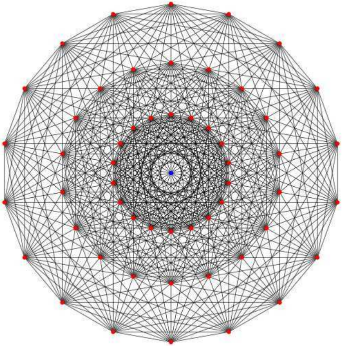
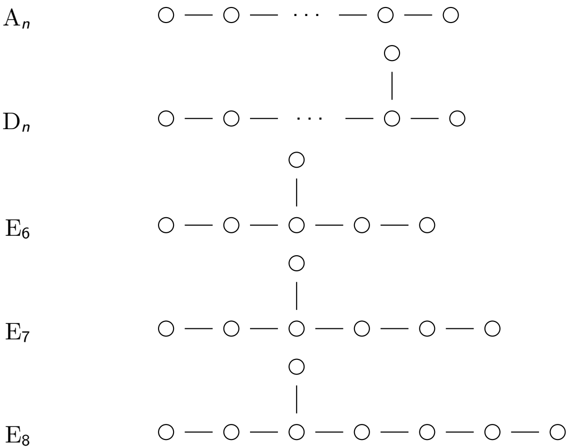


$$
\begin{tikzcd}
 & \text{Root system} \arrow[r] \arrow[dl] \arrow[ddl] & \text{Weyl group} \arrow[dl] \arrow[d, dashed] \\
\text{Macdonald representations} \arrow[ddr] & \text{7 points in } \mathbf{P}^2 \arrow[u, dashed] \arrow[r] & \text{Quartic curves} \arrow[dl] \arrow[d, dashed] \\
\text{Vinberg representations} \arrow[r] & \text{Kummer varieties} \arrow[r] & \text{Coble's quartic hypersurfaces} \arrow[dl] \\
 & \text{Göpel variety} \arrow[r] & \text{Göpel polytope}
\end{tikzcd}
$$

Start with a finite connected graph Γ. For every node $i \in \Gamma$, introduce a generator s_i and construct a group with relations

$$\begin{aligned} s_i^2 &= 1 \\ (s_i s_j)^2 &= 1 \qquad \text{if } i \text{ and } j \text{ are not adjacent} \\ (s_i s_j)^3 &= 1 \qquad \text{if } i \text{ and } j \text{ are adjacent} \end{aligned}$$

When is this group $W(\Gamma)$ finite?



These groups can be realized as groups generated by reflections in some Euclidean space.

Take the following $63 = 7 + \binom{7}{2} + \binom{7}{3}$ vectors in $\mathbf{R}^8$:

$$\begin{aligned} e_8 - e_i \qquad & (1 \le i \le 7), \\ e_i - e_j \qquad & (1 \le i < j \le 7), \\ \frac{1}{2}(e_8 + \sum_{i \in \sigma} e_i - \sum_{j \notin \sigma} e_j) \qquad & \sigma \subset \{1, 2, \dots, 7\},\ |\sigma| = 3 \end{aligned}$$

Each vector v ("positive root") gives a reflection s_v: s_v negates v and fixes the hyperplane orthogonal to it.

The group generated by all s_v is isomorphic to $W(\mathrm{E}_7)$. It's enough to take any collection of 7 positive roots in a "root basis" (e.g., $\{e_i - e_{i+1} \mid i = 1, \dots, 6\} \cup \{\frac{1}{2}(-1, -1, -1, -1, 1, 1, 1, 1)\}$)

More tangible: $W(\mathrm{E}_7) \cong \mathbf{Z}/2 \times \mathbf{Sp}_6(\mathbf{F}_2)$ (symplectic group over a finite field with 2 elements). This is a hint that $W(\mathrm{E}_7)$ is related to Siegel modular forms.

A. Cayley, *J. Reine Angew. Mathe.* **87** (1879), 165–169.

	000	100	010	110	001	101	011	111
000		236	345	137	467	156	124	257
100	237	67	136	12	157	48	256	35
010	245	127	23	68	134	357	15	47
110	126	13	78	145	356	25	46	234
001	567	146	125	247	45	17	38	26
101	147	58	246	34	16	123	27	367
011	135	347	14	57	28	36	167	456
111	346	24	56	235	37	267	457	18

For $x, y \in \mathbf{F}_2^6$, define

$$\langle x, y \rangle = x_1 y_4 + x_2 y_5 + x_3 y_6 + x_4 y_1 + x_5 y_2 + x_6 y_3.$$

The above table gives bijection between positive roots and $\mathbf{F}_2^6 \setminus 0$ (example: $247 \leftrightarrow 001110$). Two roots are orthogonal if and only if the corresponding vectors pair to 0.

Given 7 ordered points $p_1, \dots, p_7$ in general position in $\mathbf{P}^2$, do a change of coordinates so that

$$p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1], p_4 = [1:1:1].$$

In these coordinates, define $\sigma([x:y:z]) = [x^{-1} : y^{-1} : z^{-1}]$, so this gives us a birational involution

$$(p_1, \dots, p_7) \mapsto (p_1, \dots, p_4, \sigma(p_5), \sigma(p_6), \sigma(p_7))$$

on the (GIT) moduli space $(\mathbf{P}^2)^7$ of 7 ordered points in $\mathbf{P}^2$.

For $i = 1, \dots, 6$, let s_i be the involution that swaps p_i and p_{i+1}. Then $s_1, \dots, s_6, \sigma$ generates the group $W(\mathrm{E}_7)$ and this gives a birational action of $W(\mathrm{E}_7)$ on $(\mathbf{P}^2)^7$.

Given 7 points $p_1, \ldots, p_7 \in \mathbf{P}^2$ in general position, the space of cubic polynomials vanishing on them is 3-dimensional; fix a basis q_1, q_2, q_3. This gives a map

$$\begin{aligned} q\colon \mathbf{P}^2 \setminus \{p_1, \ldots, p_7\} &\to \mathbf{P}^2 \\ [x : y : z] &\mapsto [q_1(x, y, z) : q_2(x, y, z) : q_3(x, y, z)]. \end{aligned}$$

Consider the graph of q in $\mathbf{P}^2 \times \mathbf{P}^2$. The closure is a **del Pezzo surface** X and the projection $\pi_2\colon X \to \mathbf{P}^2$ is of degree 2, i.e., for almost all $x \in \mathbf{P}^2$, $\pi_2^{-1}(x)$ is 2 points. The locus where this fails is defined by a quartic polynomial.

The resulting **quartic curve** (up to change of coordinates) is independent of the $W(\mathrm{E}_7)$-orbit of $p_1, \ldots, p_7$ under the action from the last slide. It is a smooth projective curve of genus 3.

For any smooth projective curve C of genus g, the set of all isomorphism classes of line bundles on C forms a group under tensor product.

Those of degree 0 also have the structure of a g-dimensional smooth projective variety $\mathcal{J}(C)$, the **Jacobian** of C.

The **Kummer variety** $\mathcal{K}(C)$ of C is the quotient of $\mathcal{J}(C)$ by the involution $L \mapsto L^{-1}$ (the inverse map on line bundles). It naturally admits an embedding in projective space $\mathbf{P}^{2^g-1}$.

For our plane quartic, we have $g = 3$, and we will consider the Kummer variety as a subvariety of $\mathbf{P}^7$.

There is a natural subgroup of automorphisms in $\mathbf{GL}_8$ acting on $\mathcal{K}(C) \subset \mathbf{P}^7$. Let $(x_{ijk})_{i,j,k \in \mathbf{Z}/2}$ be the coordinates on $\mathbf{P}^7$.

The **(finite) Heisenberg group** H is generated by the 6 operators

$$\begin{aligned} x_{ijk} &\mapsto (-1)^i x_{ijk} & \qquad x_{ijk} &\mapsto x_{i+1,j,k} \\ x_{ijk} &\mapsto (-1)^j x_{ijk} & \qquad x_{ijk} &\mapsto x_{i,j+1,k} \\ x_{ijk} &\mapsto (-1)^k x_{ijk} & \qquad x_{ijk} &\mapsto x_{i,j,k+1} \end{aligned}$$

The Heisenberg group $\widetilde{H}$ is obtained by adding all scalar matrices. The action of $\widetilde{H}$ preserves $\mathcal{K}(C)$.

Connecting to $W(\mathrm{E}_7)$: Let $N(\widetilde{H}) = \{g \in \mathbf{GL}_8 \mid g\widetilde{H}g^{-1} = \widetilde{H}\}$ be the normalizer. Then $N(\widetilde{H})/\widetilde{H} \cong \mathbf{Sp}_6(\mathbf{F}_2)$.

Arthur Coble (1878–1966) showed that $\mathcal{K}(C)$ is the singular locus of a quartic hypersurface $\mathcal{Q}(C)$ in $\mathbf{P}^7$, and that this is the unique such quartic hypersurface with this property.

(It was later shown that $\mathcal{Q}(C)$ is the moduli space of semistable principal $\mathbf{SL}_2$ bundles on C)

By uniqueness, the equation of $\mathcal{Q}(C)$ will be an invariant of the finite Heisenberg group H. The space of invariant quartic polynomials is 15-dimensional. So this equation has the following form:

$$\begin{aligned}
F_C \quad = \quad & r\cdot(x_{000}^4+x_{001}^4+x_{010}^4+x_{011}^4+x_{100}^4+x_{101}^4+x_{110}^4+x_{111}^4)\\
&+s_{001}\cdot(x_{000}^2x_{001}^2+x_{010}^2x_{011}^2+x_{100}^2x_{101}^2+x_{110}^2x_{111}^2)\\
&+s_{010}\cdot(x_{000}^2x_{010}^2+x_{001}^2x_{011}^2+x_{100}^2x_{110}^2+x_{101}^2x_{111}^2)\\
&+s_{011}\cdot(x_{000}^2x_{011}^2+x_{001}^2x_{010}^2+x_{100}^2x_{111}^2+x_{101}^2x_{110}^2)\\
&+s_{100}\cdot(x_{000}^2x_{100}^2+x_{001}^2x_{101}^2+x_{010}^2x_{110}^2+x_{011}^2x_{111}^2)\\
&+s_{101}\cdot(x_{000}^2x_{101}^2+x_{001}^2x_{100}^2+x_{010}^2x_{111}^2+x_{011}^2x_{110}^2)\\
&+s_{110}\cdot(x_{000}^2x_{110}^2+x_{001}^2x_{111}^2+x_{010}^2x_{100}^2+x_{011}^2x_{101}^2)\\
&+s_{111}\cdot(x_{000}^2x_{111}^2+x_{001}^2x_{110}^2+x_{010}^2x_{101}^2+x_{011}^2x_{100}^2)\\
&\quad+t_{001}\cdot(x_{000}x_{010}x_{100}x_{110}+x_{001}x_{011}x_{101}x_{111})\\
&\quad+t_{010}\cdot(x_{000}x_{001}x_{100}x_{101}+x_{010}x_{011}x_{110}x_{111})\\
&\quad+t_{011}\cdot(x_{000}x_{011}x_{100}x_{111}+x_{001}x_{010}x_{101}x_{110})\\
&\quad+t_{100}\cdot(x_{000}x_{001}x_{010}x_{011}+x_{100}x_{101}x_{110}x_{111})\\
&\quad+t_{101}\cdot(x_{000}x_{010}x_{101}x_{111}+x_{001}x_{011}x_{100}x_{110})\\
&\quad+t_{110}\cdot(x_{000}x_{001}x_{110}x_{111}+x_{010}x_{011}x_{100}x_{101})\\
&\quad+t_{111}\cdot(x_{000}x_{011}x_{101}x_{110}+x_{001}x_{010}x_{100}x_{111})
\end{aligned}$$

Question: what conditions are imposed on the coefficients r, s, t?
Let $\mathcal{G}$ be the (closure) of all possible $r, s_{100}, \ldots, t_{111}$ (**Göpel variety**).

Theorem (Ren–S.–Schrader–Sturmfels)

The 6*-dimensional Göpel variety* $\mathcal{G}$ *has degree* 175 *in* $\mathbf{P}^{14}$*. The homogeneous coordinate ring of* $\mathcal{G}$ *is Gorenstein, it has the Hilbert series*

$$\frac{1+8z+36z^2+85z^3+36z^4+8z^5+z^6}{(1-z)^7},$$

and its defining prime ideal is minimally generated by 35 *cubics and* 35 *quartics. The graded Betti table of this ideal in the polynomial ring* $\mathbf{Q}[r, s_{001}, \ldots, t_{111}]$ *in* 15 *variables equals*

```
       0  1   2    3    4    5   6  7 8
total: 1 70 609 1715 2350 1715 609 70 1
    0: 1  .   .    .    .    .   .  . .
    1: .  .   .    .    .    .   .  . .
    2: . 35  21    .    .    .   .  . .
    3: . 35 588 1715 2350 1715 588 35 .
    4: .  .   .    .    .    .  21 35 .
    5: .  .   .    .    .    .   .  . .
    6: .  .   .    .    .    .   .  . 1
```

Recall that $\mathbf{Sp}_6(\mathbf{F}_2) = N(\widetilde{H})/\widetilde{H}$. So the space of coefficients r, s, t in the equation of the Coble quartic is a linear representation of $\mathbf{Sp}_6(\mathbf{F}_2)$, and hence of $W(\mathrm{E}_7) \cong \mathbf{Z}/2 \times \mathbf{Sp}_6(\mathbf{F}_2)$.

Back to the root system:

- There are 135 collections of 7 roots which are pairwise orthogonal, and $W(\mathrm{E}_7)$ acts transitively on them.
- Each collection gives a degree 7 polynomial (take the product of the corresponding linear functionals), and the linear span of these 135 polynomials is 15-dimensional.
- This gives a linear representation of $W(\mathrm{E}_7)$, which is a special instance of a **Macdonald representation**.

Ignoring the $\mathbf{Z}/2$ factor, this is the same representation as above.

Let $c_1, \ldots, c_7$ be a fixed set of pairwise orthogonal roots and do a change of coordinates to them. Then the matching of the two representations is as follows:

$$
\begin{aligned}
r &= 4c_1c_2c_3c_4c_5c_6c_7 \\
s_{001} &= c_1c_2c_7(c_3^4 - 2c_3^2c_4^2 + c_4^4 - 2c_3^2c_5^2 - 2c_4^2c_5^2 + c_5^4 - 2c_3^2c_6^2 - 2c_4^2c_6^2 - 2c_5^2c_6^2 + c_6^4) \\
&\vdots \\
t_{111} &= c_4(-c_1^4c_2^2 + c_1^2c_2^4 + c_1^4c_3^2 - c_2^4c_3^2 - c_1^2c_3^4 + c_2^2c_3^4 - c_1^4c_5^2 - 2c_1^2c_2^2c_5^2 + 2c_2^2c_3^2c_5^2 + c_3^4c_5^2 \\
&\quad + c_1^2c_5^4 - c_3^2c_5^4 + 2c_1^2c_2^2c_6^2 + c_2^4c_6^2 - 2c_1^2c_3^2c_6^2 - c_3^4c_6^2 + 2c_1^2c_5^2c_6^2 - 2c_2^2c_5^2c_6^2 + c_5^4c_6^2 \\
&\quad - c_2^2c_6^4 + c_3^2c_6^4 - c_5^2c_6^4 + c_1^4c_7^2 - c_2^4c_7^2 + 2c_1^2c_3^2c_7^2 - 2c_2^2c_3^2c_7^2 + 2c_2^2c_5^2c_7^2 - 2c_3^2c_5^2c_7^2 \\
&\quad - c_5^4c_7^2 - 2c_1^2c_6^2c_7^2 + 2c_3^2c_6^2c_7^2 + c_6^4c_7^2 - c_1^2c_7^4 + c_2^2c_7^4 + c_5^2c_7^4 - c_6^2c_7^4)
\end{aligned}
$$

So we can think of the r, s, t as functions on $\mathbf{R}^7$. Complexify this to $\mathbf{C}^7$, and we get a map on $\mathbf{P}^6$ (only defined on an open dense set)

$$
\mathbf{P}^6 \dashrightarrow \mathbf{P}^{14}
$$

$$
[c_1 : \cdots : c_7] \mapsto [r(c) : s_{100}(c) : \cdots : t_{111}(c)]
$$

The closure of the image is the Göpel variety $\mathcal{G}$.

The Macdonald perspective offers something new. Rather than take a basis of the 135 degree 7 polynomials, use all of them to get a map $\mathbf{P}^6 \dashrightarrow \mathbf{P}^{134}$.

Also, instead of treating each root as a vector in $\mathbf{R}^7$, pretend that they are all linearly independent. Then the map can be lifted to a **monomial** map

$$\mathbf{P}^{62} \dashrightarrow \mathbf{P}^{134}.$$

The closure of the image $\mathcal{T}$ is automatically a **toric variety**, that is, it has an action of a torus with a dense orbit. As such, its ideal of definition is generated by binomials, i.e., differences of monomials.

In fact, $\mathcal{G} = \mathcal{T} \cap \mathbf{P}^{14}$, so the ideal of definition of $\mathcal{G}$ (in $\mathbf{P}^{134}$ is generated by binomials and linear equations (which turn out to be trinomials).

Göpel polytope

Associated to a monomial map, we naturally get a polytope by thinking of each monomial as a 0-1 vector and taking the convex hull.

In our case, the Göpel polytope $\mathcal{P} \subset \mathbf{R}^{63}$ can be described as follows: identify each coordinate of $\mathbf{R}^{63}$ with positive roots (equivalently, $\mathbf{F}_2^6 \setminus 0$). Each vertex has 7 1's and 56 0's; the 1's correspond to 7 pairwise orthogonal roots (equivalently, a Lagrangian subspace).

<u>Question</u>: Is $\mathcal{P}$ a normal polytope? i.e., is every lattice point of $d\mathcal{P}$ (dth dilate of $\mathcal{P}$) a sum of d lattice points in $\mathcal{P}$?

Tropicalization

The Göpel variety $\mathcal{G}$ is a compactification of the moduli space of non-hyperelliptic genus 3 curves (i.e., plane quartic curves) with level structure.

From our presentation of $\mathcal{G}$ as defined by linear trinomials and cubic and quartic binomials in $\mathbf{P}^{134}$, we expect that its tropicalization behaves well, especially since $\mathcal{G}$ has a parametrization in terms of a linear map ℓ and a monomial map m:

$$\mathbf{P}^6 \xrightarrow{\ell} \mathbf{P}^{62} \overset{m}{\dashrightarrow} \mathbf{P}^{134}.$$

This approach requires calculating the "Bergman fan" of the (matroid of) the root system E_7, which is currently out of our reach.

However, this would give a hint on how to define tropical moduli spaces with level structure.

Let $\mathfrak{g}$ be a simple Lie algebra (over $\mathbf{C}$) with an order d automorphism θ. Then θ gives an eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}/d} \mathfrak{g}_i.$$

Vinberg studied the invariant theory of such decompositions. Namely, the action of $\mathrm{Group}(\mathfrak{g}_0)$ (the associated simply-connected complex Lie group) on $\mathfrak{g}_1$ behaves like the action of $\mathrm{Group}(\mathfrak{g})$ on $\mathfrak{g}$.

It makes sense to say that an element of $\mathfrak{g}$ is semisimple (diagonalizable) or nilpotent, so the same notions carry over to $\mathfrak{g}_1$. A Cartan subspace of $\mathfrak{g}_1$ is a maximal subspace consisting of pairwise commuting semisimple elements. Vinberg showed that all of them form one orbit under $\mathrm{Group}(\mathfrak{g}_0)$.

Let $\mathfrak{g}$ be the simple Lie algebra of type E_7. It has an order 2 automorphism θ such that the decomposition is

$$\mathfrak{g} = \mathfrak{sl}_8 \oplus \bigwedge^4 \mathbf{C}^8$$

Cartan subspaces $\mathfrak{h} \subset \bigwedge^4 \mathbf{C}^8$ have dimension 7.

Vinberg's theory tells us that $N(\mathfrak{h})/Z(\mathfrak{h}) \cong W(\mathrm{E}_7)$ where

$$\begin{aligned} N(\mathfrak{h}) &= \{g \in \mathrm{Group}(\mathfrak{g}_0) \mid g \cdot \mathfrak{h} = \mathfrak{h}\} \\ Z(\mathfrak{h}) &= \{g \in \mathrm{Group}(\mathfrak{g}_0) \mid g \cdot x = x \text{ for all } x \in \mathfrak{h}\}. \end{aligned}$$

(Note: if we use $\mathrm{Group}(\mathfrak{g})$ instead of $\mathrm{Group}(\mathfrak{g}_0)$, this is well known Lie theory, but in general Cartan subspaces in $\mathfrak{g}_1$ could be strictly smaller than those in $\mathfrak{g}$)

Let A be an 8-dimensional complex vector space with basis

$$a_1 = x_{000},\ a_2 = x_{100},\ a_3 = x_{010},\ a_4 = x_{110},$$
$$a_5 = x_{001},\ a_6 = x_{101},\ a_7 = x_{011},\ a_8 = x_{111}.$$

The Heisenberg group H acts on $\bigwedge^4 A$ and the space of invariants has the following basis:

$$\begin{aligned}
h_1 &= a_1 \wedge a_2 \wedge a_3 \wedge a_4 + a_5 \wedge a_6 \wedge a_7 \wedge a_8,\\
h_2 &= a_1 \wedge a_2 \wedge a_5 \wedge a_6 + a_3 \wedge a_4 \wedge a_7 \wedge a_8,\\
h_3 &= a_1 \wedge a_3 \wedge a_5 \wedge a_7 + a_2 \wedge a_4 \wedge a_6 \wedge a_8,\\
h_4 &= a_1 \wedge a_4 \wedge a_6 \wedge a_7 + a_2 \wedge a_3 \wedge a_5 \wedge a_8,\\
h_5 &= a_1 \wedge a_3 \wedge a_6 \wedge a_8 + a_2 \wedge a_4 \wedge a_5 \wedge a_7,\\
h_6 &= a_1 \wedge a_4 \wedge a_5 \wedge a_8 + a_2 \wedge a_3 \wedge a_6 \wedge a_7,\\
h_7 &= a_1 \wedge a_2 \wedge a_7 \wedge a_8 + a_3 \wedge a_4 \wedge a_5 \wedge a_6.
\end{aligned}$$

Consider $G = \mathbf{GL}_7(\mathbf{C})$ acting on $V = \bigwedge^3 \mathbf{C}^7$. There are finitely many orbits of this action, but we'll be interested in 3 of them:

- There is an invariant hypersurface of degree 7. It is the projective dual of (the affine cone over) $\mathbf{Gr}(3,7)$.
- Singular locus of the above hypersurface has codimension 4.
- Singular locus of the above orbit closure has codimension 7.

These can also be defined in a "relative" setting: if $\mathcal{E}$ is a rank 7 vector bundle on a space and $\mathcal{L}$ is a line bundle, then the total space of $\bigwedge^3 \mathcal{E} \otimes \mathcal{L}$ has subvarieties like the above
(we can define them locally where the bundle is trivial, and they don't depend on a choice of basis, so they agree on overlaps and patch together)

Consider the space $\mathbf{P}^7$ with rank 7 vector bundle $\mathcal{E} = \Omega^1_{\mathbf{P}^7}$ (cotangent bundle) and $\mathcal{L} = \mathcal{O}_{\mathbf{P}^7}(4)$. Let X_1, X_4, X_7 be the subvarieties of $\bigwedge^3 \mathcal{E} \otimes \mathcal{L}$ from the last slide.

The space of sections of $\bigwedge^3 \mathcal{E} \otimes \mathcal{L}$ is naturally isomorphic to $\bigwedge^4 \mathbf{C}^8$

Given a section $s \colon \mathbf{P}^7 \to \bigwedge^3 \mathcal{E} \otimes \mathcal{L}$, we define 3 "degeneracy" loci

$$Z_i = \{x \in \mathbf{P}^7 \mid s(x) \in X_i\}.$$

For a generic $s \in \bigwedge^4 \mathbf{C}^8$,

- $Z_4 = \mathcal{K}(C)$, Kummer variety for a plane quartic curve C,
- $Z_1 = \mathcal{Q}(C)$ is its Coble quartic hypersurface, and
- $Z_7 = \mathrm{Sing}(\mathcal{K}(C)) = 64$ points

If we take a section of the form $s = c_1 h_1 + \cdots + c_7 h_7$, then the degeneracy locus construction for the Coble quartic agrees with the parametrization given by the Macdonald representation.

This is mostly satisfactory, but elements like above are semisimple, so from the point of view of degenerations of Kummer varieties, or plane quartics, we are missing a lot of stuff (like nilpotent vectors). The GIT quotients of $\bigwedge^4 \mathbf{C}^8$ and $\mathfrak{h}$ are the same, but the stack quotients are very different!

Also, the fact that every semisimple element has the above form only works over an algebraically closed field (e.g., a rational matrix need not be diagonalizable over $\mathbf{Q}$, but could be over $\mathbf{C}$). For arithmetic questions, it is better to work over $\mathbf{Q}$ or even $\mathbf{Z}$, and then we would prefer $\bigwedge^4 \mathbf{Z}^8$ over the Cartan subspace.