Sheaf cohomology and non-normal varieties

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We're interested in the following situation (over a field K):

- V is a vector space
- X is a projective variety
- Short exact sequence of locally free sheaves over X:

$$0 \to \mathcal{S} \to V \otimes \mathcal{O}_X \to \mathcal{T} \to 0$$

- Identifying locally free sheaves with vector bundles, we have a projection map $p_1 \colon \mathcal{S} \to V$. We say that $Y = p_1(\mathcal{S})$ is **collapsing** of $\mathcal{S}$.
- Many interesting varieties (in linear algebra) can be realized as Y as above. We are interested in studying the equations and minimal free resolutions of Y.

Note that $\mathcal{O}_{\mathcal{S}}$ is a regular zero section of $p_2^*\mathcal{T}$ over $\mathcal{O}_{X\times V}$, so we have the Koszul resolution

$$\cdots \to \bigwedge^{i}(p_2^*\mathcal{T}^*) \to \bigwedge^{i-1}(p_2^*\mathcal{T}^*) \to \cdots \to \mathcal{O}_{X\times V} \to \mathcal{O}_{\mathcal{S}} \to 0.$$

Taking pushforwards, we can construct a minimal complex $\mathbf{F}_\bullet$ with

$$\mathbf{F}_i = \bigoplus_{j\geq 0} \mathrm{H}^j(X; \bigwedge^{i+j}(\mathcal{T}^*)) \otimes \mathcal{O}_V(-i-j)$$

whose homology (concentrated in non-positive degrees) is

$$\mathrm{H}_{-i}(\mathbf{F}_\bullet) = \mathrm{R}^i p_{1*}\mathcal{O}_{\mathcal{S}} = \bigoplus_{j\geq 0} \mathrm{H}^i(X; \mathrm{Sym}(\mathcal{S}^*))$$

- In particular, if $\mathrm{R}^i p_{1*}\mathcal{O}_{\mathcal{S}} = 0$ for $i > 0$, the complex $\mathbf{F}_\bullet$ would be a resolution for $p_{1*}\mathcal{O}_{\mathcal{S}}$ (assuming we could calculate the cohomology of $\bigwedge^d \mathcal{T}^*$).
- We are interested in the cases when p_1 is a desingularization for Y. Then $p_{1*}\mathcal{O}_S = \widetilde{\mathcal{O}}_Y$ is the normalization of $\mathcal{O}_Y$.
- In characteristic 0, the condition $\mathrm{R}p_{1*}\mathcal{O}_{\mathcal{S}} = \mathcal{O}_Y$ is called **rational singularities**. (In positive characteristic, one also requires that $\mathrm{R}p_{1*}\omega_{\mathcal{S}} = 0$ for $i > 0$, but we don't need this condition here.)
- So the best case is when Y has rational singularities because then we get a minimal free resolution of $\mathcal{O}_Y$.

Examples of rational singularities

- Determinantal varieties: Let V be the space of $n \times m$ matrices, or $n \times n$ (skew-)symmetric matrices. The variety of matrices with rank $\leq r$ for a given r has rational singularities.
- Type A nilpotent orbits: Let V be the space of $n \times n$ matrices. Fix a partition λ of n. The set of nilpotent matrices with Jordan normal form with Jordan blocks of sizes specified by λ is a locally closed subvariety. Its closure has rational singularities.

For Example 1: let $V = \mathrm{Hom}(E, F)$ and take X be the Grassmannian $\mathbf{Gr}(r, F)$. It has a tautological rank r subbundle $\mathcal{R} \subset F \otimes \mathcal{O}_X$. Take $\mathcal{S} = \mathcal{H}om(E, \mathcal{R})$. The minimal free resolution was calculated by Lascoux in char. 0. (Skew-)symmetry is similar.

For Example 2: X is a partial flag variety and $\mathcal{S}$ is its cotangent bundle. The equations were calculated by Weyman in char. 0.

- The next most complicated case after rational singularities would be varieties whose normalization has rational singularities, i.e., we have $\mathrm{R}^i p_{1*}\mathcal{O}_Y = 0$ for $i > 0$.
- The naive thing to do is to consider the short exact sequence

$$0 \to \mathcal{O}_Y \to \widetilde{\mathcal{O}}_Y \to C \to 0,$$

so C is a module supported on the non-normal locus of Y.
- If we are lucky, we can calculate a presentation or minimal free resolution for C, and use this to get equations or minimal free resolution for $\mathcal{O}_Y$.
- I don't know a general framework for doing this, but I will explain some examples where it can be done. Assume char. 0 from now on for simplicity of statements.

- Motivating example: Nilpotent orbits in other Lie types. Take a (semi)simple Lie group G with Lie algebra $\mathfrak{g}$. The nullcone of $\mathfrak{g}$ is the vanishing locus of all G-invariant functions on $\mathfrak{g}$, and it has finitely many G-orbits.
- Except some small cases, all non-type A ($\mathfrak{sl}_n$) Lie algebras have non-normal orbit closures.
- Not too bad: normalizations are always Gorenstein with rational singularities

Let $B_1, \dots, B_n, A$ be vector spaces of dimensions $d_1, \dots, d_n, e$. Set $\mathbf{B} = B_1 \otimes \cdots \otimes B_n$. We consider the variety

$$Y = \{\psi \in \mathrm{Hom}(\mathbf{B}, A) \mid \ker \psi \text{ contains a rank 1 tensor}\}$$

These are **hyperdeterminantal varieties**, which are the supports of the tensor complexes (as defined in Berkesch's talk).
We can take $X = \mathbf{P}(B_1) \times \cdots \times \mathbf{P}(B_n)$ and

$$\mathcal{S} = \mathcal{H}om((\mathbf{B} \otimes \mathcal{O}_X)/\mathcal{O}_X(-1, \dots, -1), A \otimes \mathcal{O}_X).$$

In general they have complicated singularities, i.e., usually $p_{1*}\mathcal{O}_{\mathcal{S}}$ has many nonzero higher direct images. So they could be a good set of examples to study since there are many parameters to tweak.

We focus on $n = 2$, $d_1 = 2$, $d_2 = d$ and $e = d_2 + 2$ so that there are no higher direct images. In this case, we study maps from the space of $2 \times d$ matrices to a vector space of dimension $d + 2$ whose kernel contains a rank 1 matrix. Alternatively: pencils of $d \times (d + 2)$ matrices containing a matrix not of full rank.
The normalization has the presentation

$$\begin{pmatrix} \bigwedge^{d+1} A^* \otimes \\ \det B_1 \otimes S^{d-1} B_1 \\ \otimes \det B_2 \otimes B_2 \end{pmatrix} \otimes \mathcal{O}_V(-d-1) \to \mathcal{O}_V \oplus \begin{pmatrix} \bigwedge^{d} A^* \otimes \\ \det B_1 \otimes S^{d-2} B_1 \\ \otimes \det B_2 \end{pmatrix} \otimes \mathcal{O}_V(-d)$$

Since everything is equivariant with respect to $G = \mathbf{GL}(A) \times \mathbf{GL}(B_1) \times \mathbf{GL}(B_2)$ and the relations are irreducible, we get the presentation matrix for C by removing $\mathcal{O}_V$ from the generators.
We can get the equations for Y in terms of representations of G.

Set $e' - 1 = \sum_{i=1}^{n}(d_i - 1)$.

- In the case $e' = e$, the hyperdeterminantal variety is an irreducible hypersurface, cut out by a **hyperdeterminant**.
- In general, the hyperdeterminantal variety is defined (set-theoretically) by the hyperdeterminants of the $d_1 \times \cdots \times d_n \times e'$-subtensors of $\mathbf{B} \otimes A$. It has codimension $e - \sum_{i=1}^{n}(d_i - 1)$.
- For $2 \times 2 \times 4$, the $2 \times 2 \times 3$ hyperminors form a 10-dimensional space of sextics. To get the radical ideal, add the determinant of $\mathbf{B} \otimes A$.

Equations of hyperdeterminantal varieties (cont.)

- For $2 \times 3 \times 5$, the $2 \times 3 \times 4$ hyperminors form a 35-dimensional space of degree 12 equations. Flatten this tensor to 6×5. Generically, such a matrix has corank 1, and the kernel element is given by the 5×5 minors. The 2×2 minors of this kernel element give (non-minimal) degree 10 equations that must vanish. For the radical ideal, we need 10 degree 9 equations

$$(\det A^*) \otimes \bigwedge^4 A^* \otimes (\det B_1)^4 \otimes B_1 \otimes (\det B_2)^3,$$

  and their meaning is not clear to me.
- For general $2 \times d \times (d+2)$, the hyperminors have degree $d(d+1)$. One needs additional degree $2d+3$ equations for the radical ideal. I can identify the representation, but their meaning is not clear to me.

- Let $L \subset U$ be vector spaces of dimensions d, n. For $s \leq d$, set

  $\mathcal{K}_{s,d,n} = \{\varphi \in \mathrm{End}(U) \mid \varphi$ preserves an s-dim. subspace of $L\}$,

  which is the **Kalman variety** introduced by Ottaviani–Sturmfels.
- To desingularize, we take $V = \mathrm{End}(U)$, $X = \mathbf{Gr}(s, L)$ and

  $$\mathcal{S} = \{(\varphi, W) \mid \varphi(W) \subseteq W\}$$

  is the subbundle of $V \otimes \mathcal{O}_X$ generated by $\mathcal{E}nd(\mathcal{R})$ and $\mathcal{H}om(U/\mathcal{R}, U)$. Then φ_1 is an isomorphism outside of $\mathcal{K}_{s+1,d,n}$.
- If $\varphi \in \mathcal{K}_{s+1,d,n}$ has distinct eigenvalues, then $p_1^{-1}(\varphi)$ is $s+1$ points. By Zariski's connectedness theorem, we see that $\mathcal{K}_{s+1,d,n}$ is the non-normal locus of $\mathcal{K}_{s,d,n}$.

Theorem (Sam)

We have exact sequences

$$0 \to \mathcal{O}_{\mathcal{K}_{1,2,n}} \to \widetilde{\mathcal{O}}_{\mathcal{K}_{1,2,n}} \to \mathcal{O}_{\mathcal{K}_{2,2,n}}(-1) \to 0.$$

$$0 \to \mathcal{O}_{\mathcal{K}_{1,3,n}} \to \widetilde{\mathcal{O}}_{\mathcal{K}_{1,3,n}} \to \widetilde{\mathcal{O}}_{\mathcal{K}_{2,3,n}}(-1) \to \mathcal{O}_{\mathcal{K}_{3,3,n}}(-3) \to 0.$$

Note that $\mathcal{K}_{d,d,n}$ is a linear subvariety. Using the above, we get the equations for $\mathcal{K}_{1,d,n}$ for $d = 2, 3$ (and free resolution when $d = 2$).

Conjecture

Set $B_s = \widetilde{\mathcal{O}}_{\mathcal{K}_{s,d,n}}(-s(s-1)/2)$*. There is a long exact sequence*

$$0 \to \mathcal{O}_{1,d,n} \to B_1 \to B_2 \to \cdots \to B_d \to 0$$

We can check this when $n = d + 1$.

Type G_2 nilpotent orbits ($1 \Leftarrow 2$)

- The normalization of any nilpotent orbit in any semisimple Lie algebra has rational singularities.
- The Lie algebra $\mathfrak{g}_2$ has 5 nilpotent orbit closures which form a chain $O(12) \geq O(10) \geq O(8) \geq O(6) \geq \{0\}$. All orbit closures are normal except $O(8)$.
- $O(6)$ is the affine cone over a homogeneous space and has coordinate ring $\bigoplus_{k\geq 0} V_{k\omega_2}$. The cokernel $\widetilde{\mathcal{O}}_{O(8)}/\mathcal{O}_{O(8)}$ is $\bigoplus_{k\geq 0} V_{\omega_1+k\omega_2}$ where $V_{\omega_1+k\omega_2}$ is in degree $k+1$, so the module structure is by Cartan multiplication.
- We can calculate the minimal free resolutions of all orbit closures. The ideal of $O(8)$ is generated by 1 quadric (Killing form), 7 cubics (V_{ω_1}), and 77 quartics ($V_{2\omega_2}$).
- These equations can be obtained from the intersection $O(3,3,2) \cap \mathfrak{g}_2$ via the embedding $\mathfrak{g}_2 \subset \mathfrak{so}_7$