Schubert complexes and degeneracy loci

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Let X be a smooth variety and $Z \subset X$ a k-dimensional subvariety. This gives a class $[Z] \in \mathrm{H}_*(X)$ in the homology of X. How can we find formulas for $[Z]$?

Naive approach: Since X is smooth, we might try to resolve $\mathcal{O}_Z$ by vector bundles on X. This gives a formula in K-theory, and the Chern character

$$\mathrm{ch}\colon \mathrm{K}(X) \otimes \mathbf{Q} \to \mathrm{H}_*(X; \mathbf{Q})$$

has the property that $\mathrm{ch}([\mathcal{O}_Z]) = [Z] + \alpha$ where $[Z] \in \mathrm{H}_k(X; \mathbf{Q})$ and α lives in lower degrees.
Probably too naive though!

Given n, m, let $\varphi = (\varphi_{i,j})$ be an $n \times m$ matrix of indeterminants.
For $r \leq \min(n, m)$, let $I_r \subset \mathbf{Z}[\varphi_{i,j}]$ be the ideal defined by the $r \times r$ minors of φ.
We can specialize the values of $\varphi_{i,j}$ to the entries in some other ring R, we'll call the resulting ideal **determinantal ideals**.

Another interpretation: let E, F be free $\mathbf{Z}$-modules of ranks m, n. Then $\varphi\colon E \to F$ is a "generic" map, and we ask when its rank is less than r.

Global version: E and F are vector bundles over X and $\varphi\colon E \to F$ is a map of bundles. Then the subvariety $Z_r = \{x \in X \mid \operatorname{rank} \varphi|_x < r\}$ is locally defined by the $r \times r$ minors of φ.

We might try the naive approach with Z_r.
First we consider $I_r \subset K[\varphi_{i,j}]$ where K is a field. Does the minimal free resolution of I_r depend on the characteristic of K? Assume that $m \geq n$ without loss of generality.

- (Eagon–Northcott) If $r = n$, then characteristic of K is irrelevant.
- Ditto if $r = n - 1$ (Akin–Buchsbaum–Weyman) or $r = n - 2$ (Hashimoto).
- (Hashimoto) If $r \leq n - 3$ and $n \geq 5$, then characteristic is important (in particular, things differ in characteristic 3).

This suggests that the naive approach might not be such a good idea.

Since X is smooth, the K-theory of vector bundles is the same as the K-theory of coherent sheaves. We can put a filtration on $\mathrm{K}(X)$ by setting $F_k\mathrm{K}(X)$ to be the subgroup generated by coherent sheaves whose support has dimension at most k. Let

$$\mathrm{gr}_k \mathrm{K}(X) = F_k\mathrm{K}(X)/F_{k-1}\mathrm{K}(X).$$

There is a map

$$\mathrm{gr}_k \mathrm{K}(X) \to \mathrm{H}_k(X)$$

that takes $[\mathcal{O}_Y]$ to $[Y]$ for any subvariety Y of dimension k.

Filtration on K-theory (continued)

Why is this useful? If M is a coherent sheaf supported on Y, then the image of $[M]$ in $\mathrm{gr}_k \mathrm{K}(X)$ is simply $(\mathrm{rank}\, M)[\mathcal{O}_Y]$ (for Y irreducible).

Slightly less naive approach: rather than try to resolve $[\mathcal{O}_Z]$ by vector bundles (since the resolution might be horrible), we might try to find a coherent sheaf supported Z which has an easier (possibly even characteristic-free) resolution by vector bundles.

Does this exist for the determinantal varieties? Yes! (under appropriate genericity conditions.)

Let $G = \mathbf{GL}_n(\mathbf{Q})$. The irreducible polynomial representations of G are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0)$ with at most n parts.

One way to construct these is as the image of a certain map

$$\bigwedge^{\lambda'_1} \otimes \cdots \otimes \bigwedge^{\lambda'_r} \to \mathrm{Sym}^{\lambda_1} \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n}.$$

where λ' is the dual partition of λ, i.e., $\lambda'_i = \{j \mid \lambda_j \geq i\}$.

This construction is functorial and makes sense over any base ring (not just $\mathbf{Q}$), i.e., we can plug in any R-module V into the above map, the image is denoted $\mathbf{S}_\lambda V$ and is a representation of $\mathbf{GL}(V)$ when V is (locally) free.

In fact, the previous definition can be "superfied". More precisely, we have $\mathbf{Z}/2$-graded versions of our favorite multilinear functors such as $\bigwedge$ and Sym, so we can construct a $\mathbf{Z}/2$-graded module $\mathbf{S}_\lambda V$ when $V = V_0 \oplus V_1$ has a $\mathbf{Z}/2$-grading.
The case of interest is a 2-term complex $\varphi\colon V_1 \to V_0$. In this case, $\mathbf{S}_\lambda V$ inherits a differential from φ, and becomes a chain complex.

Let's specialize to the case rank $V_1 = m$ and rank $V_0 = n$, $\lambda = (m-r, \dots, m-r)$ ($n-r$ times) and φ is "generic". [This will mean depth $I_{r+1}(\varphi) \geq (m-r)(n-r)$.] In this case, $\mathbf{S}_\lambda\varphi$ is acyclic and its cokernel M is a rank 1 module supported on the ideal defined by $I_{r+1}(\varphi)$.

This generalizes to the situation of vector bundles $V_1 = E$ and $V_0 = F$, and implies a formula for $[Z_r]$ in terms of the Chern roots of E and F (Thom–Porteous formula).

To a partition λ, we can associate the **Schur polynomial** $s_\lambda(x) = s_\lambda(x_1, \ldots, x_n)$ as follows. A **semistandard Young tableau** is a way of filling in the diagram of λ with the numbers $1, \ldots, n$ such that the numbers are weakly increasing along rows and along columns, with no repeats in any column. Such a filling gives a monomial in the x_i, and s_λ is the sum over all such SSYT.

$$\lambda = (5,4,2), \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 4 \\ \hline 2 & 2 & 3 & 4 & \\ \cline{1-4} 4 & 5 & & & \\ \cline{1-2} \end{array} \quad \mapsto x_1^2 x_2^4 x_3 x_4^3 x_5$$

Connection to $\mathbf{S}_\lambda V$: pick a basis $e_1, \ldots, e_n$ for V. Then $\mathbf{S}_\lambda V$ has a basis $\{v\}$ such that $\operatorname{diag}(x_1, \ldots, x_n)$ acts on v as $x_1^{m_1} \cdots x_n^{m_n}$ for some nonnegative integers $m_1, \ldots, m_n$. Summing these expressions over all such v gives $s_\lambda(x_1, \ldots, x_n)$.

We also have **double Schur polynomials** $s_\lambda(x/y)$: instead of just filling in with the numbers $1, \dots, n$, also allow $-1, \dots, -m$. The weakly increasing rule applies, along with no repeats of positive numbers in a column, and no repeats of negative numbers in any row. Such a filling gives a monomial in x and y, and the sum over all such fillings gives $s_\lambda(x/y)$.

This polynomial plays the same role for the Schur complex $\mathbf{S}_\lambda(\varphi)$ as $s_\lambda(x)$ does for the Schur functor. [We should really talk about the general linear super(group/algebra) though.]

The Thom–Porteous formula says that the dual class of $[Z_r]$ in cohomology is obtained by substituting the Chern roots of E and F for y and x, respectively, into $s_\lambda(x/y)$.

Whew!

Go back to a map of bundles $\varphi\colon E \to F$. Now assume that E has a filtration $E_1 \subset \cdots \subset E_n = E$ and F has a quotient filtration $F = F_m \twoheadrightarrow F_{m-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1$ where E_i, F_j are vector bundles of ranks i, j, and each E_{i+1}/E_i is a line bundle.

Given a permutation $w \in \Sigma_N$, define $r_w(p, q) = \#\{i \leq p \mid w(i) \leq q\}$ and define

$$D_w(\varphi) = \{x \in X \mid \operatorname{rank}(\varphi|_x\colon E_p|_x \to F_q|_x) \leq r_w(p, q)\}.$$

Fulton gave a formula for $[D_w(\varphi)]$ in terms of the Chern roots of E and F. Instead of double Schur polynomials, one gets double Schubert polynomials.

Given $w \in \Sigma_N$, its Rothe diagram is

$$D(w) = \{(i, w(j)) \mid i < j, w(i) > w(j)\}$$

(using matrix indexing notation). A **balanced labeling** is a filling of $D(w)$ with nonzero integers such that:

- No entries in the ith row are bigger than i, and no entries in the jth column are smaller than $-j$.
- No repeats of positive integers in any column, and no repeats of negative numbers in any row.
- Balanced condition: given (i, j), its hook is

  $$H(i,j) = \{(i',j) \in D(w) \mid i' \geq i\} \cup \{(i,j') \in D(w) \mid j' \geq j\}.$$

  For any (i, j), if the entries in $H(i, j)$ are rearranged in increasing order starting from top right going to bottom left, then the (i, j) entry must stay the same.

Double Schubert polynomials (continued)

$$\begin{array}{|c|c|c|}\hline 1 & 1' & 3' \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad \mapsto x_1^2 x_2 x_3 y_1 y_3$$

Each balanced labeling gives a monomial in x and y, and the sum over all such is the double Schubert polynomial $\mathfrak{S}_w(x, y)$. Schubert polynomials obtained by setting $y_i = 0$.

Is there an analogue of Schubert functors and Schubert complexes? Schubert functors introduced by Kraśkiewicz and Pragacz, definition is functorial and similar to Schur functors. So can be made into a $\mathbf{Z}/2$-graded version.

The Schubert functors have an action of the group of invertible upper triangular matrices, and the Schubert polynomials give the character of this representation. Similar story for Schubert complexes.

Back to the situation of $\varphi\colon E \to F$ where both E and F have filtrations, but now assume they are free $\mathbf{Z}$-modules, and φ consists of indeterminants. This induces ordered bases on E and F.
$w \in \Sigma_N$ is a permutation as before. Let $I_w(\varphi)$ be the ideal generated by the minors of size $r_w(p,q)+1$ inside of the upper left $p \times q$ submatrix of φ for all (p,q). Let X_w be the variety defined by $I_w(\varphi)$.

Theorem (S.)

The Schubert complex is acyclic and resolves a rank 1 module supported on X_w.

If we specialize the variables of φ, then acyclicity of the Schubert complex is controlled by the depth of $I_w(\varphi)$ in general.

Note that we introduced Schur complexes for an arbitrary partition λ, but we only used the case when λ is a rectangular shape.

It is known that Schur polynomials $s_\lambda(x_1, \ldots, x_n)$ coincide with Schubert polynomials for a certain choice of w, which depends on both λ and n. However, the double Schur polynomial is a special case of a double Schubert polynomial exactly when λ is a rectangular shape.

So the appearance of Schur complexes may just be a coincidence.

Now suppose that $F \cong E^*$ and our matrix $\varphi \colon E \to E^*$ is symmetric, i.e., $\varphi^* = \varphi$, but other than the symmetry condition, the entries of φ are indeterminants. If we let $\lambda = \rho_{n-r} = (n-r+1, n-r, \ldots, 2, 1)$ (staircase partition) then $\mathbf{S}_{\rho_{n-r}}(\varphi)$ is acyclic and its cokernel is a module (whose rank is a power of 2) supported in the variety defined by the $r \times r$ minors of φ.

This situation recovers a cohomology class formula due to Harris and Tu, and there is a similar situation for skew-symmetric matrices.

$\mathbf{S}_{\rho_{n-r}}$ is not an instance of a Schubert complex. However, $s_{\rho_{n-r}}$ does coincide with the Schur Q-functions (at least up to a power of 2).

The Schur complexes have an action of the general linear superalgebra. In analogy, the Schubert complexes have an action of a Lie superalgebra consisting of certain upper triangular matrices.

The Schur Q-functions (again up to powers of 2) are the characters of certain polynomial representations of the queer superalgebra. This is the Lie superalgebra of matrices of the form $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$. This suggests that one should really have a multilinear construction for these complexes that take into account the symmetries of the queer superalgebra.

I don't know how to do this though...

The Schur complex has a "geometric" origin. Recall, we have $\varphi\colon E \to F$ with $\dim E = n$ and $\dim F = m$. Then the variety

$$S = \{(\varphi, R) \in \mathrm{Hom}(E, F) \times \mathbf{Grass}(n-r, E) \mid \varphi(R) = 0\}$$

desingularizes Z_r via the first projection $\pi_1\colon S \to \mathrm{Hom}(E, F)$. Let $\mathcal{M} = \pi_2^*\mathcal{O}(m-r)$ where $\mathcal{O}(1)$ is the line bundle giving the Plücker embedding of $\mathbf{Grass}(n-r, E)$.

$(\pi_1)_*\mathcal{M}$ is the module supported on Z_r that the Schur complex resolves. The Schur complex itself can be constructed from a Koszul complex on $\mathrm{Hom}(E, F) \times \mathbf{Grass}(n-r, E)$ that resolves $\mathcal{M}$.

This picture should generalize to the Schubert complexes, but I haven't figured out how to do it.