Geometric approach to Littlewood inversion formulas

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- We work over the field of complex numbers $\mathbf{C}$.
- Given a partition λ, its conjugate partition is λ'.
- s_λ, e_λ, h_λ denotes Schur, elementary, and homogeneous symmetric functions
- Given a symmetric function f, let $s_{\lambda/f}$ be defined by $(s_{\lambda/f}, g) = (s_\lambda, fg)$ where $(,)$ is the Hall inner product.

- The polynomial irreps of $\mathbf{GL}(V)$ ($\dim V = m$) are indexed by partitions $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$, and are denoted $\mathbf{S}_\lambda(V)$–these can be realized as subspaces of $V^{\otimes|\lambda|}$.
- Put a nondegenerate (skew-)symmetric form ω on V.
- Suppose that ω is skew-symmetric (so $m = 2n$). The irreps of $\mathbf{Sp}(V)$ are the traceless tensors in $\mathbf{S}_\lambda(V)$: this is the intersection of the kernels of the contractions

$$v_1 \otimes \cdots \otimes v_N \mapsto \omega(v_i, v_j) v_1 \otimes \cdots \hat{v}_i \cdots \hat{v}_j \cdots \otimes v_N.$$

  Denote this irrep by $\mathbf{S}_{[\lambda]}(V)$.
- Ditto for the orthogonal group $\mathbf{O}(V)$.

Let $c^{\gamma}_{\alpha,\beta}$ be the Littlewood–Richardson coefficient, i.e., $s_\alpha s_\beta = \sum_\gamma c^{\gamma}_{\alpha,\beta} s_\gamma$.
When ω is symmetric, Littlewood showed:

$$\mathbf{S}_\lambda V \cong \bigoplus_{\mu,\nu} (\mathbf{S}_{[\mu]} V)^{\oplus c^{\lambda}_{\mu,2\nu}}$$

When ω is skew-symmetric, Littlewood showed:

$$\mathbf{S}_\lambda V \cong \bigoplus_{\mu,\nu} (\mathbf{S}_{[\mu]} V)^{\oplus c^{\lambda}_{\mu,(2\nu)'}}$$

Note that $\mathbf{S}_{[\lambda]} V$ appears with multiplicity 1, and if $\mathbf{S}_{[\mu]} V$ appears, then $\mu = \lambda$ or $|\mu| < |\lambda|$. So the branching matrix can be made upper unitriangular.

We can express the branching as saying how to write a Schur function in terms of irreducible characters of the symplectic / orthogonal group. Littlewood gave the inversion formulas. When ω is skew-symmetric,

$$s_{[\lambda]} = \sum_{i \geq 0} (-1)^i s_{\lambda/(e_i \circ e_2)}.$$

When ω is symmetric,

$$s_{[\lambda]} = \sum_{i \geq 0} (-1)^i s_{\lambda/(e_i \circ h_2)}$$

We want to find a "geometric" interpretation of these formulas in the hope that they may generalize to other groups.

For simplicity of notation, assume ω is skew-symmetric.
Consider $Y_\omega = \{\varphi\colon E \to V \mid \varphi(E) \text{ isotropic}\}$. This is a complete intersection in $\mathrm{Hom}(E, V)$ if and only if $2 \dim E \leq \dim V$, and ideal is generated by $\bigwedge^2 E$ in degree 2.

So it has the following graded minimal free resolution (set $A = \mathbf{C}[\mathrm{Hom}(E, V)]$):

$$0 \to \bigwedge^{\binom{\dim E}{2}} \bigwedge^2 E \otimes A(-2\dim E) \to \cdots \bigwedge^i \bigwedge^2 E \otimes A(-2i) \to \cdots$$
$$\to \bigwedge^2 E \otimes A(-2) \to A \to \mathbf{C}[Y_\omega] \to 0$$

Littlewood inversion formulas (cont.)

The coordinate ring of Y_ω decomposes as (Cauchy identity):

$$\mathbf{C}[Y_\omega] = \bigoplus_\lambda \mathbf{S}_\lambda(E) \otimes \mathbf{S}_{[\lambda]}(V)$$

Taking isotypic components of the Koszul complex (with respect to $\mathbf{GL}(E)$) gives a resolution of irreps for $\mathbf{Sp}(V)$ in terms of irreps for $\mathbf{GL}(V)$

$$\cdots \to \mathbf{S}_{\lambda/\bigwedge^i \bigwedge^2} V \to \cdots \to \mathbf{S}_{\lambda/\bigwedge^2} V \to \mathbf{S}_\lambda V \to \mathbf{S}_{[\lambda]} V \to 0$$

The first part is "Weyl's construction", and taking the Euler characteristic gives Littlewood's inversion formula.

This works for the orthogonal group with the appropriate changes. (Caveat: if $m = 2n$ and $\dim E = n$, then Y_ω has 2 irreducible components.)

For $G = G_2(\mathbf{C})$, there is something similar: G has an irrep V of dimension 7 which has an alternating trilinear form γ. Call a subspace R of dimension 2 **isotropic** if $\gamma(u, v, w) = 0$ for all $u, v \in R$ and $w \in V$. Can define Y_ω as before, taking $\dim E \leq 2$. We get the following resolutions:

$$\begin{aligned} 0 \to \mathbf{S}_{\mu/(4,4)} V \to (\mathbf{S}_{\mu/(3,3)} V \otimes V) \oplus \mathbf{S}_{\mu/(4,2)} V \to \\ \mathbf{S}_{\mu/(3,2)} V \otimes (\mathbf{C} \oplus V) \to \mathbf{S}_{\mu/(2,1)} V \otimes (\mathbf{C} \oplus V) \to \\ (\mathbf{S}_{\mu/(1,1)} V \otimes V) \oplus \mathbf{S}_{\mu/(2)} V \to \mathbf{S}_\mu V \to V_{(\mu_1 - \mu_2, \mu_2)} \to 0 \end{aligned}$$

The first part encodes "Weyl's construction" due to (Huang–Zhu 1999), and the Euler characteristic gives an inversion formula.

Let Σ_m be the symmetric group and let V be the m-dimensional standard representation. The right analogue of $\mathbf{S}_{[\lambda]}(V)$ is the irrep indexed by $(m - |\lambda|, \lambda_1, \dots)$ if it's a partition, and 0 otherwise. Let $\mathfrak{h}$ be the subspace of traceless diagonal matrices in $\mathrm{End}(V)$. Given n, let

$$Y = \{\varphi \colon \mathbf{C}^n \to \mathrm{End}(V) \mid \varphi(x)^2 = 0 \text{ for all } x \in \mathbf{C}^n\},$$
$$D = \{\varphi \colon \mathbf{C}^n \to \mathrm{End}(V) \mid \varphi(x) \in \mathfrak{h} \text{ for all } x \in \mathbf{C}^n\}.$$

Let $Y_D = Y \cap D$ (scheme-theoretic). Then

$$\mathbf{C}[Y_D] = \bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_{[\lambda]}(V)$$

Free resolution of Y_D over $\mathbf{C}[\mathrm{Hom}(E, \mathfrak{h})]$ should yield inversion formula ($V = \mathfrak{h} \oplus \mathbf{C}$), but seems difficult to obtain.

- The other exceptional groups are not so nice, i.e., one probably cannot resolve irreps in terms of Schur functors (at least not with the geometric methods I was using). This roughly corresponds to the fact that the map $G/B \to G/P_\alpha$ is a "twisted" orthogonal / symplectic flag variety where P_α is a certain maximal parabolic subgroup.
- However, we can still define varieties whose coordinate rings give the right analogue of the Cauchy identity if we only pay attention to a certain codimension 1 lattice of the weight lattice.

Recall that for classical groups, $Y_\omega = \{\varphi\colon E \to V \mid \omega|_{\varphi(E)} = 0\}$ is a complete intersection if and only if $2 \dim E \leq \dim V$. So when $2 \dim E > \dim V$, the complex (for ω skew-symmetric)

$$\cdots \to \mathbf{S}_{\lambda / \bigwedge^i \bigwedge^2} V \to \cdots \to \mathbf{S}_{\lambda / \bigwedge^2} V \to \mathbf{S}_\lambda V.$$

can have higher homology. Results of Koike–Terada imply that the Euler characteristic of this complex is $\pm s_{[\mu]}$ for some μ or 0.

Wenzl shows that μ is obtained from λ via a dotted Weyl group (of type D_∞ for orthogonal case and type $\mathrm{B}_\infty = \mathrm{C}_\infty$ for symplectic case) action (i.e., $\mu = (-1)^{\ell(w)} w(\lambda + \rho_m) - \rho_m$ for some ρ_m).

This is formally analogous to the Borel–Weil–Bott theorem, so we conjecture that the complex above has at most 1 nonzero homology group.