Saturation theorems for the classical groups

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- We work over the field of complex numbers $\mathbf{C}$.
- G is a reductive group (usually $\mathbf{GL}_n$ or $\mathbf{SO}_m$ or $\mathbf{Sp}_{2n}$)
- Dominant weights for G are labeled $\lambda, \mu, \nu, \ldots$.
- V_λ is irrep of G with highest weight λ
- $C_{\lambda,\mu,\nu} = \dim_{\mathbf{C}}(V_\lambda \otimes V_\mu \otimes V_\nu)^G$ (space of G-invariants)

- The irreps of $\mathbf{GL}_n$ are indexed by weakly decreasing sequences $\lambda_1 \geq \cdots \geq \lambda_n$. When $\lambda_n \geq 0$, these can be constructed using Young idempotents: $V_\lambda = e_\lambda(\mathbf{C}^n)^{\otimes|\lambda|}$. In general, twist these by powers of the determinant representation.
- The irreps of $\mathbf{Sp}_{2n}$ are the traceless tensors in V_λ: let ω be a symplectic form on $\mathbf{C}^{2n}$, the traceless tensors of $(\mathbf{C}^{2n})^{\otimes N}$ is the intersection of the kernels of the contractions

$$v_1 \otimes \cdots \otimes v_N \mapsto \omega(v_i, v_j) v_1 \otimes \cdots \hat{v}_i \cdots \hat{v}_j \cdots \otimes v_N.$$

- Ditto for the orthogonal group $\mathbf{O}_m$. The restriction to $\mathbf{SO}_m$ remains irreducible unless m is even and $\lambda_n > 0$ in which case it is the direct sum of two nonisomorphic irreps that we call V_{λ^+} and V_{λ^-}.

Saturation problems

- $C_{\lambda,\mu,\nu} = \dim_{\mathbf{C}}(V_\lambda \otimes V_\mu \otimes V_\nu)^G$ (space of G-invariants)
- easy: $C_{\lambda,\mu,\nu} > 0$ implies that $\lambda + \mu + \nu$ is in root lattice
- easy: $C_{\lambda,\mu,\nu} > 0$ implies that $C_{N\lambda,N\mu,N\nu} > 0$ for all $N > 0$.
- What about the reverse implication? Say that k is a **saturation factor** for G if $C_{N\lambda,N\mu,N\nu} > 0$ ($N > 0$, $\lambda + \mu + \nu$ in root lattice) implies that $C_{k\lambda,k\mu,k\nu} > 0$.

Theorem (Knutson–Tao, Derksen–Weyman, Kapovich–Millson)

If $G = \mathbf{GL}_n$, then $k = 1$ is a saturation factor.

Theorem (Kapovich–Millson)

Let $\theta = \sum k_i\alpha_i$ be the highest root of G. Then $k = \mathrm{lcm}(k_i)^2$ is a saturation factor.

Theorem (Belkale–Kumar)

Let $G = \mathbf{SO}_{2n+1}$ or $G = \mathbf{Sp}_{2n}$. Then $k = 2$ is a saturation factor.

Theorem (S.)

Let $G = \mathbf{SO}_{2n+1}$ or $G = \mathbf{Sp}_{2n}$ or $G = \mathbf{SO}_{2n}$. Then $k = 2$ is a saturation factor.

Actually, in the last two theorems, get something stronger: $C_{N\lambda,N\mu,N\nu} > 0$ implies $C_{2\lambda,2\mu,2\nu} > 0$ without having to assume that $\lambda + \mu + \nu$ is in the root lattice.

Why this problem?

- A problem dating back to 19th century mathematics: given Hermitian $n \times n$ matrices A, B, C such that $A + B + C = 0$, how are the eigenvalues of A, B, C related?
- Since the eigenvalues are real numbers, we can write them in decreasing order: eigenvalues of A are $\alpha_1 \geq \cdots \geq \alpha_n$, use β and γ for B and C.
- Klyachko studied this problem using geometric invariant theory and showed that the set of $(\alpha_\bullet, \beta_\bullet, \gamma_\bullet)$ form a rational polyhedral cone in $\mathbf{R}^{3n}$.
- In fact, this cone is the closure of the set

$$\{(\lambda, \mu, \nu) \in \mathbf{Q}^{3n} \mid \exists N > 0,\ \ C_{N\lambda, N\mu, N\nu} > 0\}$$

$(G = \mathbf{GL}_n)$

- Q is a directed graph without directed cycles (vertex set Q_0, arrow set Q_1)
- For $a \in Q_1$, get head $ha \in Q_0$ and tail $ta \in Q_1$: $ta \xrightarrow{a} ha$.
- Functions $d \colon Q_0 \to \mathbf{N}$ are **dimension vectors**, call set $\mathbf{N}^{Q_0}$
- **Representation variety**: for $d \in \mathbf{N}^{Q_0}$, define

$$\mathrm{Rep}(Q,d) = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbf{C}^{d(ta)}, \mathbf{C}^{d(ha)})$$

$$\mathbf{GL}_d = \prod_{x \in Q_0} \mathbf{GL}_{d(x)}$$

$$\mathbf{SL}_d = \prod_{x \in Q_0} \mathbf{SL}_{d(x)}$$

Action of $\mathbf{GL}_d$ and $\mathbf{SL}_d$ on $\mathrm{Rep}(Q,d)$ via change of basis.

- For affine variety X, let $\mathbf{C}[X]$ be its coordinate ring.
- **Semi-invariants**: $\mathsf{SI}(Q,d) = \mathbf{C}[\mathrm{Rep}(Q,d)]^{\mathbf{SL}_d}$.
- Have grading given by characters σ of $\mathbf{GL}_d$:

$$\mathsf{SI}(Q,d) = \bigoplus_{\sigma} \mathsf{SI}(Q,d)_\sigma$$

Theorem (Derksen–Weyman)

$\mathsf{SI}(Q,d)_{N\sigma} \neq 0$ *implies that* $\mathsf{SI}(Q,d)_\sigma \neq 0$.

Ingredients of proof for quiver saturation

- Pick $\alpha \in \mathbf{N}^{Q_0}$ such that

$$\sum_{x \in Q_0} \alpha(x) d(x) = \sum_{a \in Q_1} \alpha(ta) d(ha).$$

For $V \in \mathrm{Rep}(Q, \alpha)$ and $W \in \mathrm{Rep}(Q, d)$, construct map

$$d_W^V \colon \bigoplus_{x \in Q_0} \mathrm{Hom}(\mathbf{C}^{\alpha(x)}, \mathbf{C}^{d(x)}) \to \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbf{C}^{\alpha(ta)}, \mathbf{C}^{d(ha)})$$

$$(\varphi_x)_{x \in Q_0} \mapsto (\varphi_{ha} V_a - W_a \varphi_{ta})_{a \in Q_1}$$

and define $c_W^V = \det(d_W^V)$.

- If we interpret V and W as modules over the path algebra $\mathbf{C}Q$, then the kernel of d_W^V is $\mathrm{Hom}_{\mathbf{C}Q}(V, W)$ and the cokernel is $\mathrm{Ext}^1_{\mathbf{C}Q}(V, W)$.
- The function $c^V \colon W \mapsto c_W^V$ belongs to $\mathrm{SI}(Q, d)$.

- Derksen–Weyman: the ring $\mathrm{SI}(Q, d)$ is linearly spanned by functions of the form c^V.
- Define $\mathrm{Ext}^1(\alpha, d)$ to be the minimum dimension of $\mathrm{Ext}^1(V, W)$ for $V \in \mathrm{Rep}(Q, \alpha)$ and $W \in \mathrm{Rep}(Q, d)$. By construction, there exists V with $c^V \neq 0$ if and only if $\mathrm{Ext}^1(\alpha, d) = 0$.
- Schofield: For fixed d, the conditions on α for $\mathrm{Ext}^1(\alpha, d) = 0$ are a finite set of inequalities. In particular, $\mathrm{Ext}^1(N\alpha, d) = 0$ $(N > 0)$ implies that $\mathrm{Ext}^1(\alpha, d) = 0$.

Cauchy identity

Let Q be the quiver

$$\begin{tikzcd}
x_1^1 \arrow[r] & x_2^1 \arrow[r] & \cdots \arrow[r] & x_{n-1}^1 \arrow[dr] & \\
x_1^2 \arrow[r] & x_2^2 \arrow[r] & \cdots \arrow[r] & x_{n-1}^2 \arrow[r] & u = x_n^1 \\
x_1^3 \arrow[r] & x_2^3 \arrow[r] & \cdots \arrow[r] & x_{n-1}^3 \arrow[ur] &
\end{tikzcd}$$

and $d(x_i^j) = i$. Given λ, μ, ν (dominant weights for $\mathbf{GL}_n$), there is a weight σ such that $\dim_{\mathbf{C}} \mathsf{SI}(Q, d)_{N\sigma} = C_{N\lambda, N\mu, N\nu}$. Cauchy identity:

$$\mathbf{C}[\mathrm{Hom}(V, W)] = \mathrm{Sym}(V \otimes W^*) = \bigoplus_{\lambda} V_\lambda \otimes W_\lambda^*.$$

How to generalize the proof?

- To get saturation theorems for other classical groups G, need to generalize definition of quiver and find analogue of Cauchy identity when W is a vector space of dimension $2n(+1)$ with a nondegenerate form ω.
- For Cauchy identity, take $\dim V = n$ and subvariety $Y_\omega \subset \mathrm{Hom}(V, W)$ of maps whose image is isotropic. Then

$$\mathbf{C}[Y_\omega] = \bigoplus_\lambda V_\lambda \otimes W_\lambda^*,$$

where now $W_\lambda^* \cong W_\lambda$ is simple module for $\mathbf{O}_{2n(+1)}$ or $\mathbf{Sp}_{2n}$ with highest weight λ. When $G = \mathbf{SO}_{2n}$, Y_ω has two irreducible components to compensate for the fact that $W_\lambda \cong W_{\lambda^+} \oplus W_{\lambda^-}$ as $\mathbf{SO}_{2n}$-representations when $\lambda_n > 0$.

- **Symmetric quiver**: quiver Q with orientation-reversing involution τ. For each τ-fixed vertex and arrow, also fix the data of a sign $s(x) \in \{+, -\}$.
- $d \in \mathbf{N}^{Q_0}$ is symmetric if fixed by τ. Fix isomorphisms $J_x \colon \mathbf{C}^{d(x)} \to (\mathbf{C}^{d(\tau(x))})^*$. If $x = \tau(x)$, then need $J_x^T = s(x) J_x$.
- **Symmetric representation variety**: subvariety $\mathrm{SRep}(Q, d)$ of $\mathrm{Rep}(Q, d)$ "compatible with the above data".
- From above, $\bigoplus_{x \in Q_0} \mathbf{C}^{d(x)}$ has a nondegenerate form. Replace $\mathbf{GL}_d$ with the subgroup $\mathbf{G}_d$ preserving this form (and grading by Q_0), and replace $\mathbf{SL}_d$ by the commutator subgroup $\mathbf{SG}_d$ of $\mathbf{G}_d$. Replace $\mathrm{SI}(Q, d)$ by $\mathrm{SSI}(Q, d) = \mathrm{SRep}(Q, d)^{\mathbf{SG}_d}$.

Theorem (S.)

$\mathrm{SSI}(Q, d)$ *is spanned by* c^V *and their square roots (when they exist). In particular, if* $\mathrm{SSI}(Q, d)_{N\sigma} \neq 0$*, then* $\mathrm{SSI}(Q, d)_{2\sigma} \neq 0$.

Cauchy identity (again)

Let Q be the symmetric quiver

$$\begin{tikzcd}
x_1^1 \arrow[r, "a_1^1"] & \cdots \arrow[r, "a_{n-1}^1"] & x_n^1 \arrow[dr, "a_n^1"] & & \tau(x_n^1) \arrow[r, "\tau(a_{n-1}^1)"] & \cdots \arrow[r, "\tau(a_1^1)"] & \tau(x_1^1) \\
x_1^2 \arrow[r, "a_1^2"] & \cdots \arrow[r, "a_{n-1}^2"] & x_n^2 \arrow[r, "a_n^2"] & u = \tau(u) \arrow[ur, "\tau(a_n^1)"] \arrow[r, "\tau(a_n^2)"] \arrow[dr, "\tau(a_n^3)"] & \tau(x_n^2) \arrow[r, "\tau(a_{n-1}^2)"] & \cdots \arrow[r, "\tau(a_1^2)"] & \tau(x_1^2) \\
x_1^3 \arrow[r, "a_1^3"] & \cdots \arrow[r, "a_{n-1}^3"] & x_n^3 \arrow[ur, "a_n^3"] & & \tau(x_n^3) \arrow[r, "\tau(a_{n-1}^3)"] & \cdots \arrow[r, "\tau(a_1^3)"] & \tau(x_1^3)
\end{tikzcd}$$

with $d(x_i^j) = i$ and $d(u) = 2n(+1)$ and $s(u) = +1$ if $G = \mathbf{SO}_{2n(+1)}$ or $s(u) = -1$ if $G = \mathbf{Sp}_{2n}$. It seems like we want to study this symmetric quiver, but its coordinate ring contains $\mathrm{Hom}(\mathbf{C}^{d(x_n^j)}, \mathbf{C}^{d(u)})$, and we really want the coordinate ring of Y_ω appearing.

- The right fix for the previous problem is to only look at the subvariety of $\mathrm{SRep}(Q, d)$ where the compositions $\mathbf{C}^{d(x_n^j)} \to \mathbf{C}^{d(u)} \to \mathbf{C}^{d(\tau(x_n^j))}$ are 0 (this is equivalent to saying that the image of the first map is isotropic).
- This forces us to work with **quivers with relations**. New complication: The global dimension of $\mathbf{C}Q$ is 1 (i.e., $\mathrm{Ext}^2_{\mathbf{C}Q} = 0$), but the global dimension of $\mathbf{C}Q/I$ is 2. So we need analogues of Schofield's results in this setting.
- Modulo the technicalities, the outline of the proof of saturation for the orthogonal and symplectic groups is the same as the outline for the general linear group.

- **Saturation theorems for exceptional groups.** There are candidates for the varieties Y_ω when G is of exceptional type. But it is unclear how to generalize symmetric quivers.
- **Saturation theorems for stable Kronecker coefficients.** There is a collection of irreps for the infinite symmetric group indexed by partitions (of arbitrary size), first studied by Murnaghan, whose tensor product decompositions generalize those for the general linear group. There is an analogue of Y_ω in this case also, but it is a non-reduced ind-variety supported on a point. So it is unclear if quiver (or even geometric) methods are relevant.