Saturation theorems for the classical groups

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- We work over the field of complex numbers **C**.
- G is a reductive group (usually **GL**_n or **SO**_m or **Sp**_{2n})
- Dominant weights for G are labeled $\lambda, \mu, \nu, \ldots$
- V_{λ} is irrep of *G* with highest weight λ
- $C_{\lambda,\mu,\nu} = \dim_{\mathbf{C}} (V_{\lambda} \otimes V_{\mu} \otimes V_{\nu})^{\mathcal{G}}$ (space of *G*-invariants)

Representations of the classical groups

- The irreps of \mathbf{GL}_n are indexed by weakly decreasing sequences $\lambda_1 \geq \cdots \geq \lambda_n$. When $\lambda_n \geq 0$, these can be constructed using Young idempotents: $V_{\lambda} = e_{\lambda}(\mathbf{C}^n)^{\otimes |\lambda|}$. In general, twist these by powers of the determinant representation.
- The irreps of Sp_{2n} are the traceless tensors in V_λ: let ω be a symplectic form on C²ⁿ, the traceless tensors of (C²ⁿ)^{⊗N} is the intersection of the kernels of the contractions

$$v_1 \otimes \cdots \otimes v_N \mapsto \omega(v_i, v_j) v_1 \otimes \cdots \hat{v}_i \cdots \hat{v}_j \cdots \otimes v_N.$$

• Ditto for the orthogonal group \mathbf{O}_m . The restriction to \mathbf{SO}_m remains irreducible unless m is even and $\lambda_n > 0$ in which case it is the direct sum of two nonisomorphic irreps that we call V_{λ^+} and V_{λ^-} .

Saturation problems

- $C_{\lambda,\mu,\nu} = \dim_{\mathbf{C}} (V_{\lambda} \otimes V_{\mu} \otimes V_{\nu})^{G}$ (space of *G*-invariants)
- easy: $C_{\lambda,\mu,\nu} > 0$ implies that $\lambda + \mu + \nu$ is in root lattice
- easy: $C_{\lambda,\mu,\nu} > 0$ implies that $C_{N\lambda,N\mu,N\nu} > 0$ for all N > 0.
- What about the reverse implication? Say that k is a saturation factor for G if $C_{N\lambda,N\mu,N\nu} > 0$ (N > 0, $\lambda + \mu + \nu$ in root lattice) implies that $C_{k\lambda,k\mu,k\nu} > 0$.

Theorem (Knutson–Tao, Derksen–Weyman, Kapovich–Millson)

If $G = \mathbf{GL}_n$, then k = 1 is a saturation factor.

Other groups

Theorem (Kapovich–Millson)

Let $\theta = \sum k_i \alpha_i$ be the highest root of G. Then $k = \text{lcm}(k_i)^2$ is a saturation factor.

Theorem (Belkale–Kumar)

Let $G = \mathbf{SO}_{2n+1}$ or $G = \mathbf{Sp}_{2n}$. Then k = 2 is a saturation factor.

Theorem (S.)

Let $G = \mathbf{SO}_{2n+1}$ or $G = \mathbf{Sp}_{2n}$ or $G = \mathbf{SO}_{2n}$. Then k = 2 is a saturation factor.

Actually, in the last two theorems, get something stronger: $C_{N\lambda,N\mu,N\nu} > 0$ implies $C_{2\lambda,2\mu,2\nu} > 0$ without having to assume that $\lambda + \mu + \nu$ is in the root lattice.

Why this problem?

- A problem dating back to 19th century mathematics: given Hermitian n × n matrices A, B, C such that A + B + C = 0, how are the eigenvalues of A, B, C related?
- Since the eigenvalues are real numbers, we can write them in decreasing order: eigenvalues of A are α₁ ≥ · · · ≥ α_n, use β and γ for B and C.
- Klyachko studied this problem using geometric invariant theory and showed that the set of (α_●, β_●, γ_●) form a rational polyhedral cone in R³ⁿ.
- In fact, this cone is the closure of the set

$$\{(\lambda,\mu,\nu)\in \mathbf{Q}^{3n}\mid \exists N>0, \ C_{N\lambda,N\mu,N\nu}>0\}$$

 $(G = \mathbf{GL}_n)$

Quivers

- *Q* is a directed graph without directed cycles (vertex set *Q*₀, arrow set *Q*₁)
- For $a \in Q_1$, get head $ha \in Q_0$ and tail $ta \in Q_1$: $ta \xrightarrow{a} ha$.
- Functions $d: Q_0 \rightarrow N$ are dimension vectors, call set N^{Q_0}
- **Representation variety**: for $d \in \mathbb{N}^{Q_0}$, define

$$egin{aligned} \mathsf{Rep}(Q,d) &= igoplus_{a \in Q_1} \mathsf{Hom}(\mathsf{C}^{d(ta)},\mathsf{C}^{d(ha)}) \ \mathsf{GL}_d &= \prod_{x \in Q_0} \mathsf{GL}_{d(x)} \ \mathsf{SL}_d &= \prod_{x \in Q_0} \mathsf{SL}_{d(x)} \end{aligned}$$

Action of \mathbf{GL}_d and \mathbf{SL}_d on $\operatorname{Rep}(Q, d)$ via change of basis.

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- For affine variety X, let **C**[X] be its coordinate ring.
- Semi-invariants: $Sl(Q, d) = C[Rep(Q, d)]^{SL_d}$.
- Have grading given by characters σ of \mathbf{GL}_d :

$$\mathsf{SI}(Q,d) = \bigoplus_{\sigma} \mathsf{SI}(Q,d)_{\sigma}$$

Theorem (Derksen–Weyman)

 $SI(Q, d)_{N\sigma} \neq 0$ implies that $SI(Q, d)_{\sigma} \neq 0$.

Ingredients of proof for quiver saturation

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• Pick $\alpha \in \mathbf{N}^{Q_0}$ such that

$$\sum_{x\in Q_0} \alpha(x)d(x) = \sum_{a\in Q_1} \alpha(ta)d(ha).$$

For $V \in \mathsf{Rep}(Q, \alpha)$ and $W \in \mathsf{Rep}(Q, d)$, construct map

$$d_{W}^{V}: \bigoplus_{x \in Q_{0}} \operatorname{Hom}(\mathbf{C}^{\alpha(x)}, \mathbf{C}^{d(x)}) \to \bigoplus_{a \in Q_{1}} \operatorname{Hom}(\mathbf{C}^{\alpha(ta)}, \mathbf{C}^{d(ha)})$$
$$(\varphi_{x})_{x \in Q_{0}} \mapsto (\varphi_{ha}V_{a} - W_{a}\varphi_{ta})_{a \in Q_{1}}$$

and define $c_W^V = \det(d_W^V)$.

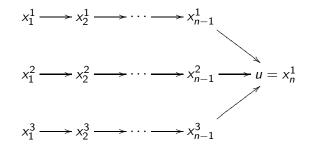
- If we interpret V and W as modules over the path algebra CQ, then the kernel of d^V_W is Hom_{CQ}(V, W) and the cokernel is Ext¹_{CQ}(V, W).
- The function $c^V \colon W \mapsto c^V_W$ belongs to SI(Q, d).

Ingredients (continued)

- Derksen–Weyman: the ring SI(Q, d) is linearly spanned by functions of the form c^V .
- Define Ext¹(α, d) to be the minimum dimension of Ext¹(V, W) for V ∈ Rep(Q, α) and W ∈ Rep(Q, d). By construction, there exists V with c^V ≠ 0 if and only if Ext¹(α, d) = 0.
- Schofield: For fixed d, the conditions on α for Ext¹(α, d) = 0 are a finite set of inequalities. In particular, Ext¹(Nα, d) = 0 (N > 0) implies that Ext¹(α, d) = 0.

Cauchy identity

Let Q be the quiver



and $d(x_i^j) = i$. Given λ, μ, ν (dominant weights for \mathbf{GL}_n), there is a weight σ such that dim_C SI(Q, d)_{N σ} = $C_{N\lambda,N\mu,N\nu}$. Cauchy identity:

$${f C}[{f Hom}(V,W)]={
m Sym}(V\otimes W^*)=igoplus_\lambda\otimes W^*_\lambda.$$

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- To get saturation theorems for other classical groups G, need to generalize definition of quiver and find analogue of Cauchy identity when W is a vector space of dimension 2n(+1) with a nondegenerate form ω .
- For Cauchy identity, take dim V = n and subvariety $Y_{\omega} \subset \text{Hom}(V, W)$ of maps whose image is isotropic. Then

$$\mathbf{C}[Y_{\omega}] = \bigoplus_{\lambda} V_{\lambda} \otimes W_{\lambda}^*,$$

where now $W_{\lambda}^* \cong W_{\lambda}$ is simple module for $\mathbf{O}_{2n(+1)}$ or \mathbf{Sp}_{2n} with highest weight λ . When $G = \mathbf{SO}_{2n}$, Y_{ω} has two irreducible components to compensate for the fact that $W_{\lambda} \cong W_{\lambda^+} \oplus W_{\lambda^-}$ as \mathbf{SO}_{2n} -representations when $\lambda_n > 0$.

Symmetric quivers

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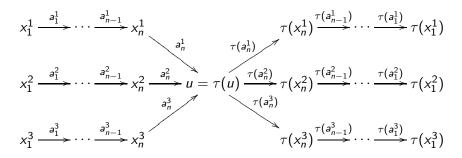
- Symmetric quiver: quiver Q with orientation-reversing involution τ. For each τ-fixed vertex and arrow, also fix the data of a sign s(x) ∈ {+,−}.
- $d \in \mathbf{N}^{Q_0}$ is symmetric if fixed by τ . Fix isomorphisms $J_x : \mathbf{C}^{d(x)} \to (\mathbf{C}^{d(\tau(x))})^*$. If $x = \tau(x)$, then need $J_x^T = s(x)J_x$.
- Symmetric representation variety: subvariety SRep(Q, d) of Rep(Q, d) "compatible with the above data".
- From above, ⊕_{x∈Q0} C^{d(x)} has a nondegenerate form. Replace GL_d with the subgroup G_d preserving this form (and grading by Q₀), and replace SL_d by the commutator subgroup SG_d of G_d. Replace SI(Q, d) by SSI(Q, d) = SRep(Q, d)^{SG_d}.

Theorem (S.)

SSI(Q, d) is spanned by c^V and their square roots (when they exist). In particular, if SSI(Q, d)_{N σ} \neq 0, then SSI(Q, d)_{2 σ} \neq 0.

Cauchy identity (again)

Let Q be the symmetric quiver



with $d(x_i^j) = i$ and d(u) = 2n(+1) and s(u) = +1 if $G = \mathbf{SO}_{2n(+1)}$ or s(u) = -1 if $G = \mathbf{Sp}_{2n}$. It seems like we want to study this symmetric quiver, but its coordinate ring contains $\operatorname{Hom}(\mathbf{C}^{d(x_n^j)}, \mathbf{C}^{d(u)})$, and we really want the coordinate ring of Y_{ω} appearing.

Quivers with relations

- The right fix for the previous problem is to only look at the subvariety of $\operatorname{SRep}(Q, d)$ where the compositions $\mathbf{C}^{d(x_n^j)} \to \mathbf{C}^{d(u)} \to \mathbf{C}^{d(\tau(x_n^j))}$ are 0 (this is equivalent to saying that the image of the first map is isotropic).
- This forces us to work with **quivers with relations**. New complication: The global dimension of CQ is 1 (i.e., $Ext_{CQ}^2 = 0$), but the global dimension of CQ/I is 2. So we need analogues of Schofield's results in this setting.
- Modulo the technicalities, the outline of the proof of saturation for the orthogonal and symplectic groups is the same as the outline for the general linear group.

- Saturation theorems for exceptional groups. There are candidates for the varieties Y_ω when G is of exceptional type. But it is unclear how to generalize symmetric quivers.
- Saturation theorems for stable Kronecker coefficients. There is a collection of irreps for the infinite symmetric group indexed by partitions (of arbitrary size), first studied by Murnaghan, whose tensor product decompositions generalize those for the general linear group. There is an analogue of Y_{ω} in this case also, but it is a non-reduced ind-variety supported on a point. So it is unclear if quiver (or even geometric) methods are relevant.