Young tableaux and Betti tables

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Our story begins with...

- K is a field.
- V is a vector space over K of dimension n.
- $A = \mathrm{Sym}(V) \cong K[x_1, \ldots, x_n]$ is the polynomial ring with the standard grading.
- All modules over A will be finitely generated and graded.

Conjecture (Boij–Söderberg).

The graded Betti table of any Cohen–Macaulay module of codimension c can be written as a positive linear combination of the graded Betti tables of modules of codimension c which have a pure resolution.

Theorem (Eisenbud–Schreyer)

It's true, plus much more!

Definitions

- Given a free resolution $\cdots \to F_1 \to F_0 \to M \to 0$ of M, it's **graded** if the differentials have degree 0. It's **minimal** if there are no constants in the matrices.
- Every module has a graded minimal free resolution, and it is unique up to isomorphism.
- If $A(-d)$ is a free A-module generated in degree d, we can write
$$F_i = \bigoplus_j A(-j)^{\oplus \beta_{i,j}}$$
for some numbers $\beta_{i,j}$. These numbers are the **graded Betti numbers** of M, and $\beta(M) = (\beta_{i,j})$ is its **graded Betti table**.
- M has a **pure resolution** if for each i, there is at most one value of j for which $\beta_{i,j} \neq 0$. Call these values d_i. Then $d = (d_0, d_1, \dots)$ is the **degree sequence** of M.

Properties of free resolutions

Theorem (Hilbert syzygy theorem)

Every module has a graded free resolution of length at most n.

Theorem (Auslander–Buchsbaum formula)

For M Cohen–Macaulay, the length of its minimal free resolution is equal to the codimension of M, i.e., $n - \dim A/\operatorname{Ann}(M)$.

Theorem (Herzog–Kühl)

If M is Cohen–Macaulay and has a pure resolution of degree sequence $d = (d_0, d_1, \ldots, d_c)$, then there exists r such that

$$\beta(M)_{i,d_i} = r \prod_{\substack{k \neq i \\ k \neq 0}} \frac{d_k - d_0}{d_k - d_i}.$$

Conjecture (Boij–Söderberg).

The graded Betti table of any Cohen–Macaulay module of codimension c can be written as a positive linear combination of the graded Betti tables of modules of codimension c which have a pure resolution.

- First question posed by Boij and Söderberg: Given $d = (d_0, d_1, \ldots, d_c)$, does there exist a module whose degree sequence is d? Necessary restriction: need $d_0 < d_1 < \cdots < d_c$.
- In characteristic 0, the first construction was given by Eisenbud, Fløystad, and Weyman using the representation theory of the general linear group (Schur functors).
- Eisenbud and Schreyer later showed that they exist for any characteristic.

Now we assume the characteristic of K is 0.

- Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, let $\mathbf{S}_\lambda(V)$ denote the Schur functor corresponding to λ. It is an irreducible representation of $\mathbf{GL}(V)$ of highest weight λ.
- Our polynomial ring is a natural representation of $\mathbf{GL}(V)$:

$$A = \mathrm{Sym}(V) = \bigoplus_{k \geq 0} \mathrm{Sym}^k V = \bigoplus_{k \geq 0} \mathbf{S}_{(k,0,\dots,0)}(V).$$

- Given $\mathbf{S}_\lambda(V)$, we can turn it into an A-module by tensoring with A. Pieri's rule:

$$A \otimes_K \mathbf{S}_\lambda(V) = \bigoplus_{\mu} \mathbf{S}_\mu(V)$$

where μ ranges over all partitions obtained from λ by adding a vertical (!?) strip. [Convention: I draw λ the way Englishmen would draw the transpose λ'.]

Since the Pieri decomposition of $A \otimes \mathbf{S}_\lambda(V)$ is multiplicity free, we get inclusions $\varphi\colon \mathbf{S}_\mu(V) \to A \otimes \mathbf{S}_\lambda(V)$ which are unique up to a choice of constant. So we can extend this to a degree 0 map of A-modules:

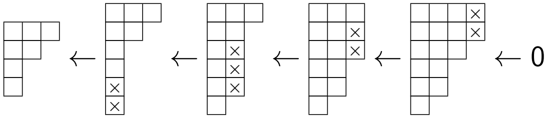
$$\begin{aligned} A(-|\mu/\lambda|) \otimes \mathbf{S}_\mu(V) &\to A \otimes \mathbf{S}_\lambda(V) \\ p(x) \otimes v &\mapsto p(x) \otimes \varphi(v). \end{aligned}$$

Call these **Pieri maps**.

Modules with pure resolutions can be constructed as the cokernels of certain Pieri maps.

Pieri resolutions

Abbreviate $A \otimes \mathbf{S}_\lambda(V)$ by the Young diagram of λ. For $n = 4$:



Degree sequence: $(0, 2, 5, 7, 9)$

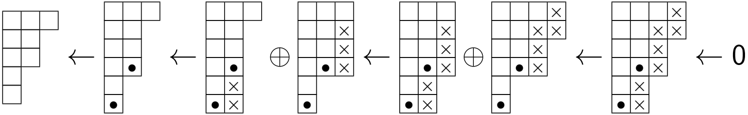
Theorem (Eisenbud–Fløystad–Weyman)

If μ is obtained from λ by adding boxes in the first column, then the cokernel of $A(-|\mu/\lambda|) \otimes \mathbf{S}_\mu(V) \to A \otimes \mathbf{S}_\lambda(V)$ has a pure resolution.

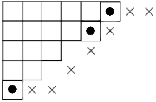
We can guess what the resolution looks like as in the example above. Since the differentials are equivariant, the fact that it is a complex is immediate. The exactness is more delicate.

- Proof of Eisenbud, Fløystad, and Weyman uses the Borel–Weil–Bott theorem.
- Simpler proof given by S.–Weyman using explicit matrix presentations of Pieri maps due to Olver.
- Given the theorem of Eisenbud, Fløystad, and Weyman, the next natural thing to consider is the cokernel of $A(-|\mu/\lambda|) \otimes \mathbf{S}_\mu(V) \to A \otimes \mathbf{S}_\lambda(V)$ without requiring that μ is obtained from λ by adding boxes in the first column.

Pieri resolutions



The terms are described by "critical boxes" (marked by $\times$):



The admissible subsets are unions of intervals of critical boxes whose leftmost point is next to a framed box. The ith term in the resolution is a direct sum of ways to add admissible subsets of size $i-1$.

- Proof by S.–Weyman uses the "horseshoe lemma" for building a resolution of an extension of two modules.
- Now what about cokernels of

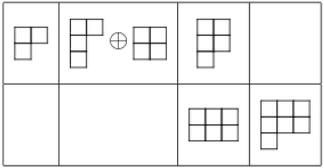
$$\bigoplus_j A(-|\mu^j/\lambda|) \otimes \mathbf{S}_{\mu^j}(V) \to A \otimes \mathbf{S}_\lambda(V)?$$

Can also construct resolutions using mapping cones, but they are not necessarily minimal. This construction gives minimal resolutions in the case that each of the column indices of the μ^j/λ are pairwise disjoint. It would be interesting to describe the terms of the minimal resolution explicitly.

Equivariant Betti diagrams

Does an equivariant version of the Boij–Söderberg decompositions hold?

- Betti tables would contain characters instead of ranks of modules.



- Instead of rational numbers, we would use ratios of characters.
- What is the right analogue of a positive rational number? One guess: ratios of Schur positive symmetric functions.

- Set $S\mathbf{Q}$ to be the field of fractions of the symmetric functions $\mathbf{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$.
- $S\mathbf{Q}_{\geq 0}$ are those elements which can be written as ratios of Schur positive symmetric functions.
  These are all subtraction-free expressions starting from the Schur polynomials.
- They're not well-behaved:

$$s_4 - s_{3,1} = \frac{s_3(s_4 - s_{3,1})}{s_3} = \frac{s_7}{s_3} \quad (n = 2)$$

Proposition

If A is any monomial positive symmetric function, then there exists a Schur polynomial s_λ such that As_λ is Schur positive.

Proof.

Follows from determinantal expression for Schur polynomials. $\square$

This is far from a necessary condition:

$$\frac{s_5^3(s_4 - s_{3,1} - s_{2,2})}{s_5^3} = \frac{1}{s_5^3}(s_{19} + 2s_{18,1} + 2s_{17,2} + 2s_{16,3} + 3s_{15,4} \\ +4s_{14,5} + 2s_{13,6} + s_{11,8} + 2s_{10,9}) \quad (n = 2)$$

Can we characterize subtraction-free Schur positive symmetric functions? Describe it as a cone?

Let M be a finite length equivariant A-module with equivariant minimal free resolution $\mathbf{F}_\bullet$.

Strong version: Does there exist degree sequences $d^1, \ldots, d^r$ and representations $W, W_1, \ldots, W_r$ such that $W \otimes \mathbf{F}_\bullet$ has a filtration of subcomplexes whose quotients are isomorphic to $W_i \otimes \mathbf{F}(d^i)_\bullet$, where the $\mathbf{F}(d^i)_\bullet$ is a pure resolution of degree d^i and the isomorphism is of equivariant complexes?

Weak version: Is it true that $\beta(M)$ can be written as a subtraction-free Schur positive linear combination of equivariant pure Betti tables?

Theorem (S.–Weyman)

If $\beta(M)$ is an equivariant Betti table of a finite length module M which is pure in all degrees except possibly one, then $\beta(M)$ is a Schur positive linear combination of pure Betti tables.

Corollary

Every pure equivariant Betti table of a finite length module is a Schur positive scalar multiple of a Betti table arising from the EFW construction.

Eisenbud–Schreyer proof of Boij–Söderberg conjectures

Since the Betti diagrams form a cone, it's enough to find equations for their facets.
For a cochain complex of free A-modules
$E : 0 \to E^1 \to E^2 \to \cdots \to E^n \to 0$, set

$$\gamma_{i,d}(E) = \dim_K \mathrm{H}^i(E)_d,$$

$$\langle \beta, \gamma \rangle_{c,\tau} = \sum_{\substack{i,j,k \\ j \leq i \\ j < \tau \text{ or } j \leq i-2}} (-1)^{i-j} \beta_{i,k}\gamma_{j,-k} + \sum_{\substack{k,\varepsilon \\ 0 \leq \varepsilon \leq 1 \\ k \leq c+\varepsilon}} (-1)^{\varepsilon} \beta_{\tau+\varepsilon,k}\gamma_{\tau,-k}$$

When β is the Betti table of a minimal free resolution, this number is nonnegative (!)
The facet-defining equations come from complexes which are linear monads for **supernatural vector bundles** on projective space.

A vector bundle $\mathcal{E}$ on projective space is **natural** [Hartshorne–Hirschowitz] if $\mathcal{E}(d)$ has at most one nonzero cohomology group for all $d \in \mathbf{Z}$.
A natural vector bundle is **supernatural** [Eisenbud–Schreyer] if the roots of the Hilbert polynomial $P_{\mathcal{E}}(d) = \chi(\mathcal{E}(d))$ are distinct integers.

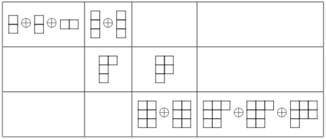
We can do all of the above setup in the equivariant case too!
One problem: finding facet-defining equations isn't enough because we don't have "Schur positive convex geometry."

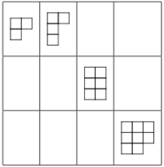
Eisenbud and Schreyer also proved that the cone of Betti tables has a natural triangulation.
Boij and Söderberg guessed its existence, and gave a simple algorithm for writing down the decomposition of a Betti table:

$$\begin{bmatrix} 2 & 1 & . \\ . & 1 & . \\ . & 1 & . \\ . & . & . \\ . & . & 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 5 & 6 & . \\ . & . & . \\ . & . & . \\ . & . & . \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} 7/6 & . & . \\ . & 1 & . \\ . & 1 & . \\ . & . & . \\ . & . & 5/6 \end{bmatrix}$$

$$\begin{bmatrix} 7/6 & . & . \\ . & 1 & . \\ . & 1 & . \\ . & . & . \\ . & . & 5/6 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2 & . & . \\ . & 3 & . \\ . & . & . \\ . & . & . \\ . & . & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & . & . \\ . & . & . \\ . & 1 & . \\ . & . & . \\ . & . & 1/2 \end{bmatrix}$$

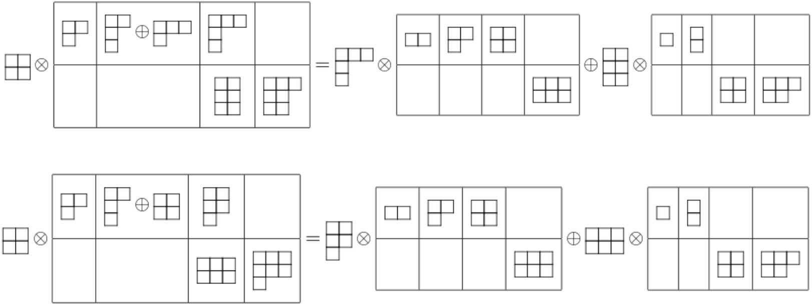
The obvious analogue of this algorithm **fails** in the equivariant setting:





Using the second table, we cannot pick a multiple to clear an entry of the first table and leave something subtraction-free Schur positive.

Some cases do work:



Does it work for Pieri resolutions in general?

What else?

- Multilinear constructions of pure resolutions in positive characteristic?
- Multigraded resolutions?
- Cone of cohomology tables for other varieties? Two natural generalizations of projective space: toric varieties and homogeneous spaces.

References

- David Eisenbud, Gunnar Fløystad, and Jerzy Weyman, The existence of pure free resolutions
- David Eisenbud and Frank-Olaf Schreyer, Betti numbers of graded modules and cohomology of vector bundles
- Steven V Sam and Jerzy Weyman, Pieri resolutions for classical groups

Pieri maps can be calculated using the package `PieriMaps` in Macaulay 2.