Math 184, Spring 2023
Homework 5
Due: Wednesday, May 24 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences.
I'll put some hints at the very end.

(1) Let u_n be the number of integer partitions of n that only use the parts 3 and 4.
   (a) Write $U(x) = \sum_{n \geq 0} u_n x^n$ as a rational function.
   (b) Use the method in Example 6.2.5 from the notes to get a linear recurrence relation for u_n. Make sure to state the relevant initial conditions.

(2) For $n > 0$, let a_n be the number of integer partitions of n such that every part appears at most twice, and let b_n be the number of integer partitions of n such that no part is divisible by 3. Set $a_0 = b_0 = 1$. Show that $a_n = b_n$ for all n.

(3) Consider the alphabet consisting of the 3 symbols

$$\texttt{(} \qquad \texttt{)} \qquad \texttt{0}.$$

Call a word **balanced** if, when deleting the `0`'s, the result is a balanced set of parentheses (empty word allowed). Let a_n be the number of balanced words of length n. For instance, $a_4 = 9$ and the set of balanced words of length 4 are:

`0000`, `()00`, `(0)0`, `(00)`, `0()0`, `0(0)`, `00()`, `()()`, `(())`.

   (a) Prove that (a_n) satisfies the following recursive formula:

$$a_n = a_{n-1} + \sum_{i=0}^{n-2} a_i a_{n-2-i} \qquad \text{(for } n \geq 2\text{)}.$$

   and $a_0 = a_1 = 1$.
   (b) Use the technique in the proof of Corollary 6.3.3 and the text following it to find a simple formula for $A(x) = \sum_{n \geq 0} a_n x^n$, i.e., find a quadratic polynomial that $A(x)$ is a root of and then use the quadratic formula to solve for $A(x)$. *You do not need to solve for a closed formula for a_n.*

(4) Show that the following two quantities are counted by the nth Catalan number and list/draw the 5 examples when $n = 3$:
   (a) The number of paths from $(0, 0)$ to $(2n, 0)$ using the steps $(1, 1)$ and $(1, -1)$ which never go below the x-axis (touching x-axis is ok).
   (b) The number of integer partitions $(\lambda_1, \ldots, \lambda_n)$ such that $n \geq \lambda_i \geq n - i + 1$ for all $i = 1, \ldots, n$.

(5) (a) We have n distinguishable telephone poles. We want to paint each one either red, blue, or green such that #(red poles) is even and #(blue poles) is odd. How many ways can this be done?
   (b) Continuing with the situation in (a), we add the colors white and yellow, but with the rule that #(white poles) + #(yellow poles) must be an odd number. How many ways can this be done?

Extra practice problems (don't turn in)

(6) Let n be a positive integer and let a_n be the number of different ways to pay n dollars using only 1, 2, 5, 10, 20 dollar bills in which at most three 20 dollar bills are used. Express $A(x) = \sum_{n\geq 0} a_n x^n$ as a rational function.

(7) The goal of this problem is to derive the formula for the Catalan numbers using an idea called the *reflection principle*. This completely avoids formal power series. First we need some definitions.

A forward path from $(0,0)$ to (n,n) is **good** if it never goes strictly above the diagonal line $x = y$. Any other forward path is **bad**. From class, the number of good forward paths is the nth Catalan number. We denote paths as sequences $(v_1, \dots, v_{2n})$ where each v_i is either the vector $(1,0)$ or $(0,1)$.

(a) Given a bad path $(v_1, \dots, v_{2n})$ from $(0,0)$ to (n,n), let r be the smallest index such that $v_1 + \cdots + v_r$ is above the line $x = y$, i.e., the second coordinate is strictly bigger than the first coordinate. Create a new path $(w_1, \dots, w_{2n})$ by

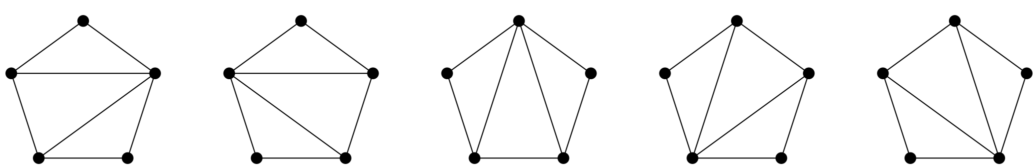
$$w_i = \begin{cases} v_i & \text{if } 1 \leq i \leq r \\ (1,1) - v_i & \text{if } r+1 \leq i \leq 2n \end{cases},$$

i.e., w agrees with v for the first r steps, and we swap all of the remaining steps. Show that w is a forward path from $(0,0)$ to $(n-1, n+1)$.

(b) In (a) we defined a function $\{\text{bad forward paths from } (0,0) \text{ to } (n,n)\} \to \{\text{forward paths from } (0,0) \text{ to } (n-1,n+1)\}$. Show that this function is a bijection.

(c) The previous parts show that the number of good forward paths is a difference of two quantities, each of which is the number of forward paths. Use the previous formula for forward paths to get a new derivation for Catalan numbers.

(8) Let n be a positive integer. Show that the number of ways of triangulating (i.e., drawing diagonals between vertices that do not intersect except at vertices so that the regions are all triangles) a convex polygon with $(n+2)$ vertices is the Catalan number C_n. By convention, the triangle has exactly one triangulation and here are the 5 triangulations of a pentagon:



Hints

2: Adapt the proof of Theorem 6.2.4. The identity $1 + x^i + x^{2i} = \dfrac{1 - x^{3i}}{1 - x^i}$ will help.

3a: Adapt the proof of Theorem 6.3.2.

4: The connection between Catalan numbers and forward paths discussed in lecture is helpful here.

5b: There are two cases: #white is odd and #yellow is even (or the other way around). Alternatively, you can treat white and yellow together as a single structure.