Math 184, Spring 2023
Homework 4
Due: **Wednesday**, May 10 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences.

(1) If a formula is used from class, state which one you are using.

(a) What is the coefficient of $x_1^5 x_3^2 x_4^2$ in $(x_1 + x_2 + x_3 + x_4 + x_5)^9$?

(b) What is the coefficient of x^{15} in $\dfrac{1}{(2-x)^2}$?

(c) Let α be a scalar. What is the coefficient of x^{12} in $\dfrac{2+2x^3}{(1+\alpha x)^5}$?

(d) Let α be a scalar. What is the coefficient of x^5 in $\dfrac{\sqrt{1-\alpha x}}{(1-x)^4}$?

(2) (a) Define a sequence by

$$a_0 = 1, \qquad a_1 = 3, \qquad a_n = 8a_{n-1} - 16a_{n-2} + 3^n \qquad \text{for } n \geq 2.$$

Find a closed formula for a_n.

(b) Define a sequence by

$$b_0 = 2, \qquad b_1 = 1, \qquad b_n = 5b_{n-1} - 6b_{n-2} + n \qquad \text{for } n \geq 2.$$

Find a closed formula for b_n.

(3) Define a sequence by

$$a_0 = -1, \qquad a_1 = 3, \qquad a_2 = 1, \qquad a_n = 3a_{n-2} + 2a_{n-3} + n^2 \qquad \text{for } n \geq 3.$$

Write $A(x) = \sum_{n \geq 0} a_n x^n$ as a rational function (= a polynomial divided by another polynomial) in x. You do not need to solve for a closed formula for a_n.

(4) Let $S(n,k)$ be the Stirling number of the second kind. For each $k \geq 1$, define the ordinary generating function

$$\mathbf{S}_k(x) = \sum_{n \geq 0} S(n,k)x^n = S(0,k) + S(1,k)x + S(2,k)x^2 + \cdots$$

(a) For $k \geq 2$, translate the identity from lecture

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k) \qquad (\text{for } n \geq k)$$

into an identity involving $\mathbf{S}_k(x)$ and $\mathbf{S}_{k-1}(x)$.

(b) Use the identity you found in (a) and induction on k to show that for all $k \geq 1$:

$$\mathbf{S}_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

(5) You want to build a stack of blocks that is n feet high. You have 3 different kinds (unlimited of each): green blocks are 1 foot high, while red and blue blocks are 2 feet high. Blocks of the same color are considered indistinguishable. Let a_n be the number of ways to stack these blocks.

Find a linear recurrence relation and initial conditions satisfied by a_n. You don't need to solve for a closed formula.