Math 184, Spring 2023
Homework 3
Due: **Wednesday**, May 3 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences. I'll put some hints at the very end.

(1) Use Stirling numbers to write x^5 and x^6 as a linear combination of falling factorials.

(2) Let n be a positive integer. Find simple formulas for $S(n+2, n)$ and $S(n+3, n)$.

(3) (a) Let Y_n be the collection of *set* partitions of $[n]$ such that every block has size 2 or 3 and let $y_n = |Y_n|$. If $n \geq 4$, prove that

$$y_n = (n-1)y_{n-2} + \binom{n-1}{2} y_{n-3}.$$

(b) Let Z_n be the set of *integer* partitions λ of n such that every entry of λ is either 2 or 3 and let $z_n = |Z_n|$. If $n \geq 3$, prove that

$$z_n = \begin{cases} z_{n-3} + 1 & \text{if } n \text{ is even} \\ z_{n-3} & \text{if } n \text{ is odd} \end{cases}.$$

(4) (a) Evaluate $\sum_{i=0}^{n} \binom{n}{i} 3^{n-i}$.

(b) Evaluate $\sum_{i=0}^{n} \binom{n}{i} (-1)^i 3^{n-i}$.

(c) Evaluate $\sum_{\substack{0 \leq i \leq n \\ i \text{ even}}} \binom{n}{i} 3^{n-i}$ (the sum is over i from 0 to n such that i is even).

(d) Evaluate $\sum_{\substack{0 \leq i \leq n \\ i \text{ even}}} i \binom{n}{i} 4^{n-i}$ (the sum is over i from 0 to n such that i is even).

(5) A "forward path" in the xy-plane is a sequence of steps of the form $\vec{(1,0)}$ and $\vec{(0,1)}$ (i.e., going one unit to the right or one unit up). Let a, b be non-negative integers.

(a) How many forward paths are there from $(0,0)$ to (a,b)?

(b) Let $S_{a,b}$ be the set of integer partitions λ (no restriction on $|\lambda|$) such that $\ell(\lambda) \leq b$ and $\lambda_1 \leq a$. Find and describe a bijection (and its inverse) between $S_{a,b}$ and the set of forward paths from $(0,0)$ to (a,b).

Extra practice problems (don't turn in)

(6) Fix positive integers n, m, k. By comparing the coefficients of x^k of $(x+1)^n \cdot (x+1)^m$ and $(x+1)^{n+m}$, prove that

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}.$$

(7) Let $F(n)$ be the number of all set partitions of $[n]$ such that every block has size ≥ 2. Prove that

$$B(n) = F(n) + F(n+1),$$

where $B(n)$ is the nth Bell number.

(8) (a) Using the multinomial theorem, compare the coefficients of both sides of the equation $(x+y+z)(x+y+z)^n = (x+y+z)^{n+1}$ to get a generalization of Pascal's identity for multinomial coefficients.

(b) Do the same thing with k variables for general k.

Hints

2: First think about all of the possible ways to break up $n+2$ (or $n+3$) objects into n nonempty blocks and then handle each possible case separately.

3a: Given a set partition, consider how big the block containing n is.

3b: Given a partition λ, consider two cases based on if $\lambda_1 = 2$ or $\lambda_1 = 3$.

4c: The sum under consideration is the average of the sums in (a) and (b).

5a: Every forward path is a sequence of length $a+b$ consisting of a "right"s and b "up"s.

5b: Given a forward path, consider the region bounded by it together with the lines $x = 0$ and $y = b$.