Math 184, Spring 2023
Homework 2
Due: Friday, Apr. 21 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences. I'll put some hints at the very end.

(1) (a) A classroom has 100 seats and 80 students. How many different seating arrangements are there? (Students cannot share a seat.)
(b) A neighborhood has 10 houses and each needs to painted one of three colors: white, brown, or orange. How many ways can we choose colors subject to the constraint that there has to be at least two orange houses?

(2) Let n be a positive integer. Below, A, B, C will denote subsets of $[n]$.
(a) How many pairs (A, B) satisfy $A \cup B = [n]$?
(b) How many triples (A, B, C) satisfy $A \cap B = \emptyset$ and $B \cap C = \emptyset$?
(c) How many triples (A, B, C) satisfy $A \cap B = \emptyset$ and $B \cap C \neq \emptyset$?

(3) Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 80.$$

For each of the following conditions, how many solutions are there? (Each part is an independent problem, don't combine the conditions.)
(a) The x_i are all positive even integers.
(b) The x_i are all positive odd integers.
(c) x_1, x_2, x_3, x_4 are positive even integers and x_5, x_6 are positive odd integers.

(4) We consider some variations of standard Poker hands. Start with a standard deck of cards (4 suits, 13 values, so 52 cards in total).
(a) How many ways can we choose 8 cards so that we have two triples and a pair? i.e., 3 of the cards have the same value, another 3 cards have the same value, and the remaining 2 cards also have the same value.
(b) How many ways can we choose 6 cards so that at most 2 suits appear? (i.e., either all cards have the same suit or at worst, there are 2 different suits so that each card has one or the other).

(5) Let n and k be positive integers.
(a) Assume $n \geq k - 1$. Let A_n be the set of k-element subsets of $[n - k + 1]$ and let B_n be the set of k-element subsets of $[n]$ that have no consecutive elements (if $i \in S$ then $i + 1 \notin S$). Carefully describe a bijection between A_n and B_n. (Note if n is too small, both sets are empty but that won't really matter too much.)
(b) Let's call a subset S "spaced out" if any two different elements are always at least 3 apart from each other (in symbols: if $i, j \in S$ and $i \neq j$, then $|i - j| \geq 3$). How many spaced out k-element subsets of $[n]$ are there?

Extra practice problems (don't turn in)

(6) Describe a bijection between {compositions of n} and {subsets of $[n-1]$}.

(7) Fix an integer $n \geq 2$. Call a composition $(a_1, \ldots, a_k)$ of n **doubly even** if the number of a_i which are even is also even (i.e., there could be no even a_i, or 2 of them, or 4, or ...).

Show that the number of doubly even compositions of n is 2^{n-2}.

For example, if $n = 4$, then here are the 4 doubly even compositions of 4:

$$(2,2), \qquad (3,1), \qquad (1,3), \qquad (1,1,1,1).$$

Hint: Given a composition $\alpha = (a_1, \ldots, a_k)$, define another composition $\Phi(\alpha)$ by

$$\Phi(\alpha) = \begin{cases} (1, a_1 - 1, a_2, a_3, \ldots, a_k) & \text{if } a_1 > 1 \\ (a_2 + 1, a_3, \ldots, a_k) & \text{if } a_1 = 1 \end{cases}.$$

(in both cases, we didn't do anything to $a_3, \ldots, a_k$). Show that Φ defines a bijection between the set of doubly even compositions of n and the set of compositions of n which are not doubly even.

(8) How many ways can we choose 7 cards so that we have 2 pairs (different values) and a triple?

(9) An "alternator" is a choice of 7 cards such that the values are all distinct, and when they are put in increasing order, the suits alternate (the first, third, fifth, and last card have the same suit S_1, and the second, fourth, and sixth card have the same suit S_2, and $S_1 \neq S_2$). How many are there?

For this problem, A can either be considered the largest or the smallest value (as long as one of the possibilities works).

Hints

(2)c: The answer to b is helpful for solving c.

(3)a: Write $x_i = 2y_i$.

b: Write $x_i = 2y_i + 1$.

(5)a: Suppose that $S = \{s_1, \ldots, s_k\}$ has no consecutive values and that we've written the elements in order: $s_1 < s_2 < \cdots < s_k$. Consider subtracting from them like we did when studying multisets: $s_1, s_2 - 1, s_3 - 2, \ldots, s_k - k + 1$.

b: The method is going to be very similar to part a, though some details change. The similarity is that "no consecutive elements" means that if $i, j \in S$ and $i \neq j$, then $|i - j| \geq 2$.