Specht modules

$\Bbbk$ = field, $|\lambda| = n$

M^λ = permutation rep. on λ-tabloids

$\cong \mathrm{Ind}_{\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}}^{\mathfrak{S}_n} \Bbbk$

t = λ-tableau

C_t = subgroup of $\mathfrak{S}_n$ which preserves set of entries of each column of t

signed column sum $\quad \kappa_t = \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma)\sigma \in \Bbbk[\mathfrak{S}_n]$

polytabloid $\quad e_t = \kappa_t \cdot \{t\} \in M^\lambda$

Ex. $\lambda = (3,2)$, $t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$

$$e_t = \left\{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 1 & 5 & 3 \\ \hline 4 & 2 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 4 & 5 & 3 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}\right\} \in M^{3,2}$$

Specht module $\quad S^\lambda$ is span of $\{e_t \mid t \ \lambda\text{-tableau}\}$

Ex. · $\lambda = (n)$, $M^{(n)} = \Bbbk$ since all λ-tableau equivalent

$\Rightarrow S^{(n)} = \Bbbk$

· $\lambda = (n-1,1)$ There n λ-tabloids: relevant info is what number is in second row, $M^{(n-1,1)}$ is n-dim'l, let X_i be λ-tabloid w/ i in second row. $M^{(n-1,1)} \cong \Bbbk^n$ (the permutation rep of $\mathfrak{S}_n$)

if $t = \begin{array}{|c|c|}\hline j & \cdots \\ \hline i \\ \cline{1-1}\end{array} \rightsquigarrow e_t = X_i - X_j$

$$\Rightarrow S^{(n-1,1)} = \{\alpha_1 X_1 + \cdots + \alpha_n X_n \mid \alpha_1 + \cdots + \alpha_n = 0\}$$
(standard rep)

· $\lambda = (1^n)$: There are $n!$ λ-tabloids, $M^{(1^n)} = \Bbbk[\mathfrak{S}_n]$

All e_t are equal up to sign, $S^{(1^n)}$ = sgn rep.

Lemma. S^λ is an $\mathfrak{S}_n$-subrepresentation of M^λ. Furthermore, as a $\Bbbk[\mathfrak{S}_n]$-module S^λ is generated by any e_t, t λ-tableau.

Pf. For $\sigma \in \mathfrak{S}_n$, λ-tableau t, we have

$$C_{\sigma t} = \sigma C_t \sigma^{-1},$$

so $\sigma \kappa_t \sigma^{-1} = \kappa_{\sigma t}$.

$\Rightarrow$ $\sigma \cdot e_t = \sigma \kappa_t \{t\} = \kappa_{\sigma t} \sigma \{t\} = \kappa_{\sigma t}\{\sigma t\} = e_{\sigma t}$.

$\Rightarrow$ $\mathfrak{S}_n$ preserves S^λ, and given any e_t, can get all of them using permutations. $\square$

Lemma. λ, μ be partitions of n. $t_1 = \lambda$-tableau, $t_2 = \mu$-tableau

Suppose $\kappa_{t_1}\{t_2\} \neq 0$. Then $\lambda \trianglerighteq \mu$.

Furthermore, if $\lambda = \mu$, then $\kappa_{t_1}\{t_2\} = \pm e_{t_1}$.

Pf. Lemma. $\lambda, \mu \in \mathrm{Par}(n)$. Let t_1 be λ-tableau, t_2 be μ-tableau. Suppose for every i, the numbers in ith row of t_2 are in different columns of t_1. Then $\lambda \trianglerighteq \mu$.

Let a, b be numbers in same row of t_2. Then $(1 - (a,b))\{t_2\} = 0$.

Suppose a, b in same column of t_1. Then $(a,b) \in C_{t_1}$,

pick coset reps $\sigma_1, \dots, \sigma_k$ for $C_t / \{1, (a,b)\}$.

Then $\kappa_{t_1} = \sum_{i=1}^{k} \mathrm{sgn}(\sigma_i)\sigma_i (1 - (a,b)) \Rightarrow \kappa_{t_1}\{t_2\} = 0$

Lemma $\Rightarrow$ $\lambda \trianglerighteq \mu$. (Since a, b arbitrary)

Now suppose $\lambda = \mu$. Then $\exists\, \sigma \in C_{t_1}$ so that $\sigma\{t_1\} = \{t_2\}$: every pair a, b of numbers in same row of t_2 are in different columns of t_1

So, pick σ so that a, b in t_1 get moved to same row where they are in t_2.

$$\Rightarrow \quad \kappa_{t_1}\{t_2\} = \kappa_{t_1}\sigma\{t_1\} = \operatorname{sgn}(\sigma)\,\kappa_{t_1}\{t_1\} = \pm e_{t_1} \qquad \square$$

<u>Cor.</u> For $u \in M^\lambda$ and λ-tableau t, $\kappa_t u$ is scalar multiple of e_t.

<u>Pf.</u> u is linear combination of $\{t'\}$, and $\kappa_t\{t'\}$ is a scalar multiple of e_t. $\square$

Define symmetric bilinear form $\langle , \rangle$ on M^λ by

$$\langle \{t_1\}, \{t_2\} \rangle = \begin{cases} 1 & \text{if } t_1 \sim t_2 \\ 0 & \text{else} \end{cases}$$

extend linearly in each argument.

This is $\mathfrak{S}_n$-equivariant: $\langle \sigma v_1, \sigma v_2 \rangle = \langle v_1, v_2 \rangle$ $\forall \sigma \in \mathfrak{S}_n$, $v_1, v_2 \in M^\lambda$.

<u>Lemma.</u> For $u, v \in M^\lambda$, λ-tableau t, we have

$$\langle \kappa_t u, v \rangle = \langle u, \kappa_t v \rangle.$$

<u>Pf.</u>
$$\langle \kappa_t u, v \rangle = \sum_{\sigma \in C_t} \operatorname{sgn}\sigma \langle \sigma u, v \rangle = \sum_{\sigma \in C_t} \operatorname{sgn}\sigma \langle u, \sigma^{-1} v \rangle$$
$$= \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma) \langle u, \sigma v \rangle = \langle u, \kappa_t v \rangle. \qquad \square$$

Given subspace $V \subseteq M^\lambda$. Define $V^\perp = \{ v \in M^\lambda \mid \langle v, w \rangle = 0 \ \forall w \in V \}$.

<u>Thm</u> (submodule theorem). Let $U \subseteq M^\lambda$ be $\mathfrak{S}_n$-subrep. Then either $S^\lambda \subseteq U$ <u>or</u> $U \subseteq (S^\lambda)^\perp$.

<u>Pf.</u> Pick $u \in U$. For λ-tableau t, $\kappa_t u$ is scalar multiple of e_t. If $\exists\, u \in U$, λ-tableau t s.t. $\kappa_t u \neq 0$, then $e_t \in U$

$\Rightarrow S^\lambda \subseteq U.$

Otherwise, $\forall u \in U$, λ-tableau t we have $\kappa_t u = 0$.

$\Rightarrow 0 = \langle \kappa_t u, \{t\} \rangle = \langle u, \kappa_t \{t\} \rangle = \langle u, e_t \rangle$

$\Rightarrow u \in (S^\lambda)^\perp \Rightarrow U \subseteq (S^\lambda)^\perp.$ $\square$

Cor. $S^\lambda / (S^\lambda \cap (S^\lambda)^\perp)$ is either 0, or absolutely irred. rep.

Pf. By submodule thm, any subrep of S^λ is either all of S^λ or contained in $S^\lambda \cap (S^\lambda)^\perp$. $\Rightarrow S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$ is irreducible (or 0).

Fact: If V is finite-dim vector space w/ symmetric bilinear form $\langle , \rangle$ and basis $v_1, \dots, v_m$. Then

$$\dim V/V^\perp = \operatorname{rank} \left(\langle v_i, v_j \rangle \right)_{i,j=1,\dots,m}$$

Apply w/ $V = S^\lambda$, $\langle , \rangle$ = restriction of pairing on M^λ to S^λ.

$\Rightarrow \dim S^\lambda / (S^\lambda \cap (S^\lambda)^\perp)$ stays same if we enlarge field. $\square$

Thm. If char $k = 0$, then S^λ is irred. rep of $\mathfrak{S}_n$.

Pf. Suppose $\mathbb{k} = \mathbb{Q}$. Then $\langle u, u \rangle > 0$ for all $u \in S^\lambda$.

$\Rightarrow S^\lambda \cap (S^\lambda)^\perp = 0 \Rightarrow S^\lambda$ is absolutely irreducible.

$\Rightarrow S^\lambda$ irred. over any field containing $\mathbb{Q}$. $\square$

Lemma. Suppose $\exists$ $\mathfrak{S}_n$-equivariant map $\theta : M^\lambda \to M^\mu$ s.t. $\theta|_{S^\lambda} \neq 0$. Then $\lambda \trianglerighteq \mu$. Furthermore, if $\lambda = \mu$, then $\theta|_{S^\lambda}$ is a scalar multiple of the inclusion map.

Pf. Let t be λ-tableau. Then $\theta(e_t) \neq 0$. Then $\theta(e_t) = \theta(\kappa_t \{t\})$

$= \kappa_t \theta(\{t\})$

$\kappa_t \theta(\{t\})$ is linear combination of $\kappa_t\{t'\}$ where t' are μ-tabloids.

By previous lemma, $\kappa_t\{t'\} \neq 0 \Rightarrow \lambda \trianglerighteq \mu$ and $\kappa_t\{t'\} = \pm e_t$.

$\Rightarrow \theta(e_t)$ is a nonzero scalar multiple of e_t $\square$

Cor. Let $\mathrm{char}(\mathbb{k}) = 0$. If $\lambda \neq \mu$, then $S^\lambda \not\cong S^\mu$.
Furthermore, S^λ appears w/ multiplicity 1 in decomposition of M^λ as a direct sum of irreducible representations.

Pf. Suppose $S^\lambda \cong S^\mu$. Consider composition:

$$\underbrace{M^\lambda \longrightarrow S^\lambda \xrightarrow{\cong} S^\mu \subset M^\mu}_{\theta} \Rightarrow \lambda \trianglerighteq \mu.$$

projection map, exists by $\mathrm{char}(\mathbb{k})=0$

Similarly, have $M^\mu \to S^\mu \cong S^\lambda \subset M^\lambda \Rightarrow \mu \trianglerighteq \lambda$

$\Rightarrow \lambda = \mu$.

If S^λ appears w/ multiplicity ≥ 2, then $\exists$ nonzero map $S^\lambda \to M^\lambda$ which is not a scalar multiple of the inclusion.
Compose w/ $M^\lambda \to S^\lambda$ to get $\theta: M^\lambda \to M^\lambda$ which is nonzero on S^λ and not scalar multiple of inclusion $\to\leftarrow$ $\square$

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dim S^λ? can we find linearly independent subset of polytabloids?

What are character values of σ on S^λ?