Partitions

A partition of non-negative integer n is a sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ s.t. $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$ (integers) s.t. $\lambda_1 + \dots + \lambda_r = n$.

adding 0's at the end will be considered same partition.

$\ell(\lambda) =$ # of nonzero entries, length of λ

$|\lambda| = n$ size of λ

Shorthand: omit commas, so $(1,1,1,1)$ same as 1111

exponential notation 1111 same as 1^4

$\mathrm{Par}(n) =$ set of partitions of n, $p(n) = |\mathrm{Par}(n)|$.

Convention: $\mathrm{Par}(0) = \{(0)\}$, $p(0) = 1$

Ex.

$\mathrm{Par}(1) = \{1\}$

$\mathrm{Par}(2) = \{2, 1^2\}$

$\mathrm{Par}(3) = \{3, 21, 1^3\}$

$\mathrm{Par}(4) = \{4, 31, 211, 2^2, 1^4\}$

$\mathrm{Par}(5) = \{5, 41, 32, 2^2 1, 31^2, 21^3, 1^5\}$

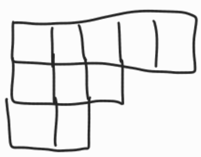
For each i, $m_i(\lambda) =$ multiplicity of i in λ

$=$ # of times i appears in λ.

Young diagrams give way to draw partitions

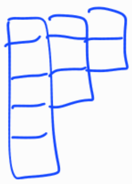
Draw λ_i boxes left-justified in row i, rows listed top to bottom

Ex. $\lambda = (5,3,2)$ $\quad Y(\lambda) =$



Transpose of λ is λ^T: flip Young diagram across diagonal

$Y(\lambda^T) =$



$\lambda^T = (3,3,2,1,1)$

$\lambda^T_i = \#\{ j \mid \lambda_j \geq i \}$

Note: $(\lambda^T)^T = \lambda$

Partial orderings

- $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i.
- dominance order: $\lambda \leq \mu$ if $\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i$ for all i.

If $|\lambda| = |\mu|$, then $\lambda \leq \mu \Leftrightarrow \lambda^T \geq \mu^T$

$\hookrightarrow$ transpose gives order-reversing involution on $\mathrm{Par}(n)$

- lexicographic order: If $|\lambda| = |\mu|$. then $\lambda \leq^R \mu$ if either $\lambda = \mu$, or:
  $\exists i$ s.t. $\lambda_1 = \mu_1, \dots, \lambda_{i-1} = \mu_{i-1}$ and $\lambda_i < \mu_i$. [total ordering]

If $\lambda \leq \mu \Rightarrow \lambda \leq^R \mu$

[Suppose $\lambda_1 = \mu_1, \dots, \lambda_i = \mu_i$ but $\lambda_{i+1} \neq \mu_{i+1}$,

Note: $\lambda_1 + \dots + \lambda_{i+1} \leq \mu_1 + \dots + \mu_{i+1} \Rightarrow \lambda_{i+1} < \mu_{i+1}$]

Lemma. let $a_{\lambda,\mu}$ be integers indexed by partitions of size n.

Assume that:
- $a_{\lambda\lambda} = 1 \quad \forall \lambda$
- $a_{\lambda\mu} \neq 0 \Rightarrow \mu \leq \lambda$

For any ordering of $\mathrm{Par}(n)$, $(a_{\lambda\mu})_{\lambda,\mu}$ is invertible (over $\mathbb{Z}$)

i.e., has $\det = \pm 1$.

Same conclusion if instead we have $\cdot\ a_{\lambda\lambda^T} = 1$, $\cdot\ a_{\lambda\mu} \neq 0 \Rightarrow \mu \leq \lambda^T$

Pf. Pick lexicographic ordering on $\mathrm{Par}(n) \rightsquigarrow (a_{\lambda\mu})$ matrix.

It has 1's on diagonal and is lower-triangular

$\Rightarrow \det(a_{\lambda\mu}) = 1$

Any other ordering differs by permutations, so det differs at most by a sign. Proof essentially same for second statement. $\square$

Tabloids

n = positive integer, $\lambda = (\lambda_1, \ldots, \lambda_r)$ partition of n.

A *λ-tableau* is a filling of boxes of $Y(\lambda)$ w/ $1, \ldots, n$ (each appearing once). $\mathfrak{S}_n$ acts on λ-tableau by permuting labels.

[There are $n!$ many λ-tableau]

Given λ-tableaux t_1, t_2, write $t_1 \sim t_2$ if t_2 obtained from t_1 by exchanging entries within their rows.

$\sim$ is equivalence relation, *λ-tabloid* is an equivalence class.

$\lambda = (3,2)$
$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 \\ \cline{1-2}\end{array}$$

[Note: there are $\dfrac{n!}{\lambda_1! \lambda_2! \cdots \lambda_r!}$ many λ-tabloids]

Given λ-tableau t, let $\{t\}$ be its equivalence class.

$\mathfrak{S}_n$ also acts on set of λ-tabloids.

This action is transitive, stabilizer of a λ-tabloid is isomorphic to $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r}$

Lemma. $\lambda, \mu \in \mathrm{Par}(n)$. Let t_1 be λ-tableau, t_2 be μ-tableau. Suppose for every i, the numbers in ith row of t_2 are in different columns of t_1. Then $\lambda \trianglerighteq \mu$.

Pf. There are μ_1 entries in first row of t_2. They are in different columns of $t_1 \Rightarrow Y(\lambda)$ has at least μ_1 columns $\Rightarrow \lambda_1 \geq \mu_1$.

Next, there are μ_2 entries in second row of t_2. Delete all entries from t_1 that don't come from first two rows of t_2. Now shift remaining entries to top of their columns. These values now belong to first two rows of $Y(\lambda)$

$\implies \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$

This argument extends to any $i \implies \lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$.

$\implies \lambda \geq \mu$. $\square$