Group algebra

k = field, G = finite group

group algebra is $k[G]$ which is k-vector space w/ basis $\{e_g \mid g \in G\}$ and multiplication $e_g e_h = e_{gh}$

$k[G]$ is associative and has unit e_{1_G}.

Claim: $\left\{ \begin{array}{c} \text{Representations} \\ \text{of } G \text{ over } k \end{array} \right\} \xleftrightarrow{\simeq} \left\{ \begin{array}{c} \text{left } k[G] \\ \text{modules} \end{array} \right\}$

Given $\rho : G \to GL(V)$, define $k[G]$-module structure on V by

$$\left(\sum_{g \in G} \alpha_g g\right)(v) := \sum_{g \in G} \alpha_g \, \rho(g)(v)$$

$v \in V$, $\alpha_g \in k$

Given left $k[G]$-module M, the function $\sigma_g : M \to M$, $m \to e_g \cdot m$ is invertible linear operator for each $g \in G$.

Define $\rho : G \to GL(M)$, $g \to \sigma_g$.

Restriction and Induction

Setup: $H \subseteq G$ subgroup

Given rep. $\rho : G \to GL(V)$, get restriction $\rho : H \to GL(V)$

call this $\mathrm{Res}^G_H(\rho)$ (or $\mathrm{Res}^G_H(V)$)

Note: restriction of class functions also makes sense

Given rep: $\rho : H \to GL(V)$, we can define induced representation $\mathrm{Ind}^G_H(\rho)$ (or $\mathrm{Ind}^G_H(V)$), a representation of G as follows:

Interpret V as a left $k[H]$-module

Think of $\Bbbk[G]$ as a right $\Bbbk[H]$-module by $e_g \cdot e_h = e_{gh}$ for $g \in G, h \in H$

[right R-module : $(m \cdot r) \cdot r' = m \cdot (rr')$]

General construction: R ring, M = right R-module, N = left R-module

Define tensor product $M \otimes_R N$ to be abelian group spanned by symbols $m \otimes n$ $(m \in M, n \in N)$ modulo relations:

$$\left.\begin{array}{ll} ① & (m+m') \otimes n = m \otimes n + m' \otimes n \\ ② & m \otimes (n+n') = m \otimes n + m \otimes n' \\ ③ & mr \otimes n = m \otimes rn \end{array}\right\} \begin{array}{l} m, m' \in M \\ n, n' \in N \\ r \in R \end{array}$$

In our case, take $R = \Bbbk[H]$, $M = \Bbbk[G]$, $N = V$

$$\Bbbk[G] \otimes_{\Bbbk[H]} V =: \mathrm{Ind}_H^G(V) \leftarrow \text{left } \Bbbk[G]\text{-module}$$

Given $g \in G$: $e_g \cdot \left(\sum_i e_{g_i} \otimes w_i \right) = \sum_i e_{gg_i} \otimes w_i$

$g_i \in G, w_i \in W$

Properties: $\cdot$ $\mathrm{Ind}_H^G(V)$ is a $\Bbbk$-vector space

$\cdot$ If $g_1, \ldots, g_r$ are left coset reps. for G/H and $v_1, \ldots, v_n$ basis for V,

then $\left\{ e_{g_i} \otimes v_j \;\middle|\; \begin{array}{l} i=1,\ldots,r \\ j=1,\ldots,n \end{array} \right\}$ basis for $\mathrm{Ind}_H^G(V)$

$$\Rightarrow \dim \mathrm{Ind}_H^G(V) = |G/H| \cdot \dim(V)$$

Computations: Fix $g_1, \ldots, g_r$ coset reps, $v_1, \ldots, v_n$ basis

Given $g \in G$, $\exists\, k$ s.t. $gg_i \in g_k H \Rightarrow g_k^{-1} g g_i \in H$

$$g \cdot (e_{g_i} \otimes v_j) = e_{g_k} \cdot (g_k^{-1} g g_i) \otimes v_j = e_{g_k} \otimes (g_k^{-1} g g_i) \cdot v_j$$

Ex. X = set w/ transitive G-action

$\hookrightarrow$ $\forall x, y \in X$ $\exists g \in G$ s.t. $g \cdot x = y$

Fix $x \in X$, let $H = \{g \in G \mid g \cdot x = x\}$ its stabilizer

left-action of G on G/H is isomorphic to X as G-sets

$$gH \longrightarrow g \cdot x$$

$\Rightarrow$ $\mathbb{k}[G/H] \cong \mathbb{k}[X]$, and $\mathbb{k}[G/H] \cong \mathrm{Ind}_H^G(\mathbb{k})$

$\hookrightarrow$ trivial rep. of H $\square$

Induction is transitive: given $K \leq H \leq G$, rep. V of K, have $\mathrm{Ind}_K^G(V) \cong \mathrm{Ind}_H^G(\mathrm{Ind}_K^H(V))$.

Prop. $g_1, \ldots g_r$ coset reps for G/H. For $g \in G$:
V rep. of H

$$\chi_{\mathrm{Ind}_H^G(V)}(g) = \sum_{\substack{1 \leq i \leq r \\ \text{s.t. } g_i^{-1} g g_i \in H}} \chi_V(g_i^{-1} g g_i)$$

Pf. Pick basis $v_1, \ldots v_n$ for V, use basis $\{e_{g_i} \otimes v_j\}$ for $\mathrm{Ind}_H^G V$

Consider subspace $\langle e_{g_i} \rangle \otimes V$, the span of $\{e_{g_i} \otimes v_j \mid j = 1, \ldots, n\}$

In matrix form, g on $\mathrm{Ind}_H^G(V)$ is block matrix w.r.t. these subspaces.

$$\begin{array}{c} \\ \langle e_{g_1} \rangle \otimes V \\ \vdots \\ \langle e_{g_r} \rangle \otimes V \end{array} \begin{array}{c} \begin{array}{ccc} \langle e_{g_1} \rangle \otimes V & \cdots & \langle e_{g_r} \rangle \otimes V \end{array} \\ \left(\begin{array}{c|c|c} & & \\ \hline & \ddots & \\ \hline & & \end{array} \right) \end{array}$$

Nonzero diagonal blocks come from i s.t. $g_i^{-1} g g_i \in H$

In that case, $g \cdot (e_{g_i} \otimes v_j) = e_{g_i} \otimes (g_i^{-1} g g_i) \cdot v_j$

i.e., that block is matrix of $g_i^{-1} g g_i$ on V

$$\Rightarrow \chi_{\mathrm{Ind}_H^G(V)}(g) = \sum_{\substack{1 \le i \le r \\ \text{s.t. } g_i^{-1} g g_i \in H}} \chi_V(g_i^{-1} g g_i). \qquad \square$$

Extend this formula to any class function for $f \in CF(H)$,

$\mathrm{Ind}_H^G(f) \in CF(G)$ given by (for $g \in G$):

$$\mathrm{Ind}_H^G(f)(g) = \sum_{\substack{1 \le i \le r \\ \text{s.t. } g_i^{-1} g g_i \in H}} f(g_i^{-1} g g_i).$$

$\rightsquigarrow$ linear function: $\mathrm{Ind}_H^G : CF(H) \longrightarrow CF(G)$

also have $\mathrm{Res}_H^G : CF(G) \to CF(H)$.

<u>Thm</u> (Frobenius reciprocity). Given $\varphi \in CF(H)$, $\psi \in CF(G)$,

$$(\mathrm{Ind}_H^G(\varphi), \psi)_G = (\varphi, \mathrm{Res}_H^G(\psi))_H$$

<u>Pf.</u> $g_1, \ldots, g_r$ coset reps for G/H.

$$(\mathrm{Ind}_H^G(\varphi), \psi)_G = \frac{1}{|G|} \sum_{g \in G} \mathrm{Ind}_H^G(\varphi)(g) \overline{\psi(g)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{1 \le i \le r \\ \text{s.t. } g_i^{-1} g g_i \in H}} \varphi(g_i^{-1} g g_i) \overline{\psi(g_i^{-1} g g_i)} \qquad (*)$$

belong to H

How many times does $h \in H$ appear in this sum?

i.e., how many pairs (g_i, g) satisfy $g_i^{-1} g g_i = h$?

$\rightarrow$ $g = g_i h g_i^{-1}$ $\rightarrow$ r solutions for any $h \in H$.

$$(*) = \frac{1}{|G|} \sum_{h \in H} r \cdot \varphi(h) \overline{\psi(h)}$$

$$= \frac{1}{|H|} \sum_{h \in H} \varphi(h) \overline{\psi(h)} = \left(\varphi, \mathrm{Res}_H^G(\psi) \right)_H .$$

<u>Rmk</u>: Frob. reciprocity follows formally from properties of tensor products.