Schur Functors

If $\rho : GL(V) \to GL(W)$ is representation,

then $\rho(g)\rho(h) = \rho(gh) \quad \forall g, h \in GL(V)$.

Examples: $W = V^{\otimes d}$, $\mathrm{Sym}^d V$, $\overset{d}{\Lambda} V$, $D^d V$, ...

Note: for any linear map $f : U \to V$,

get linear map $f^{\otimes d} : U^{\otimes d} \to V^{\otimes d}$

$$\sum u_1 \otimes \cdots \otimes u_d \to \sum f(u_1) \otimes \cdots \otimes f(u_d)$$

and for any $g : V \to W$, get $\boxed{(g \circ f)^{\otimes d} = g^{\otimes d} \circ f^{\otimes d}}$

Informally: $(-)^{\otimes d}$ is a functor: it outputs vector space for every vector space and respects composition of linear maps (and identity maps)

$\mathrm{Sym}^d(-)$, $\overset{d}{\Lambda}(-)$, $D^d(-)$ are also functors.

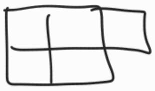
Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be partition,
$\mu = \lambda^T = (\mu_1, \ldots, \mu_s)$

Given vector space V, let $S_\lambda V$ (Schur functor applied to V) is the image of the composition:

$$\overset{\mu_1}{\Lambda} V \otimes \overset{\mu_2}{\Lambda} V \otimes \cdots \otimes \overset{\mu_s}{\Lambda} V \longrightarrow V^{\otimes \mu_1} \otimes V^{\otimes \mu_2} \otimes \cdots \otimes V^{\otimes \mu_s}$$

$$\xrightarrow{\text{rearrangement}} V^{\otimes \lambda_1} \otimes V^{\otimes \lambda_2} \otimes \cdots \otimes V^{\otimes \lambda_r}$$

$$\longrightarrow \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_r} V$$

Ex. $\lambda = (3,2)$, $\mu = (2,2,1)$ 

$$\Lambda^2 V \otimes \Lambda^2 V \otimes V \longrightarrow V^{\otimes 2} \otimes V^{\otimes 2} \otimes V$$
$$\longrightarrow V^{\otimes 3} \otimes V^{\otimes 2}$$
$$\longrightarrow \mathrm{Sym}^3 V \otimes \mathrm{Sym}^2 V$$

element of $\Lambda^2 V \otimes \Lambda^2 V \otimes V$ is a sum of elements of the form

$(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5$, $v_i \in V$

comultiplication: $v_1 \wedge v_2 \longrightarrow v_1 \otimes v_2 - v_2 \otimes v_1$

$v_3 \wedge v_4 \longrightarrow v_3 \otimes v_4 - v_4 \otimes v_3$

$v_5 \longrightarrow v_5$

$$(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5 \longrightarrow \begin{array}{|c|c|c|} \hline v_1 & v_3 & v_5 \\ \hline v_2 & v_4 \\ \cline{1-2} \end{array} - \begin{array}{|c|c|c|} \hline v_2 & v_3 & v_5 \\ \hline v_1 & v_4 \\ \cline{1-2} \end{array}$$
$$- \begin{array}{|c|c|c|} \hline v_1 & v_4 & v_5 \\ \hline v_2 & v_3 \\ \cline{1-2} \end{array} + \begin{array}{|c|c|c|} \hline v_2 & v_4 & v_5 \\ \hline v_1 & v_3 \\ \cline{1-2} \end{array}$$

$$\longrightarrow v_1 v_3 v_5 \otimes v_2 v_4 - v_2 v_3 v_5 \otimes v_1 v_4$$
$$- v_1 v_4 v_5 \otimes v_2 v_3 + v_2 v_4 v_5 \otimes v_1 v_3$$

Key properties: • $S_\lambda(-)$ is a functor: given linear map $f: U \to V$, get $S_\lambda(f): S_\lambda U \to S_\lambda V$ s.t. $S_\lambda(g \circ f) = S_\lambda g \circ S_\lambda f$.
and $S_\lambda(id) = id$.

- $S_\lambda(V)$ is a $GL(V)$-representation for all vector spaces V.
- $S_\lambda V = 0$ if $\ell(\lambda) > \dim V$ since $\Lambda^d V = 0$ if $d > \dim V$.
(and $\mu_1 = \ell(\lambda)$)

Extreme cases: • If $\lambda = (d)$, then $\mu = (1, \dots, 1)$

get $\underbrace{V \otimes \cdots \otimes V}_{d} \to \underbrace{V \otimes \cdots \otimes V}_{d}$

$\to V^{\otimes d} \to \mathrm{Sym}^d V$

so $S_{(d)} V = \mathrm{Sym}^d V$

• If $\lambda = (\underbrace{1, \dots 1}_{d})$, $\mu = (d)$

get $\bigwedge^d V \longrightarrow V^{\otimes d}$

$\longrightarrow \underbrace{V \otimes \cdots \otimes V}_{d} \xrightarrow{id} V \otimes \cdots \otimes V$

so $S_{(1^d)} V = \bigwedge^d V$

Let $e_1, \dots e_n$ be a basis for V.

Let T be a tableau of shape λ w/ values $1, \dots, n$.

$\rightsquigarrow$ vector $(e_{T_{1,1}} \wedge e_{T_{2,1}} \wedge \cdots \wedge e_{T_{\mu_1,1}}) \otimes \cdots \otimes (e_{T_{1,s}} \wedge \cdots \wedge e_{T_{\mu_s,s}})$

$\cap$

$\bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_s} V$

Let e_T be its image in $S_\lambda V$

Ex. $T = \begin{matrix} 1 & 2 & 1 \\ 3 & 5 & \end{matrix}$ vector $= (e_1 \wedge e_3) \otimes (e_2 \wedge e_5) \otimes e_1$

Thm. $\{e_T \mid T$ Semistandard Young tableau of shape $\lambda\}$ is a basis for $S_\lambda V$.

$\implies \dim S_\lambda V = \#\mathrm{SSYT}$ of shape λ w/ values $1, \dots, n$.