Multilinear Algebra

$\mathbb{k}$ = field, V = finite dim vector space

$d > 0$ integer, $V^{\otimes d} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{d}$

$$V^{\otimes 0} = \mathbb{k}$$

Given $\sigma \in \mathfrak{S}_d$, $\sigma \cdot \left(\sum v_1 \otimes \cdots \otimes v_d \right) = \sum v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$

(this gives right action of $\mathfrak{S}_d$ on $V^{\otimes d}$)

$GL(V)$ = invertible linear operators on V

For $g \in GL(V)$, $g \cdot \left(\sum v_1 \otimes \cdots \otimes v_d \right) = \sum g v_1 \otimes \cdots \otimes g v_d$

These actions commute.

The <u>dth symmetric power</u> of V is quotient of $V^{\otimes d}$ by subspace spanned by $\{ v - \sigma \cdot v \mid v \in V^{\otimes d}, \sigma \in \mathfrak{S}_d \}$.

This quotient is $GL(V)$-representation.

<u>Notation</u>: $\mathrm{Sym}^d V$ or $S^d V$

Given $v_1 \otimes \cdots \otimes v_d$ where $v_i \in V$, let $v_1 \cdots v_d$ denote its image in $\mathrm{Sym}^d V$. In general, for any $\sigma \in \mathfrak{S}_d$, we have

$$v_1 \cdots v_d = v_{\sigma(1)} \cdots v_{\sigma(d)}$$

Intuition: $\mathrm{Sym}^d V$ = space of degree d polynomials

For $g \in GL(V)$, $g \cdot (v_1 \cdots v_d) = (g v_1) \cdots (g v_d)$.

Let $e_1, \ldots, e_n$ be basis for V.

Then $\{ e_{i_1} \otimes \cdots \otimes e_{i_d} \mid i_1, \ldots, i_d \in \{1, \ldots, n\} \}$ is basis for $V^{\otimes d}$.

<u>Claim</u>: $\{ e_{i_1} \cdots e_{i_d} \mid 1 \le i_1 \le i_2 \le \cdots \le i_d \le n \}$ is basis for $\mathrm{Sym}^d V$

<u>Pf.</u> Spanning is easy

Pick nonzero scalars $\alpha_1, \ldots, \alpha_n$. Let $g = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} \in GL(\mathbb{k}^n) = GL_n(\mathbb{k})$

$g \cdot e_{i_1} \cdots e_{i_d} = \alpha_{i_1} \cdots \alpha_{i_d} e_{i_1} \cdots e_{i_d}$.

So $e_{i_1} \cdots e_{i_d}$ are eigenvectors. If $\mathbb{K}$ is infinite, then we can pick $\alpha_1, \ldots, \alpha_n$ so that $\alpha_{i_1} \cdots \alpha_{i_d}$ are all distinct.

If not, we can extend coefficients (this does not affect linear independence).

Consider linear functional on $V^{\otimes d}$ that sends each $e_{i_1} \otimes \cdots \otimes e_{i_d}$ to 1. This factors through to linear functional on $\mathrm{Sym}^d V$ that values 1 on each $e_{i_1} \cdots e_{i_d}$ (so they're $\neq 0$).

$\rightsquigarrow$ nonzero eigenvectors w/ distinct eigenvalues are linearly independent $\square$

$\Longrightarrow$ $\dim \mathrm{Sym}^d V = \binom{n+d-1}{d}$, $n = \dim V$.

Given $d, e \geq 0$, we have "multiplication" map

$$\mu : \mathrm{Sym}^d V \otimes \mathrm{Sym}^e V \longrightarrow \mathrm{Sym}^{d+e} V$$
$$(v_1 \cdots v_d) \otimes (w_1 \cdots w_e) \longmapsto v_1 \cdots v_d w_1 \cdots w_e$$

$GL(V)$-equivariant.

The <u>dth exterior power</u> of V is quotient of $V^{\otimes d}$ by subspace spanned by $\{ v_1 \otimes \cdots \otimes v_d \mid \exists i \neq j \text{ s.t. } v_i = v_j \}$.

<u>Notation:</u> $\bigwedge^d V$, This is $GL(V)$-representation.

Given $v_1 \otimes \cdots \otimes v_d$, $v_i \in V$, let $v_1 \wedge \cdots \wedge v_d$ be its image in $\bigwedge^d V$. For $g \in GL(V)$,

$$g \cdot (v_1 \wedge \cdots \wedge v_d) = (g v_1) \wedge \cdots \wedge (g v_d).$$

Rmk. Note: $v_1 \wedge \dots \wedge v_d$ is skew-symmetric in the sense that

$$v_1 \wedge \dots \wedge v_d = \operatorname{sgn}(\sigma)\, v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(d)} \quad \text{for any } \sigma \in \mathfrak{S}_d.$$

When $d=2$:

$$0 = (v_1+v_2)\wedge(v_1+v_2) = \cancel{v_1 \wedge v_1} + v_1\wedge v_2 + v_2 \wedge v_1 + \cancel{v_2 \wedge v_2}$$

$$\Rightarrow \quad v_1 \wedge v_2 = -v_2 \wedge v_1$$

We could define $\bigwedge^d V$ as quotient by $\{v - \operatorname{sgn}(\sigma)\sigma v\}$ if $\operatorname{char} k \neq 2$:

for any $v \in V$, $v\wedge v = -v \wedge v \Rightarrow 2 v \wedge v = 0$

Pick basis $e_1, \dots, e_n$ for V. Then

$\{e_{i_1} \wedge \dots \wedge e_{i_d} \mid 1 \le i_1 < \dots < i_d \le n\}$ is basis for $\bigwedge^d V$.

$\Rightarrow \dim \bigwedge^d V = \binom{n}{d}$ where $n = \dim V$

(note: this is 0 if $d > \dim V$.)

Multiplication map:

$$\mu: \bigwedge^d V \otimes \bigwedge^e V \longrightarrow \bigwedge^{d+e} V$$

$$(v_1 \wedge \dots \wedge v_d) \otimes (w_1 \wedge \dots \wedge w_e) \mapsto v_1 \wedge \dots \wedge v_d \wedge w_1 \wedge \dots \wedge w_e$$

μ is $GL(V)$-equivariant.

Comultiplication map:

$$\Delta: \bigwedge^{d+e} V \longrightarrow \bigwedge^d V \otimes \bigwedge^e V$$

$$v_1 \wedge \dots \wedge v_{d+e} \longrightarrow \sum_{I} \operatorname{sgn}(I, I^c)\, v_I \otimes v_{I^c}$$

$I \leftarrow$ subsets of $1, \dots, d+e$ of size d

$v_I = v_{i_1} \wedge \dots \wedge v_{i_d}$ if $i_1 < \dots < i_d$ are elements of I

$v_{I^c} = \dots$

$\mathrm{sgn}(I, I^c) \in \{1, -1\}$ satisfies $v_I \wedge v_{I^c} = \mathrm{sgn}(I, I^c) v_1 \wedge \dots \wedge v_{d+e}$.

Δ is $GL(V)$-equivariant

Δ is "coassociative" compositions are same:

$$\begin{tikzcd} \bigwedge^{d+e+f} V \arrow[r] \arrow[dr] & \bigwedge^{d+e} V \otimes \bigwedge^{f} V \arrow[r] & \bigwedge^{d} V \otimes \bigwedge^{e} V \otimes \bigwedge^{f} V \\ & \bigwedge^{d} V \otimes \bigwedge^{e+f} V \arrow[ur] & \end{tikzcd}$$

If we iterate Δ $d-1$ times, get to

$$\bigwedge^{d} V \longrightarrow V^{\otimes d} \quad (\text{since } \bigwedge^{1} V = V)$$

$$v_1 \wedge \dots \wedge v_d \longmapsto \sum_{\sigma \in \mathfrak{S}_d} (\mathrm{sgn}\, \sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}.$$

This is injective, so $\bigwedge^d V$ is a subrepresentation of $V^{\otimes d}$.

The composition $\bigwedge^d V \longrightarrow V^{\otimes d} \longrightarrow \bigwedge^d V$

is $d!$ times identity on $\bigwedge^d V$.

So if char $k = 0$, or larger than d, then $\bigwedge^d V$ is a direct summand of $V^{\otimes d}$.

The image is subspace $\{v \in V^{\otimes d} \mid v = \mathrm{sgn}\,\sigma\,(\sigma \cdot v)\ \forall \sigma \in \mathfrak{S}_d\}$.

The dth divided power is subspace $\{v \in V^{\otimes d} \mid v = \sigma v\ \forall \sigma \in \mathfrak{S}_d\}$

denoted $D^d V$, $GL(V)$-subrep.

- $D^d V$ is generally not isomorphic to $\mathrm{Sym}^d V$ (but have same dim). Holds if char $k=0$ or larger than d.
- In general, $D^d(V^*) \cong (\mathrm{Sym}^d V)^*$ as $GL(V)$-representations

For $d=1$, $V^{\otimes 1}=V$ is irreducible representation of $GL(V)$.

For $d=2$, if $\operatorname{char} k \neq 2$, we can decompose $V^{\otimes 2}$:

$$V^{\otimes 2} \cong \bigwedge^2 V \oplus \operatorname{Sym}^2 V$$

$$(v \otimes w) \longrightarrow \left(\frac{v\otimes w - w \otimes v}{2}, \frac{v \otimes w + w \otimes v}{2} \right)$$

For $d=3$, there's more to $V^{\otimes 3}$ than just $\bigwedge^3 V$ & $\operatorname{Sym}^3 V$:

[if $\dim V = 2$, then $\dim V^{\otimes 3} = 8$
but $\bigwedge^3 V = 0$ and $\dim \operatorname{Sym}^3 V = 4$]