Counting Semistandard Young Tableaux

$s_\lambda(\underbrace{1,\dots,1}_{k}) = \#$ of SSYT of shape λ and values $1,\dots,k$.

$[= \dim S_\lambda(\mathbb{C}^k)$ ← irreducible rep of $GL_k(\mathbb{C})]$

<u>Thm</u>. $s_\lambda(\underbrace{1,\dots,1}_{k\ 1\text{'s}}) = \prod_{1\le i<j\le k} \frac{\lambda_i-\lambda_j+j-i}{j-i}$

<u>Pf</u>. let $x_1,\dots,x_k$ be variables.

$$s_\lambda(x_1,\dots,x_k) = \frac{\det\left(x_i^{\lambda_j+k-j}\right)_{i,j=1,\dots,k}}{\det\left(x_i^{k-j}\right)_{i,j=1,\dots,k}}$$

Set $x_i = q^{i-1}$ (q new variable)

$$s_\lambda(1,q,\dots,q^{k-1}) = \frac{\det\left(q^{i(\lambda_j+k-j)}\right)}{\det\left(q^{i(k-j)}\right)}$$

← Vandermonde det's

$$= \frac{\prod_{1\le i<j\le k}\left(q^{\lambda_i+k-i}-q^{\lambda_j+k-j}\right)}{\prod_{1\le i<j\le k}\left(q^{k-i}-q^{k-j}\right)}$$

$$= \prod_{1\le i<j\le k}\left(\frac{q^{\lambda_i+k-i}-q^{\lambda_j+k-j}}{q^{k-i}-q^{k-j}}\right)$$

← divide by q^{k-j}, pull out q^{λ_j}

$$= \prod_{1\le i<j\le k} q^{\lambda_j}\cdot\frac{q^{\lambda_i-\lambda_j-i+j}-1}{q^{j-i}-1}$$

plug in $q=1$:

$$= \prod_{1\le i<j\le k}\frac{\lambda_i-\lambda_j+j-i}{j-i}.$$

□

Cor. k fixed. let $n\lambda = (n\lambda_1, n\lambda_2, \dots n\lambda_k)$.

Then $n \to s_{n\lambda}(1,\dots,1)$ is a polynomial in n.

Given $(i,j) \in Y(\lambda)$, $c(i,j) = j-i$ **content**

Ex.

0	1	2	3	4
-1	0	1	2	
-2	-1	0		

contents.

Thm. (Hook-content formula).

$$s_\lambda(1,\dots,1) = \prod_{(i,j)\in Y(\lambda)} \frac{k + c(i,j)}{h(i,j)}$$

Pf. Let $n = |\lambda|$, $l_i = \lambda_i + k - i$. **Recall:** $f^\lambda = \dfrac{n!}{\prod_{(i,j)\in Y(\lambda)} h(i,j)}$

$$= \frac{n!}{l_1! \cdots l_k!} \prod_{1\le i<j\le k} (l_i - l_j)$$

RHS:

$$\prod_{(i,j)\in Y(\lambda)} \frac{k+j-i}{h(i,j)} = \prod_{(i,j)\in Y(\lambda)} (k+j-i) \cdot \frac{1}{l_1! \cdots l_k!} \prod_{1\le i<j\le k} (l_i - l_j) \qquad (*)$$

contents in row i of $Y(\lambda)$: $\quad -i+1 \quad -i+2 \quad \cdots \quad -i+\lambda_i$

product over boxes in row i: $\quad (k-i+1)(k-i+2)\cdots(k-i+\lambda_i) = \dfrac{l_i!}{(k-i)!}$

$$(*) = \prod_{i=1}^{k} \frac{\cancel{l_i!}}{(k-i)!} \cdot \frac{1}{\cancel{l_1! \cdots l_k!}} \prod_{1\le i<j\le k} (l_i - l_j)$$

$$= \frac{\prod_{1\le i<j\le k} (l_i - l_j)}{(k-1)!(k-2)! \cdots 1!\,0!}$$

$$l_i - l_j = \lambda_i + k - i - \lambda_j - k + j$$
$$= \lambda_i - \lambda_j + j - i$$

$$= \prod_{1\le i<j\le k} \frac{\lambda_i-\lambda_j+j-i}{j-i}.$$

Also: $(k-1)!\,(k-2)!\,\dots\,1!$

$$= \prod_{1\le i<j\le k} (j-i)$$

$\square$

Cor. Function $k \longrightarrow s_\lambda(\underbrace{1,\dots,1}_{k})$ is a polynomial of degree $|\lambda|$.

Thm. $s_\lambda(1,\dots,1) = \det\left(\binom{k+\lambda_i-i+j-1}{k-1}\right)_{i,j=1}^{n}$

where $n \ge \ell(\lambda)$.

Pf. Jacobi-Trudi: $s_\lambda = \det\left(h_{\lambda_i-i+j}\right)_{i,j=1}^{n}$

$h_d(\underbrace{1,\dots,1}_{k})$ = # monomials in $x_1,\dots,x_k$ of degree d $= \binom{d+k-1}{d} = \binom{d+k-1}{k-1}$ $\square$

encode $x_1^{p_1}\dots x_k^{p_k}$ as sequence of 0's and 1's

where p_1 0's followed by 1 followed by

p_2 0's followed by 1 ...

p_k 0's