Multiplying Schur functions

(1) $a_{\nu+\rho}\, e_\mu(x_1,\dots,x_n) = \sum_\lambda K_{\lambda^T/\nu^T,\mu}\, a_{\lambda+\rho}.$

(2) $s_\lambda(x_1,\dots,x_n) = \dfrac{a_{\lambda+\rho}}{a_\rho}$

<u>Cor.</u> $s_\nu e_\mu = \sum_\lambda K_{\lambda^T/\nu^T,\mu}\, s_\lambda$

<u>Pf.</u> Divide (1) by a_ρ to get

$$s_\nu(x_1,\dots,x_n)\, e_\mu(x_1,\dots,x_n) = \sum_\lambda K_{\lambda^T/\nu^T,\mu}\, s_\lambda(x_1,\dots,x_n)$$

This holds for any $n \gg 0$, hence holds for $n=\infty$. $\square$

<u>Cor.</u> $s_\nu h_\mu = \sum_\lambda K_{\lambda/\nu,\mu}\, s_\lambda$

<u>Pf.</u> Apply ω to previous corollary:

$$s_{\nu^T} h_\mu = \sum_\lambda K_{\lambda^T/\nu^T,\mu}\, s_{\lambda^T}. \qquad \square$$

<u>Thm.</u> For any $f \in \Lambda$, we have

$$\langle f s_\nu, s_\lambda\rangle = \langle f, s_{\lambda/\nu}\rangle$$

<u>Pf.</u> Consider $f = h_\mu$.

$$\langle h_\mu s_\nu, s_\lambda\rangle = \sum_\alpha K_{\alpha/\nu,\mu}\langle s_\alpha, s_\lambda\rangle = K_{\lambda/\nu,\mu}$$

By definition, $s_{\lambda/\nu} = \sum_\mu K_{\lambda/\nu,\mu}\, m_\mu$

$$\langle h_\mu, s_{\lambda/\nu}\rangle = \sum_\alpha K_{\lambda/\nu,\alpha}\langle h_\mu, m_\alpha\rangle = K_{\lambda/\nu,\mu} \qquad \square$$

Since $\{s_\lambda\}$ is basis for Λ, $\exists$ integers $c^\lambda_{\mu\nu}$ s.t.

$$s_\mu s_\nu = \sum_\lambda \underbrace{c^\lambda_{\mu\nu}}_{} s_\lambda$$

→ Littlewood-Richardson coefficients

Immediate properties:

① If $c^\lambda_{\mu\nu} \neq 0$, then $|\lambda| = |\mu| + |\nu|$.

② $c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu}$

③ $c^\lambda_{\mu\nu} = c^{\lambda^T}_{\mu^T,\nu^T}$.

$\Longrightarrow$ $c^\lambda_{\mu\nu} = \langle s_\mu s_\nu, s_\lambda \rangle = \langle s_\mu, s_{\lambda/\nu} \rangle$

$\Longrightarrow$ $s_{\lambda/\nu} = \sum_\mu c^\lambda_{\mu\nu} s_\mu$.

Pieri rules:

Given $\nu \subseteq \lambda$, say λ/ν is a horizontal strip if all boxes of $Y(\lambda/\nu)$ appear in different columns.

... vertical strip if all boxes in different rows

Thm. (Pieri rule). ① If $\mu = (1^k)$, then

$$c^\lambda_{1^k,\nu} = \begin{cases} 1 & \text{if } |\lambda| = |\nu| + k \text{ \& } \lambda/\nu \text{ is vertical strip} \\ 0 & \text{else} \end{cases}$$

i.e., $s_\nu s_{1^k} = \sum_\lambda s_\lambda$ ← λ s.t. λ/ν vertical strip of size k.

② If $\mu = (k)$, then $c^\lambda_{(k),\nu} = \begin{cases} 1 & \text{if } |\lambda| = |\nu| + k \text{ \& } \lambda/\nu \text{ is horizontal strip} \\ 0 & \text{else.} \end{cases}$

i.e., $s_\nu s_k = \sum s_\lambda$ ← λ s.t. λ/ν horizontal strip of size k.

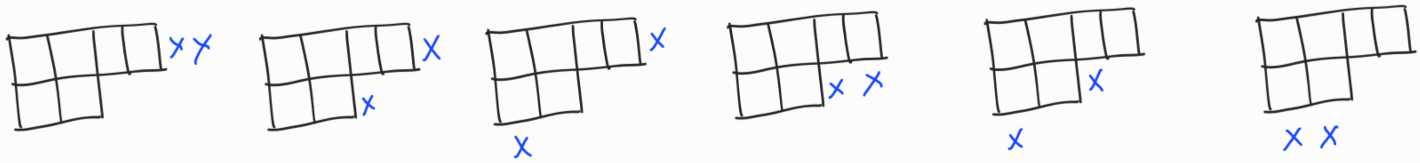
Pf. $s_{1^k} = e_k$, so $s_\nu s_{1^k} = \sum_\lambda K_{\lambda^T/\nu^T,(k)} s_\lambda$

ways to fill $Y(\lambda^T/\nu^T)$ w/ k 1's s.t. it is SSYT.

$$= \begin{cases} 1 & \text{if } \lambda^T/\nu^T \text{ is horizontal strip of size } k \\ 0 & \text{else} \end{cases}$$

□

Ex. $\nu = (4,2)$, $k = 2$ $\quad s_{(4,2)} s_{(2)} = ?$



$$s_{42} s_2 = s_{62} + s_{53} + s_{521} + s_{44} + s_{431} + s_{422}$$

Cor. $$s_1^n = \sum_{\substack{\lambda \\ \text{s.t. } |\lambda| = n}} f^\lambda s_\lambda$$ where f^λ = # standard Young tableaux of shape λ

Pf. Pieri rule tells us: $$s_1^n = \sum_{\substack{\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)} \\ \text{s.t. } |\lambda^{(i)}| = i}} s_{\lambda^{(n)}} = \sum_\lambda f^\lambda s_\lambda .$$

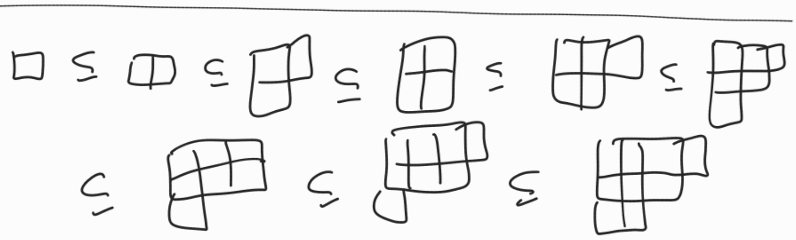
How many chains exist w/ given $\lambda^{(n)}$?

Given $\lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(n)}$, build SYT by putting i in unique box of $\lambda^{(i)} \setminus \lambda^{(i-1)}$.

Conversely, given SYT of shape λ, let $\lambda^{(i)}$ be partition whose Young diagram is all boxes w/ value $\leq i$. □

1	2	5	8
3	4	7	
6	9		

$\Leftrightarrow$



Thm. $\omega(s_{\lambda/\nu}) = s_{\lambda^T/\nu^T}$

Pf. $\langle s_{\mu^T}, s_{\lambda^T/\nu^T}\rangle = \langle s_{\mu^T} s_{\nu^T}, s_{\lambda^T}\rangle$

$= \langle \omega(s_{\mu^T} s_{\nu^T}), \omega(s_{\lambda^T})\rangle$

$= \langle s_\mu s_\nu, s_\lambda\rangle$

$= \langle s_\mu, s_{\lambda/\nu}\rangle$

$= \langle \omega(s_\mu), \omega(s_{\lambda/\nu})\rangle$

$= \langle s_{\mu^T}, \omega(s_{\lambda/\nu})\rangle$.

$\Rightarrow \langle s_{\mu^T}, \underbrace{s_{\lambda^T/\nu^T} - \omega(s_{\lambda/\nu})}_{\text{pairs to 0 w/ all of } \Lambda}\rangle = 0 \quad \forall \mu.$

$\Rightarrow \quad s_{\lambda^T/\nu^T} = \omega(s_{\lambda/\nu}).$ $\square$

Jacobi-Trudi identity.

Thm. Pick $\mu \subseteq \lambda$ w/ $n \geq \ell(\lambda)$. Set $h_i = 0$ for $i < 0$. Then

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i - \mu_j - i + j}\right)_{i,j=1,\dots,n}$$

$$= \det\left(e_{\lambda_i^T - \mu_j^T - i + j}\right)_{i,j=1,\dots,n}$$

Pf. Note: if we use $n+1$ instead, last row of matrix is

$(0 \ 0 \cdots 0 \ 1)$

Fix μ, work in N variables where $N \geq n$.

$$\sum_\lambda s_{\lambda/\mu}(x) s_\lambda(y) = \sum_\lambda \sum_\nu c^\lambda_{\mu\nu} s_\nu(x) s_\lambda(y)$$

$$= \sum_\nu s_\nu(x) s_\mu(y) s_\nu(y)$$

$$= s_\mu(y) \prod_{1 \leq i,j \leq N} (1 - x_i y_j)^{-1}$$

$$= s_\mu(y) \sum_\nu h_\nu(x)\, m_\nu(y)$$

Multiply both sides by $a_\rho(y)$ to get

$$\sum_\lambda s_{\lambda/\mu}(x)\, a_{\lambda+\rho}(y) = a_{\mu+\rho}(y) \sum_\nu h_\nu(x)\, m_\nu(y).$$

$$= \left(\sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma)\, \sigma(y^{\mu+\rho}) \right) \sum_{\alpha \in \mathbb{Z}_{\geq 0}^N} h_\alpha(x)\, y^\alpha$$

$$= \sum_{\sigma \in \mathfrak{S}_N} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^N} \operatorname{sgn}(\sigma)\, h_\alpha(x)\, y^{\alpha + \sigma(\mu+\rho)}$$

Take coefficient of $y^{\lambda+\rho}$ of both sides

$$\left[\begin{array}{l} \alpha + \sigma(\mu+\rho) = \lambda+\rho \\ \alpha = \lambda+\rho-\sigma(\mu+\rho) \end{array} \right]$$

$$s_{\lambda/\mu}(x) = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma)\, h_{\lambda+\rho-\sigma(\mu+\rho)}$$

$$= \det\left(h_{\lambda_i - \mu_j - i + j} \right)_{i,j=1,\dots N}$$

$\Rightarrow$ get same identity w/ size n matrix in any # of variables

Let $N \to \infty$, see that size n matrix gives correct value. $\square$