Determinant formula for Schur polynomials

Fix positive integer n.

$\alpha = (\alpha_1, \alpha_2 \dots \alpha_n) \in \mathbb{Z}_{\geq 0}^n$

$$a_\alpha = \det\left(x_i^{\alpha_j}\right)_{i,j=1,\dots,n} = \det\begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & & & \\ \vdots & & & \\ x_n^{\alpha_1} & & \cdots & x_n^{\alpha_n} \end{pmatrix}$$

a_α is skew-symmetric: for any $\sigma \in \mathfrak{S}_n$, $\sigma \cdot a_\alpha = \mathrm{sgn}(\sigma)\, a_\alpha$

$\rho = (n-1, n-2, \dots, 1, 0)$

Lemma. (a) $\prod_{1 \leq i < j \leq n} (x_i - x_j)$ divides every skew-symmetric polynomial in $x_1, \dots x_n$.

(b) $a_\rho = \prod_{1 \leq i < j \leq n} (x_i - x_j)$

Pf. (a) Suppose $f(x_1, \dots, x_n)$ is skew-symmetric.

Then $(i, j) f = -f$, so substitution $x_i \to x_j$ gives 0,

so $(x_i - x_j)$ divides f. $\Rightarrow$ Since i, j arbitrary,

$\prod_{i<j} (x_i - x_j)$ divides f.

(b) $\deg \prod_{1 \leq i < j \leq n} (x_i - x_j) = \binom{n}{2}$

$$a_\rho = \det\begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & & & & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{pmatrix}$$

$\Rightarrow \deg a_\rho = (n-1) + (n-2) + \dots + 1 + 0 = \binom{n}{2}$

$\Rightarrow$ a_ρ and $\prod_{i<j}(x_i - x_j)$ have same degree.

Since latter divides a_ρ, they are equal up to scalar.

coeff. of $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ is 1 in both, so they're equal. $\square$

$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$

Given $\nu \subseteq \lambda$, let $K_{\lambda/\nu, \mu}$ be #SSYT of shape λ/ν and type μ

Lemma. let μ, ν be partitions w/ $\ell(\mu) \le n$, $\ell(\nu) \le n$. Then

$$a_{\nu+\rho}\, e_\mu(x_1, \ldots, x_n) = \sum_\lambda K_{\lambda^T/\nu^T, \mu}\, a_{\lambda+\rho}.$$

Pf. Claim: The coefficient of $x^{\lambda+\rho}$ in $a_{\nu+\rho} e_\mu$ is $K_{\lambda^T/\nu^T, \mu}$.

To get monomial in $a_{\nu+\rho} e_{\mu_1} e_{\mu_2} \cdots e_{\mu_n}$, we pick monomial x^β in $a_{\nu+\rho}$ and monomials $x^{\alpha(i)}$ in e_{μ_i} and multiply together.

We need to pick all ways so that $\beta + \alpha(1) + \cdots + \alpha(n) = \lambda + \rho$.

Note: ① Each partial product $a_{\nu+\rho} e_{\mu_1} \cdots e_{\mu_r}$ is skew-symmetric.

② Also, each x^γ appearing in skew-symmetric polynomial w/ nonzero coefficient satisfies all γ_i distinct.

③ There exists unique permutation σ so that β is in order. Since $\alpha(1), \ldots \alpha(n)$ are 0-1 vectors, if $x^{\beta + \alpha(1) + \cdots + \alpha(n)}$ has nonzero coefficient in $a_{\nu+\rho} e_\mu$, then σ is unique permutation so that $\sigma(\beta + \alpha(1) + \cdots + \alpha(n))$ is in order.

$\Rightarrow$ β needs to be decreasing to end up w/ $\lambda + \rho$.

and also $\beta + \alpha(1) + \cdots + \alpha(r)$ decreasing for all $r = 1, \ldots, n$.

$\Rightarrow$ Forces $\beta = \nu + \rho$.

If $\gamma = (\gamma_1, \ldots \gamma_n)$ is non-negative and strictly decreasing, then $\gamma - \rho$ is partition.

$\Rightarrow$ get sequence of partitions

$$\underset{\gamma(0)}{\nu} \subseteq \underset{\gamma(1)}{\nu + \alpha(1)} \subseteq \underset{\gamma(2)}{\nu + \alpha(2)} \subseteq \cdots \subseteq \underset{\gamma(n)}{\nu + \alpha(1) + \cdots + \alpha(n)} = \lambda$$

In each step, difference of two partitions has no 2 boxes in same row.

Every choice of monomials $\alpha(1), \ldots, \alpha(n)$ is in bijection w/ sequences $\gamma(0) \subseteq \gamma(1) \subseteq \cdots \subseteq \gamma(n)$ where $\gamma(0) = \nu$, $\gamma(n) = \lambda$

$Y(\gamma(i+1)/\gamma(i))$ consists of μ_i boxes, no 2 in same row.

Given this, label boxes of $Y(\lambda/\nu)$ w/ i if it appears in $Y(\gamma(i+1)/\gamma(i))$

Take transpose, we get SSYT of shape λ^T/ν^T of type μ.

$\Rightarrow$ There are $K_{\lambda^T/\nu^T, \mu}$ many ways. ✓

Consider difference

$$a_{\nu+\rho} e_\mu - \sum_\lambda K_{\lambda^T/\nu^T, \mu} \, a_{\lambda+\rho}$$

If $\lambda' \neq \lambda$, then coeff. of $x^{\lambda+\rho}$ in $a_{\lambda'+\rho}$ is 0

$\Rightarrow$ for all partitions, coeff of $x^{\lambda+\rho}$ is 0 in difference.

$\Rightarrow$ difference is 0. $\square$

Cor. $\lambda \in \mathrm{Par}$,

$$s_\lambda(x_1, \dots x_n) = \frac{a_{\lambda+\rho}}{a_\rho}.$$

pf. Take $\nu = \phi$ in previous formula:

$$e_\mu = \sum_\lambda K_{\lambda^T,\mu} \frac{a_{\lambda+\rho}}{a_\rho}$$

But also, $h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda$, apply ω:

$$e_\mu = \sum_\lambda K_{\lambda^T,\mu} s_\lambda.$$

s_λ form basis, so $(K_{\lambda^T,\mu})$ is invertible.

$$\Longrightarrow \frac{a_{\lambda+\rho}}{a_\rho} = s_\lambda.$$

$\square$