Irreducible representations

Def. A <u>subrepresentation</u> of V is a subspace $W \subseteq V$ s.t. $\rho_V(g)w \in W$ $\forall g \in G, w \in W$. We can define $\rho_W : G \to GL(W)$ by $\rho_W(g) = \rho_V(g)|_W$.

If $f : V \to V'$ is G-equivariant, $\Rightarrow$ $\ker f$ is a subrep of V, image f is subrep of V'.

Def. A (nonzero) rep V is <u>irreducible</u> if it has no nonzero subreps other than V. V is <u>reducible</u> if not irreducible.

Irreducible $\Rightarrow$ Indecomposable

<u>Recall:</u> If $W \subseteq V$ is subspace then a complement to W is subspace W' s.t. $W \cap W' = 0$ & $W + W' = V$.

If this is the case, $V \cong W \oplus W'$, and denote by $V = W \oplus W'$.

$$\left\{ \begin{array}{c} \text{complements to} \\ W \end{array} \right\} \overset{\cong}{\longleftrightarrow} \left\{ \begin{array}{c} \text{projections } \pi \text{ on} \\ V \text{ w/ image } W \end{array} \right\} \qquad \begin{array}{l} \pi^2 = \pi \\ \& \ \pi : V \to V \end{array}$$

$\ker \pi \longleftarrow \pi$

$W' \longrightarrow$ every $v \in V$ is uniquely of form $v = w + w'$, $w \in W$, $w' \in W'$, define $\pi(v) = w$

<u>Lemma.</u> Suppose $\operatorname{char} k = 0$ <u>or</u> $\operatorname{char} k = p > 0$ & $p \nmid |G|$.

<u>Then</u> given subrep $W \subseteq V$, $\exists$ subrep $U \subseteq V$ s.t. $V = W \oplus U$.

<u>Pf.</u> First pick any complement W' to W. $\rightsquigarrow$ projection π

Define $\psi : V \to V$ by

$$\psi = \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ \pi \circ \rho_V(g)^{-1}$$

Claim: $\psi^2 = \psi$ & image $\psi = W$, and $\rho_V(h) \circ \psi \circ \rho_V(h)^{-1} = \psi$ $\forall h \in G$.

$$\rho_V(h)\circ\psi\circ\rho_V(h)^{-1} = \frac{1}{|G|}\sum_{g\in G}\rho_V(h)\rho_V(g)\,\psi\,\rho_V(g)^{-1}\rho_V(h)^{-1}$$
$$= \frac{1}{|G|}\sum_{g\in G}\rho_V(hg)\,\psi\,\rho_V(hg)^{-1}$$
$$= \frac{1}{|G|}\sum_{g\in G}\rho_V(g)\,\psi\,\rho_V(g)^{-1} = \psi$$

$\Rightarrow$ ψ is G-equivariant: $\rho_V(h)\psi = \psi\rho_V(h)$ $\forall h\in G$

Pick $v\in V$ and $g\in G$. Then $\pi(\rho_V(g)^{-1}v)\in W$

W subrep $\Rightarrow$ $\rho_V(g)\pi\rho_V(g)^{-1}v\in W$

$\Rightarrow$ $\psi(v)\in W$

$\Rightarrow$ image $\psi\subseteq W$

Pick $v\in W$. Then $\rho_V(g)^{-1}v\in W$, so $\pi(\rho_V(g)^{-1}v) = \rho_V(g)^{-1}v$.

$\Rightarrow$ $\rho_V(g)\pi\rho_V(g)^{-1}v = v$

$\Rightarrow$ $\psi(v) = \frac{1}{|G|}\sum_{g\in G}\rho_V(g)\pi\rho_V(g)^{-1}v = \frac{1}{|G|}\cdot|G|\cdot v = v$

$\Rightarrow$ image $\psi = W$. & $\psi^2 = \psi$. ✓

Set $U = \ker\psi$. Since ψ is G-equivariant, U is subrep. $\square$

Rmk. ① Any 1-dim rep is irreducible.

② Take $V = \mathbb{k}^2$, $\rho_V(g) = \mathrm{id}$ $\forall g\in G$.

V reducible since any 1-dim subspace is a subrep.

Any other 1-dim subspace is complement, so choice generally not unique.

Lemma $\Rightarrow$ (If char $\mathbb{k}=0$ or char $\mathbb{k}\nmid|G|$ then indecomposable $\Rightarrow$ irreducible.)

Note: $G = \{1, z\}\cong\mathbb{Z}/2$, $\rho(z) = \begin{bmatrix}1&1\\0&1\end{bmatrix}$ when char $\mathbb{k}=2$

is example of indecomposable, reducible rep.

Thm (Schur's lemma) Let V, W be irred. reps of G.
$\varphi: V \to W$ be G-equivariant.

① Either φ is an isomorphism or $\varphi = 0$.

② If $\mathbb{k}$ alg. closed, and $V=W$, then φ is scalar multiple of identity.

③ If $\mathbb{k}$ alg. closed, then $\dim \mathrm{Hom}_G(V,W) \leq 1$.

Pf. ① $\ker \varphi \subseteq V$ is subrep $\Rightarrow$ $\ker \varphi = 0$ or $\ker \varphi = V$.

If $\ker \varphi = V \Rightarrow \varphi = 0$

Else If $\ker \varphi = 0$, then φ is injective & $V \cong \mathrm{image}\, \varphi$

$\mathrm{image}\, \varphi \subseteq W$ is subrep $\Rightarrow$ $\mathrm{image}\, \cancel{\varphi = 0}$ or $\mathrm{image}\, \varphi = W$
$\Downarrow$
φ is an isom.

② Pick eigenvalue λ of φ, consider $\varphi - \lambda\, \mathrm{id}_V$, which is also G-equivariant, and has nonzero kernel. ① $\Rightarrow$ $\varphi = \lambda \cdot \mathrm{id}_V$.

③ Suppose $\mathrm{Hom}_G(V,W) \neq 0$. Pick nonzero $\varphi: V \to W$ G-equivariant.
① $\Rightarrow$ φ isom, so can identify V & W ② $\Rightarrow$ φ is scalar multiple of identity $\Rightarrow$ $\dim \mathrm{Hom}_G(V,W) = 1$. $\square$

Example Take $\mathbb{k} = \mathbb{R}$, $G = \mathbb{Z}/4 = \{1, z, z^2, z^3\}$, $V = \mathbb{R}^2$
$\rho(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ gives rep. In fact, irreducible: $\rho(z)$ has no eigenvectors (over $\mathbb{R}$). Since G is abelian, $\rho(z)$ commutes w/ $\rho(g)$ $\forall g \in G$, so $\rho(z): V \to V$ is G-equivariant, but not scalar mult. of identity!

[If we extend scalars to $\mathbb{C}$, V becomes reducible]

Def. V is <u>absolutely irreducible</u> if it stays irreducible whenever $\mathbb{k}$ is extended to a larger field.

Thm (Maschke). Suppose char $\Bbbk = 0$ or char $\Bbbk = p > 0$ & $p \nmid |G|$.

Every rep of G is isomorphic to a direct sum of irreducible reps.

Furthermore, decomposition is unique: if $V \cong V_1 \oplus \cdots \oplus V_r$ ← irreducible

$\cong V_1' \oplus \cdots \oplus V_s'$ ← nonzero

then $r = s$ & $\exists$ permutation σ s.t. $V_i \cong V'_{\sigma(i)}$.

Pf. Induction on $\dim V$. If $\dim V = 1$, then V is irred. ✓

Suppose $\dim V > 1$, V reducible. Pick nonzero subrep $W \subseteq V$ s.t. $W \neq V$

By lemma, $\exists$ $U \subseteq V$ subrep s.t. $V \cong W \oplus U$.

$\Rightarrow$ $\dim W < \dim V$, $\dim U < \dim V$.

By induction, both U, W are isom. to direct sum of irred. reps.
combine them.

Uniqueness: Suppose $V \cong V_1 \oplus \cdots \oplus V_r \cong V_1' \oplus \cdots \oplus V_s'$

Consider $id: V \to V$. Consider maps $V_i \to V \to V_j'$

Schur $\Rightarrow$ each map is either isom. or 0.

let $W_1, \ldots W_a$ be all of the V_i which are isom. to V_1

let $W_1', \ldots W_b'$ be all of the V_j' which are isom. to V_1

$\Rightarrow$ restriction of identity: $V \to V$ to

$W_1 \oplus \cdots \oplus W_a \longrightarrow W_1' \oplus \cdots \oplus W_b'$ must be an isom.

$\Rightarrow$ $a = b$. Remove these summands and repeat. $\square$